(sec 13.5 of AGT) 9/29

Frugality in Auctions and Mechanism Design: (some sort of profit maximization)

Setting: The auctioneer is a buyer who wants to purchase goods or services.
Agents are seller who have costs for providing the good or service

The auctioneer's goal is to maximize the social welfare, but how much he should pay? $\boxed{\text{Frugality of a mechanism is the amount by which we over pay.}}$

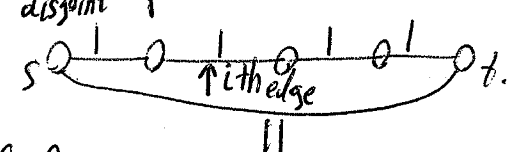
note that for a single good, in Vickrey's auction payment = second cheapest price

In general, we might have more complicated settings. e.g.

Path auction: given a graph (network), the auctioneer wants to buy an s-t path. Each edge is owned by a different agent. The auctioneer will try to buy the shortest path (maximize the social welfare). note that here each edge has some internal cost to do the tranfer say.

OR spanning tree (ST) auction: the same setting but the auctioneer wants to buy a spanning tree instead. $\boxed{\text{For [redacted] single-item auctions we can show Vickrey is optimal in term of profit maximization}}$

We can use VGG, but when VCG or any other incentive compatible mechanism achieve a total payment of at most the second cheapest solution?



shortest path: 4

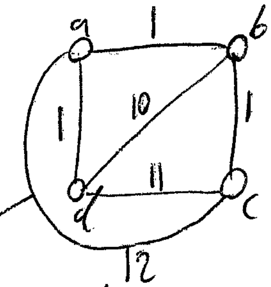
VCG pays: $\underbrace{11}_{\text{second option}} - (4 - \underbrace{1}_{\text{not counting ith edge}}) = 8$ to ith edge

In total, VCG pays $4 \times 8 = 32$
while the second shortest path is 11, thus we pay $\frac{32}{11}$ X more

MST: 3

VCG pays to ad: $12-(3-1)=10$
to ab: $12-(3-1)=10$
to bc: $13-(3-1)=11$

Second MST which is edge disjoint:
$10+11+12=33$
The ratio is $\frac{31}{33}$



A Frugal Mechanism should minimize the total payment (e.g. path auc. was bad, while MST auc. was good but is it TRUE in general?

VCG for path auction can be as bad as $\Omega(n)$ where n is the length of the path. (why? see the example above)

Is this the flaw of VCG or any incentive-compatible Mechanism has this problem? we show indeed this the problem for every incentive-compatible mech.

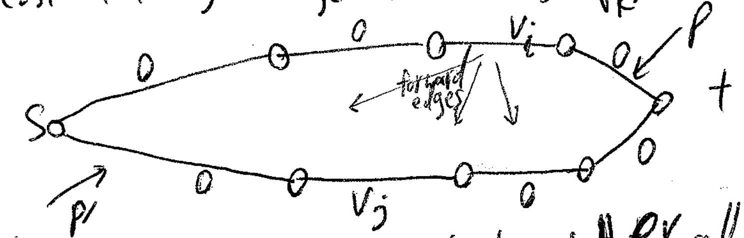
Thm: For any incentive compatible mechanism M and any graph G with two vertex disjoint s-t paths P and P', there is a valuation profile v such that M pays an $\Omega(\sqrt{|P|\,|P'|})$ factor more than the cost of the second cheapest path.

