Frugality in Auctions and Mechanism Design: (some sort of profit maximization)

Setting: The auctioneer is a buyer who wants to purchase goods or services.
Agents are sellers who have costs for providing the good or service.

The auctioneer's goal is to maximize the social welfare, but how much should he pay? Frugality of a mechanism is the amount by which we overpay.

Note that for a single good, Vickrey's auction payment = second cheapest price.

In general, we might have more complicated settings. E.g.,
Path auction: given a graph (network), the auctioneer wants to buy an s-t path.
Each edge is owned by a different agent. The auctioneer will try to buy the shortest path (maximize the social welfare). Note that here each edge has some internal cost to do the transfer, say.

OR spanning tree auction: the same setting, but the auctioneer wants to buy a spanning tree instead. For single-item auctions, we can show Vickrey is optimal in terms of profit maximization.

We can use VCG, but when VCG or any other incentive-compatible mechanism achieve a total payment of at most the second cheapest solution?

Shortest path: 4
VCG pays: 11 - (4 - 1) = 8, not counting the edge.

MST: 3
VCG pays total: 12 - (3 - 1) = 10
  to ab: 12 - (3 - 1) = 10
  to bc: 13 - (3 - 1) = 11

In total, VCG pays 4 * 8 = 32 while the second shortest path is 11, thus we pay 32 / 11 more.

A Frugal Mechanism should minimize the total payment (e.g., path auc. was bad, while MST auc. was good, but is it true in general?)

VCG for path auction can be as bad as \( O(n) \), where \( n \) is the length of the path.

Is this the flaw of VCG or any incentive-compatible mechanism has this problem? We show indeed this the problem for every incentive-compatible mech.
Thm: For any incentive compatible mechanism \( M \) and any graph \( G \) with two vertex disjoint \( S-T \) paths \( P \) and \( P' \), there is a valuation profile \( v \) such that \( M \) pays on \( S(\{P, P'\}) \) factor more than the cost of the second cheapest path.

Corollary: There exists a graph for which any incentive compatible mechanism has a worst-case \( S(k) \) factor overpayment. [Recall that two disjoint paths each length \( \frac{n}{2} \).]

**Proof:** Let \( k = |P| \) and \( k' = |P'| \). Ignore all edges not in \( P \) or \( P' \) by setting their cost to infinity. Define \( V_{ij} \) as follows: The cost of the \( i \)th edge of \( P \) is \( v_i = \frac{1}{\sqrt{k}} \) and all other edges cost zero.

Graph: 

![Diagram of graph](image)

In the instance \( V_{ij} \), note that \( M \) on \( V_{ij} \) must select either all edges in path \( P \) or all edges in path \( P' \) as winners (since we have only two edge disjoint path \( P \) and \( P' \) as options). Define directed bipartite graph \( G' = (P, P', E) \) on edges in paths \( P \) and \( P' \) as follows. For any pair of vertices \((i, j)\) in the bipartite graph, there is either a directed edge \((i, j)\) in \( E' \) saying \( M \) on \( V_{ij} \) selecting path \( P \) (called forward edges) or a directed edge \((j, i)\) denoting \( M \) on \( V_{ij} \) selecting path \( P \) (called backward edges). Note that \( |E'| = k' \cdot n \cdot \log k \). W.l.o.g assume \( E' \) has more forward edges and thus at least \( \frac{k'}{2} \) forward edges. Since there are \( k \) edges in path \( P \), there must be one edge \( i \) with at least \( \frac{k}{2} \) forward edges. Let \( F(i) \) represent neighbors of \( i \) in the bipartite graph with \( |F(i)| \geq \frac{k}{2} \). Now consider the valuation profile \( V_{io} \) in which the cost of \( i \)th edge of \( P \) is \( v_i = \frac{1}{\sqrt{k}} \) and all other edges cost zero.

By the definition of \( F(i) \), to any \( j \in F(i) \), \( M \) on instance \( V_{ij} \) selects path \( P' \).

Since \( M \) is incentive compatible, its allocation rule must be monotone, i.e., if agent \( j \) is selected when bidding \( v_{ij} \) it must be selected when bidding 0 (otherwise agent \( j \) would have positive utility). Therefore \( M \) selects \( P' \) on \( V_{io} \). It is called weak monotonicity (WMON).
In addition, for $j \in F(i)$, the payment should be at least $\frac{1}{\sqrt{k}}$, since when the valuation profile is $V_{i,j}$, the payment should be at least $\frac{1}{\sqrt{k}}$ (otherwise, $j$ will receive negative utility). By the direct characterization of incentive compatible mechanisms, we know when other bidder valuations and the outcome are the same, the payment should also be the same. So payment for $j$ is at least $\frac{1}{\sqrt{k}}$ when the valuation profile is $V_{i,j}$. So on $V_{i,j}$, the total payment of $M$ is at least $F(i) \times \frac{1}{\sqrt{k}} \geq \frac{\sqrt{k}}{2}$. Remember that the second cheapest path is $p$ with cost $\frac{1}{\sqrt{k}}$. Therefore, the overpayment is at least $\frac{\sqrt{k}}{2}$ as desired.

