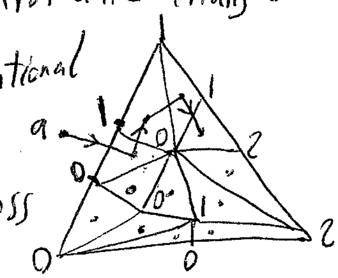
10/27

# Complexity of Finding Nash Equil., PPAD-Completeness ①

Nash's Theorem (1951): Every finite game has a mixed Nash equilibrium.
Can we find it efficiently (in polynomial-time) or It is NP-complete?
(still we don't know, we only know it is PPAD-complete).

Proof of Nash's theorem is based on Brouwer's fixedpoint theorem. Stating that every continuous function $f$ from the $n$-dimensional unit ball to itself has a fixed point, a point $x$ such that $f(x)=x$.
Nash's proof is a clever reduction of the existence of a mixed equilibrium to the existence of such a fixed point.
Brouwer's theorem is well-known for its non-constructive nature and finding it was known to be hard problem (1989), but recent proofs shows Nash is (PPAD-complete) precisely as hard as fixed Brouwer's fixed point.

Sperner's Lemma: A triangle and its triangulation is given. Each of the vertices of the big triangle has a unique color and each vertex on an edge of the big triangle can only have the color of one of the two colors of the end points of its edge. In any such scenario there exists a trichromatic triangle.



Pf: First there is an edge of colors 0-1 on the side 01 (two-dimentional Sperner). Add a vertex a out side and one vertex inside each small triangle. Start from a and always only go across edges that are colored 0-1. Since the triangles that we start and we see have no color 2, then they should be 0,1,1 or 0, 0,1. So if ~~start~~ enter with 0-1, there is always another edge 0,1 that we exit. But since we see each triangle once and their number is limited (why?) we should see a triangle of degree one (which is monocromatic). To avoid unlimited case and be more formal, in the graph above connect every two triangles (indeed vertices corresponding to triangles) with a common edge 0-1. vertex a has odd degree and any other vertex has even degree if there is no monocromatic triangle which is impossible, since # of vertices of odd degree in any undirected graph is even. ∎

Application of Sperner's Lemma to the proof of two dimensional Brouwer fixed-point theorem: Brouwer's theorem in 2D can be interpreted (by homotopy) as saying that a continuous mapping from a triangular region T to itself must have a fixed point. Suppose the corners of T are points (vectors) $x_0, x_1, x_2$. Due to convexity we can express each $x \in T$ uniquely as a weighted averages of the corners: $x = a_0x_0 + a_1x_1 + a_2x_2$ where $\sum a_i = 1$ and each $a_i \geq 0$. So we can specify $x$ by its vector of coefficients $a = (a_0, a_1, a_2)$. Now define sets $S_0, S_1, S_2$ for each mapping $f$ by saying that $a \in S_i$ if $a_i' \leq a_i$ where $f(a) = a'$. Because the coefficients of each point sum to one, every point in T belongs to at least one of the sets, and a point belongs to all three sets if and only if it is a fixed point for f. We want to show that the three sets have a common point.

Given an arbitrary simplicial subdivision (triangulation) of T, for each node a choose a label i such that $a \in S_i$. Note that the points on the edge of T opposite $x_i$ have ith coordinate 0. Since their ith coordinate cannot decrease under f, we can choose some label different from i for each point on that edge. Note that $x_i \in S_i$ since coefficient of $x_i$ is one (max) when we write $x_i$ in terms of $x_1, x_2, x_3$. So the resulting labeling is proper and we can apply Sperner's lemma to obtain a completely labeled cell. We can repeat the process using triangulation with smaller and smaller cells, we obtain a sequence of smaller and smaller completely labeled triangles; call them $\{x_i, y_i, z_i\}$ receiving labels 0, 1, 2, respectively. In each $S_i$, we obtain an infinite sequence of points. The rest is topology to say since f is continuous $\{x_i\}, \{y_i\}, \{z_i\}$ converges to $x_{ik}, y_{ik}, z_{ik}$ and since the distances between them approches zero (since triangles are inside each others, $x_{ik}, y_{ik}, z_{ik}$ converges to the same point which belongs to all $S_0, S_1, S_2$ and we are done ☒

Thus Sperner's lemma $\Longrightarrow$ Fixed point Brouwer's Theorem $\xRightarrow{\text{thus}}$ Nash's Theorem.

Note that in the proof of Sperner's lemma, the graph has a very simple "path-like" structure: All vertices (except the source) have either one or two edges incident upon them — the important point is that there is definitely at least one known endpoints of the path (i.e., the source vertex of odd-degree in the outside. We must conclude that there is another end point of degree one, i.e. the triangle with all three colors.

We can make the graph directed such that starting from the source vertex we can assign a direction to its incident edges, at most one coming in and at most one going out and do it such that it is consistent from one vertex to another. Indeed the existence proof of Nash's theorem (for two player games though something similar holds for the the general case as well) has the following abstract structure. A directed graph is defined on vertices of the polytope where all strategies are easily recognizable and represented). Each one of these vertices has indegree and outdegree at most one; therefore the graph is a set of paths and cycles (even simpler that Sperner's lemma). By necessity there is one vertex with no incoming edges and one outgoing edge, called a standard source (in the polytope of 2-player Nash, the all-zero vertex). We must conclude that there must be a sink; a nash equilibrium. Any such proof suggest a simple algorithm for finding a solution: start from the standard source, and follow the path until you find a sink (in case of 2-player Nash is called the Lemke-Howson algorithm).