- Corollary: There exists a graph for which any incentive compatible mechanism has a worst-case $\Omega(n)$ factor overpayment. $\boxed{\text{Pf: say two disjoint path each of length } \frac{n}{2}}$

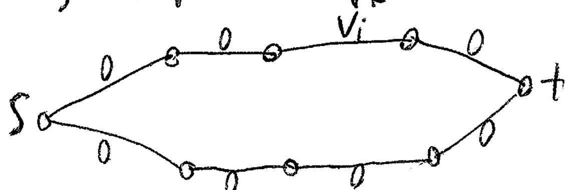
Pf: let $k=|P|$ and $k'=|P'|$. Ignore all edges not in P or P' by setting their cost to infinity. We define V_{ij} as follows: The cost of ith edge of P is $v_i = \frac{1}{\sqrt{k}}$. The cost of the jth edge of P' is $v_j = \frac{1}{\sqrt{k'}}$ and all other edges cost zero.

Graph :

instance V_{ij}



Note that M on V_{ij} must select either all edges in path P or all edges in path P' as winners (since we have only two edge disjoint path P and P' as options). Define directed bipartite graph $G'=(P, P', E')$ on edges in path P and P' as follows. For any pair of vertices (i,j) in the bipartite graph, there is either a directed edge (i,j) in E' saying M on V_{ij} selecting path P' (called forward edges) or a directed edge (j,i) denoting M on V_{ij} selecting path P (called backward edges). Note that $|E'| = kk'$. W.l.o.g assume E' has more forward edges and thus at least $\frac{kk'}{2}$ forward edges. Since there are k edges in path P, there must be one edge i with at least $\frac{k'}{2}$ forward edges ~~in the neighbors of~~ in the bipartite graph. Let $F(i)$ represent neighbors of i with $|F(i)| \geq \frac{k'}{2}$. Now consider the valuation profile V_{i0} in which the cost of ith edge of P is $v_i = \frac{1}{\sqrt{k}}$ and all other edges cost zero.



By the definition of $F(i)$, for any $j \in F(i)$, M on instance V_{ij} selects path P'. Since M is incentive compatible, its allocation rule must be monotone, i.e., if agent j is selected when bidding v_j, it must be selected when bidding 0 (otherwise agent of 0 price bid v_j and has positive utility). Therefore M selects P' on V_{i0}.

It is called weak monotonicity (WMON)

In addition, for $j \in F(i)$, the payment should be at least $\frac{1}{\sqrt{K'}}$, since when the valuation profile is V_{ij}, the payment should be at least $\frac{1}{\sqrt{K'}}$ (otherwise, j will receive negative utility). By the direct characterization① of incentive compatible mechanisms, we know when other bidders valuations and the outcome are the same, the payment should also be the same. So payment for j is at least $\frac{1}{\sqrt{K'}}$ when the valuation profile is V_{i0}. So on V_{i0}, the total payment of M is at least $F(i) \times \frac{1}{\sqrt{K'}} \geq \frac{\sqrt{K}}{2}$. Remember that the second cheapest path is P with cost $\frac{1}{\sqrt{K}}$. Therefore, the overpayment ratio is at least $\frac{\sqrt{K}K}{2}$ as desired. ∎

> ①: Direct characterization of incentive compatible Mechanisms:
> A mechanism is incentive compatible if and only if:
> 1) the payment P_i does not depend on V_i, but only on alternative (outcome) chosen $f(V_i, V_{-i})$
> 2) The mechanism optimizes for each player. That is for every V_i, we have that $f(V_i, V_{-i}) \in \arg\max_a (V_i(a) - P_i)$ where the quantification is over all alternatives in the range of $f(\cdot, V_{-i})$.

Thus NO incentive compatible mechanism is more frugal than VCG in the worst case.

However we can show:

Thm: The total VCG cost for spanning tree auction is at most the cost of the second cheapest disjoint spanning tree.

even more generally:

Thm: VCG has frugality ratio $\left(\frac{\text{Payment}}{\text{cost of second disjoint solution}}\right)$ one if and only if the feasible sets of the set system are the bases of a matroid.

The proof is involved and omitted. (see References in section 13.5 of AGT book)

In the rest, we are talking more on Profit Maximization in Mechanism design, called optimal mechanism design in economics.

we even get rid of truthfulness.

