Network Cournot Competition

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The VI Problem

- Given a set $\mathcal{K} \subseteq \mathbb{R}^n$ and a mapping $\mathbf{F} : \mathcal{K} \to \mathbb{R}^n$, the VI problem VI$(\mathcal{K}, \mathbf{F})$ is to find a vector $\mathbf{x}^\star \in \mathcal{K}$ such that

$$(\mathbf{y} - \mathbf{x}^\star)^T \mathbf{F}(\mathbf{x}^\star) \geq 0 \qquad \forall \mathbf{y} \in \mathcal{K}.$$

- Let SOL$(\mathcal{K}, \mathbf{F})$ denote the solution set of VI$(\mathcal{K}, \mathbf{F})$.

Why do We Care?

- In general games are hard to solve.
- Potential Games with convex potential functions are exceptions.
- But we don't really care about potential functions.

Why do We Care?

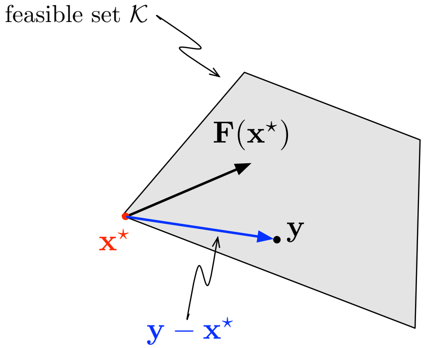
- The gradient of potential function gives marginal utilities for a game.
  - I.e., how do utilities at a point vary with player's strategy at a point.
- Jacobian of the gradient is called Hessian.
- Convex potential games are interesting because Hessian of a convex function is a symmetric positive definite matrix. Such games and associated functions have very nice properties.

Why do We Care?

- It turns out that we don't need symmetry of the Hessian.
- When we relax this condition, the variation of utilities can no longer be captured by a single potential function.
- However, as long as Jacobian of marginal utilities is positive semi-definite, all the nice properties of convex potential games are maintained.
- Equilibria of games can be represented (and solved) by Monotone Variational Inequalities.
- We use this fact to generalize results for an important market model.
  - Later half of this presentation.

Geometrical Interpretation

- A feasible point $\mathbf{x}^\star$ that is a solution of the VI$(\mathcal{K}, \mathbf{F})$: $\mathbf{F}(\mathbf{x}^\star)$ forms an acute angle with all the feasible vectors $\mathbf{y} - \mathbf{x}^\star$



Convex Optimization as a VI

- Convex optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in \mathcal{K} \end{array}$$

  where $\mathcal{K} \subseteq \mathbb{R}^n$ is a convex set and $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function.

- Minimum principle: The problem above is equivalent to finding a point $\mathbf{x}^\star \in \mathcal{K}$ such that

$$(\mathbf{y} - \mathbf{x}^\star)^T \nabla f(\mathbf{x}^\star) \geq 0 \quad \forall \mathbf{y} \in \mathcal{K} \quad \Longleftrightarrow \quad \text{VI}(\mathcal{K}, \nabla f)$$

  which is a special case of VI with $\mathbf{F} = \nabla f$.

VI's are More General

- It seems that a VI is more general than a convex optimization problem only when $\mathbf{F} \neq \nabla f$.
- But is it really that significative? The answer is affirmative.
- The VI$(\mathcal{K}, \mathbf{F})$ encompasses a wider range of problems than classical optimization whenever $\mathbf{F} \neq \nabla f$ ($\Leftrightarrow$ $\mathbf{F}$ has not a symmetric Jacobian).
- Some examples of relevant problems that can be cast as a VI include NEPs, GNEPs, system of equations, nonlinear complementary problems, fixed-point problems, saddle-point problems, etc.

System of Equations

- In some engineering problems, we may not want to minimize a function but instead finding a solution to a system of equations:

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}.$$

- This can be cast as a VI by choosing $\mathcal{K} = \mathbb{R}^n$.
- Hence,

$$\mathbf{F}(\mathbf{x}) = \mathbf{0} \qquad \Longleftrightarrow \qquad \mathsf{VI}(\mathbb{R}^n, \mathbf{F}).$$

Non-linear Complementarity Problem

- The NCP is a unifying mathematical framework that includes linear programming, quadratic programming, and bi-matrix games.
- The NCP($\mathbf{F}$) is to find a vector $\mathbf{x}^\star$ such that

$$\text{NCP}(\mathbf{F}): \qquad \mathbf{0} \le \mathbf{x}^\star \perp \mathbf{F}(\mathbf{x}^\star) \ge \mathbf{0}.$$

- An NCP can be cast as a VI by choosing $\mathcal{K} = \mathbb{R}^n_+$:

$$\text{NCP}(\mathbf{F}) \qquad \Longleftrightarrow \qquad \text{VI}(\mathbb{R}^n_+, \mathbf{F}).$$

KKT Conditions

- Suppose that the (convex) feasible set $\mathcal{K}$ of VI$(\mathcal{K}, \mathbf{F})$ is described by a set of inequalities and equalities

$$\mathcal{K} = \{\mathbf{x} : \mathbf{g}(\mathbf{x}) \leq \mathbf{0},\ \mathbf{h}(\mathbf{x}) = \mathbf{0}\}$$

  and some constraint qualification holds.