1. Direct characterization of incentive compatible mechanisms:
   A mechanism is incentive compatible if and only if:
   1) the payment $p_i$ does not depend on $V_i$, but only on alternative outcomes chosen for $V_i$.
   2) The mechanism optimizes for each player. That is for every $V_i$, we have $f(V_i, V_i) = \text{argmax}_{a} (V_i(a) - p_i)$ where the quantification is over all alternatives in the range of $f(\cdot, V_i)$.

Thus no incentive compatible mechanism is more frugal than VCG in the worst case.

However we can show:

Thm: The total VCG cost for spanning tree auction is at most the cost of the second cheapest disjoint spanning tree.

Even more generally:

Thm: VCG has frugality ratio (Payment \over Cost of second disjoint solution) one if and only if the feasible sets of the set system are the bases of a matroid.

The proof is involved and omitted. (See References in section 13.5 of the book.)

In the rest, we are talking more on profit maximization in mechanism design, called optimal mechanism design in economics. We even get rid of truthfulness.

The topic has been considered by Guruswami et al. in CS.
Thm: There is a simple \( \log n + \log m \) approximation for envy-free pricing when bidders are single-minded and items are available in unlimited supply (where \( n \) is the number of bidders and \( m \) is the number of items) and we price items individually.

pf: We consider pricings in which all items are priced the same. The candidate prices are \( q_i = \frac{v_i}{15_i} \) (agent \( i \) wants set \( S_i \) with value \( v_i \) and buys it if the price is less than \( q_i \)).

Assume consumers are ordered \( q_1 \geq q_2 \geq \ldots \geq q_n \). If all items are priced at \( q_i \), then the seller profit is \( R_i = \sum_{1 \leq j \leq i} 15_j \cdot \frac{v_i}{15_i} \). By rearranging we have \( v_i = \frac{15_i}{15_j} R_i \). Note that the price is too much for \( j > i \). Because the algorithm chooses the price \( R \) maximizing profit, we have that \( R_i \leq R \) for all \( i \), and thus \( \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} \frac{15_i}{15_j} R_i \leq R \sum_{i=1}^{n} \frac{15_i}{15_j} \leq R \sum_{j=1}^{m} \frac{15_j}{15_j} \leq R \ln(\sum_{i=1}^{n} 15_i) \).

\( \sum v_i \) is a trivial upper bound on the optimum, so the theorem follows because \( \sum 15_i \leq nm \).

The analysis of this algorithm is tight. However, the algorithm seems very simple. Can we do better?

Indeed we showed no (essentially):

The unique coverage problem:
- Given a universe \( U \) of \( n \) elements,
- Given a collection \( S \) of subsets of \( U \).
- Find a sub-collection \( S' \), a subset of \( S \), which maximizes the number of elements that are uniquely covered, i.e., appear in exactly one set of \( S' \).

There is a simple \( O(\log n) \) approximation for unique coverage based on the ideas above for the pricing problem.
However, we show surprisingly:

Thm: The unique coverage problem is hard to approximate within a factor better than $O(\log^2 n)$, unless $NP$ has a sub-exponential algo.

Indeed the problem is $O(\log^5 n)$ hard or even $O(\log n)$ hard under stronger but plausible complexity assumptions.

The proof of Thm is involved though.

From pricing to unique coverage: (unique coverage is a special case)

- Each set $S_i$ maps to an item $I_i$.
- Each element of $E_i$ of the universe $U$ maps to a bager $b_i$.
- For edges, bager $b_i$ has a valuation $1$ for one bundle $B_i$, namely the set of items $I_j$ that corresponding to sets $S_j$ containing $E_i$.

The zero-one pricing of items is corresponding to unique coverage.

The zero-one case is indeed corresponding to the general fractional case. (maybe with some constant approximation).

Two well-known special cases: highway problem: we have $n$ items (highway segments) $1, 2, \ldots n$ and each customer (driver) has a desired bundle that consists interval $[i, j]$. Lots of effort, $O(\log n)$ because of the general case, $O(\log n)$ in 2010 and finally PTAS in SODA'11. The problem is strongly $NP$-hard though.

The graph vertex problem: we have all bundles $|S_i| = 2$. For bipartite graphs we can get 2 and for general graph 4 (by ignoring half of edges.) Indeed it seems 4 is tight for combinatorial approaches.

It is 2-hard assuming Unique Game Conjecture and at least $\Omega(\log \log n)$ otherwise (APRX'10).

In summary there are lots of interesting problems for profit maximization esp. considered in computer science due to online auctions.