The topic first has been considered by Guruswami et al. in CS.

we consider a natural case study here.

Thm: There is simple $\log n + \lg m$ approximation for ~~only free~~ pricing when bidders are single-minded and items are available in unlimited supply (where n is the number of bidders and m is the number of items) and we price items individually)

Pf: we consider pricings in which all items are priced the same.
The candidate prices are $q_i = \frac{v_i}{|S_i|}$ (agent i wants set S_i with value v_i and buys it only if the price is less than v_i)

Assume consumers are ordered $q_1 \geq q_2 \geq \cdots \geq q_n$. If all items are priced at q_i, then the seller profit is $R_i = \sum_{1 \leq j \leq i} |S_j| \cdot \frac{v_i}{|S_i|}$. By rearranging we have $v_i = \frac{|S_i| R_i}{\sum_{j=1}^{i} |S_j|}$. Note that the price is too much for $j > i$. Because the algorithm chooses the price R maximizing profit, we have that $R_i \leq R$ for all i and thus

$$\boxed{\frac{|S_i|}{\Sigma^i} = \frac{|S_i|}{|S_i| + \Sigma^{i-1}} = \sum_{k=1}^{|S_i|} \frac{1}{|S_i| + \Sigma} \leq \sum_{k=1}^{|S_i|} \frac{1}{k + \Sigma}}$$

why?

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} \frac{|S_i| R_i}{\sum_{j=1}^{i} |S_j|} \leq R \sum_{i=1}^{n} \sum_{k=1}^{|S_i|} \frac{1}{k + \sum_{j=1}^{i-1} |S_j|} \leq R \cdot \ln(\sum_i |S_i|$$

$\sum v_i$ is a trivial upper bound on the optimum, so the theorem follows because $\sum_i |S_i| \leq nm$. ∎

The analysis of this algorithm is tight. However, the algorithm seem very simple. Can we do better?

Indeed we showed no (essentially):

The unique coverage problem:

- Given a universe U of n elements,
- Given a collection S of subsets of U.
- find a sub-collection S', a subset of S, which maximizes the number of elements that are uniquely covered, i.e., appear in exactly one set of S'.

There is a simple $O(\lg n)$ approximation for unique coverage based on ideas above for the pricing problem.

However, we show surprisingly:

<u>Thm</u>: The unique coverage problem is hard to approximate within a factor better than $O(\log^c n)$, $0<c<1$, unless NP has a sub-exponential algo.

Indeed the problem is $O(\log^{1/3} n)$ hard or even $O(\log n)$ hard under stronger but plausible complexity assumptions.

The proof of Thm is involved though.

From pricing to Unique coverage. (unique coverage is a special case)

→ Each set S_i maps to an Item I_i

→ Each element of e_i of the Universe U maps to a buyer b_i.

→ for edges, buyer b_i has a valuation 1 for once bundle B_i, namely the set of items I_j that corresponding to sets S_j containing e_i.

The zero-one pricing of items is corresponding to Unique Coverage

The zero-one case is indeed corresponding to the general / fractional case. (maybe with some constant approximation).

Two well-known special cases: <u>highway problem</u>: we have n items, (highway segments) $1, 2, \ldots n$ and each customer (driver) has a desired bundle that consists interval $[i, i']$. lots of effort, $O(\log n)$ because of the general case, $O\left(\frac{\log n}{\log\log n}\right)$ in 2010 and finally PTAS in SODA'11. The problem is strongly NP-Hard though.

The graph vertex problem: we have all bundles $|S_i| = 2$. For bipartite graphs we can get 2 and for generall graph 4 (by ignoring half of edges.) Indeed it seems 4 is tight for combinatorial approaches. It is 2-hard assuming Unique Game Conjecture and at least $\frac{17}{16}$ hard otherwise (APPROX 2010).

In summary there are lots of Interesting Problems for Profit Maximization esp. considered in Computer Science due to online auctions.