Unfortanately this is not an efficient algorithm because the number of vertices in the graph is exponentially large (and indeed this happens in 2-player Nash). Indeed besides Nash, there is a host of of other computational problems such as Sperner's with exponentially large set of vertices or finding an approximate Brouwer's fixed point in which whose solution space can be set up as the set of all sinks (and all non standard) sources in a directed graph defined on a finite but exponentially large set of vertices with each vertex has indegree and out degree at most one in which 1) telling a vertex 2) its neighbors and 3) telling which one is the predecessor and successor (direction of each edge) are computationally easy and we are given an standard source and we want to find a sink or any non-standard source.

All these problems comprise the complexity class called PPAD. Solving a PPAD problem is to telescope the long path and arrive at a sink fast without exhaustive search. We do not know PPAD belongs to P or not

IN case of NP, we have a useful notion of difficulty - NP-completeness. Similarly we have PPAD-completeness, meaning all problems in PPAD ~~reduces~~ can be reduced this problem and if we can solve one of these PPAD-complete problems efficiently we can solve all. Nash, Brouwer, Sperner, finding Arrow-Debreu equilibria and many more are PPAD-complete. However PPAD completeness is weaker evidence of intractability that NP-completeness, and thus it could be easily that PPAD = P ≠ NP.

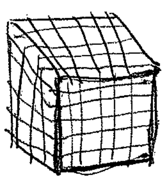
There ~~Some~~ somehow compelling evidence that PPAD ≠ P otherwise some non-trivial results happen but then so what? :)

The rest we discuss PPAD-completeness of Nash

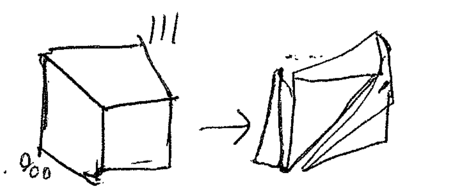
Note that the proof of Sperner is PPAD-complete is relatively easy because the problem is essentially the same as the directed graph problem.

We can generalize Sperners to 3 and more dimensions.

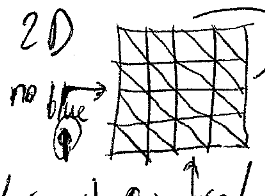
in 3-dimensions



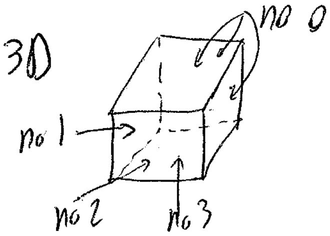
- first we divide each dimension into integer multiples of $2^{-m}$ for some integer m, each called cubelets.
- then simplicization of each cubelet into six tetrahedra all having corner 000, 111.




legal coloring in 2D




in 3D




Sperner's lemma: for a legal coloring (no vertex of the same color as the face) there is an all chromatic simplex.

So it can be proved that 3D-Sperner is PPAD complete (reduction to large directed graph).

Next we define the Brouwer fixed point for 3D:

We convert coloring of the cube to the direction of the displacement vector $f(x)-x$.

color 0 (3-face color) $\longrightarrow \delta_0 = (-1,-1,-1)\,2^{-m}$

color 1 $\rightarrow \delta_1 = (1,0,0)\,2^{-m}$ color 2 $\rightarrow \delta_2 = (0,1,0)\,2^{-m}$ color 3 $\rightarrow \delta_3 = (0,0,1)\,2^{-m}$.

More precisely, we consider a discrete version of the Brower fix point theorem. It is presented in terms of a function $\phi$ from the three-dimensional unit cube to itself. The cube is subdivided into $2^m$ equal cublets and the function needed only be described at all cubelet centers. At a cubelet center $x$, $\phi(x)$ can take four values $x+\delta_i, i=0,..,3$, where $\delta_i$ are defined before. We want to find a fixed point which is defined here to be any internal cublet corner point such that among its eight adjacent cublets, all four possible displacement $\delta_{i=0,..,3}$ are present.

Essentially the proof of PPAD-completeness for this discrete Brouwer is the proof of PPAD for 3D Sperner (colors go to displacement).

Finally we can reduce Brower for Nash with many but constant # players. All these player have just two strategies 0 and 1; therefore we can think of any mixed strategy of a player as a number in [0,1]. There are three players called leaders who cordinate a point in the cube. others will respond by analyzing these coordinates to identify the cubelet wherein this point lies and by computing (by a simulation of a boolean circuit) the displacements $\delta_i$ at the cubelet and adjacent cublets. The resulting choices by the player will incentivize the 3 leaders to change their mixed strategy - unless the point is a fixed point of $\phi$, in which case the three players will not change strategies and we are at a Nash equilibrium. First these could be simulated by many players, then 4 players, then 3-player and conjectured 2-player is indeed in P. finally the conjecture was wrong and even 2-player Nash is PPAD-complete.