- Then VI$(\mathcal{K}, \mathbf{F})$ is equivalent to its KKT conditions:

$$\begin{aligned} \mathbf{0} &= \mathbf{F}(\mathbf{x}) + \nabla \mathbf{g}(\mathbf{x})^T \boldsymbol{\lambda} + \nabla \mathbf{h}(\mathbf{x})^T \boldsymbol{\nu} \\ \mathbf{0} &\leq \boldsymbol{\lambda} \perp \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \\ \mathbf{0} &= \mathbf{h}(\mathbf{x}). \end{aligned}$$

KKT Conditions

- To derive the KKT conditions it suffices to realize that if $\mathbf{x}$ is a solution to VI$(\mathcal{K}, \mathbf{F})$ then it must solve the following convex optimization problem and vice-versa:

$$\begin{array}{ll} \underset{\mathbf{y}}{\text{minimize}} & \mathbf{y}^T \mathbf{F}(\mathbf{x}^\star) \\ \text{subject to} & \mathbf{y} \in \mathcal{K}. \end{array}$$

(Otherwise, there would be a point $\mathbf{y}$ with $\mathbf{y}^T \mathbf{F}(\mathbf{x}^\star) < \mathbf{x}^{\star T} \mathbf{F}(\mathbf{x}^\star)$ which would imply $(\mathbf{y} - \mathbf{x}^\star)^T \mathbf{F}(\mathbf{x}^\star) < 0$.)

- The KKT conditions of the VI follow from the KKT conditions of this problem noting that the gradient of the objective is $\mathbf{F}(\mathbf{x}^\star)$.

Primal-Dual Representation

- We can now capitalize on the KKT conditions of VI$(\mathcal{K}, \mathbf{F})$ to derive an alternative representation of the VI involving not only the primal variable $\mathbf{x}$ but also the dual variables $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$.

- Consider VI$(\tilde{\mathcal{K}}, \tilde{\mathbf{F}})$ with $\tilde{\mathcal{K}} = \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p$ and

$$\tilde{\mathbf{F}}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \begin{bmatrix} \mathbf{F}(\mathbf{x}) + \nabla \mathbf{g}(\mathbf{x})^T \boldsymbol{\lambda} + \nabla \mathbf{h}(\mathbf{x})^T \boldsymbol{\nu} \\ -\mathbf{g}(\mathbf{x}) \\ \mathbf{h}(\mathbf{x}) \end{bmatrix}.$$

- The KKT conditions of VI$(\tilde{\mathcal{K}}, \tilde{\mathbf{F}})$ coincide with those of VI$(\mathcal{K}, \mathbf{F})$. Hence, both VIs are equivalent.

Primal-Dual Representation

- VI($\mathcal{K}, \mathbf{F}$) is the original (primal) representation whereas VI($\tilde{\mathcal{K}}, \tilde{\mathbf{F}}$) is the so-called primal-dual form as it makes explicit both primal and dual variables.
- In fact, this primal-dual form is the VI representation of the KKT conditions of the original VI.

Monotonicity is Like Convexity

- Monotonicity properties of vector functions.
- Convex programming - a special case: monotonicity properties are satisfied immediately by gradient maps of convex functions.
- In a sense, role of monotonicity in VIs is similar to that of convexity in optimization.
- Existence (uniqueness) of solutions of VIs and convexity of solution sets under monotonicity properties.

Definitions

- A mapping $\mathbf{F} : \mathcal{K} \rightarrow \mathbb{R}^n$ is said to be

  (i) *monotone* on $\mathcal{K}$ if

  $$(\mathbf{x} - \mathbf{y})^T (\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})) \geq 0, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$$

  (ii) *strictly monotone* on $\mathcal{K}$ if

  $$(\mathbf{x} - \mathbf{y})^T (\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})) > 0, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K} \text{ and } \mathbf{x} \neq \mathbf{y}$$

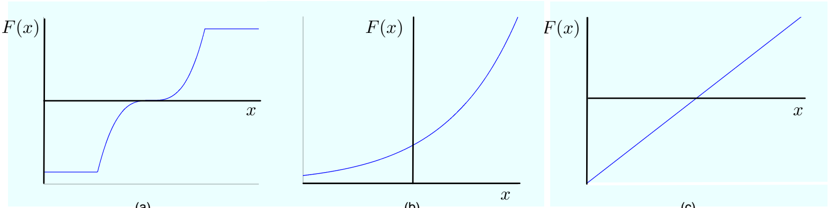
  (iii) *strongly* monotone on $\mathcal{Q}$ if there exists constant $c_{\text{sm}} > 0$ such that

  $$(\mathbf{x} - \mathbf{y})^T (\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})) \geq c_{\text{sm}} \| \mathbf{x} - \mathbf{y} \|^2, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$$

  The constant c_{sm} is called strong monotonicity constant.

Examples

- Example of (a) monotone, (b) strictly monotone, and (c) strongly monotone functions:

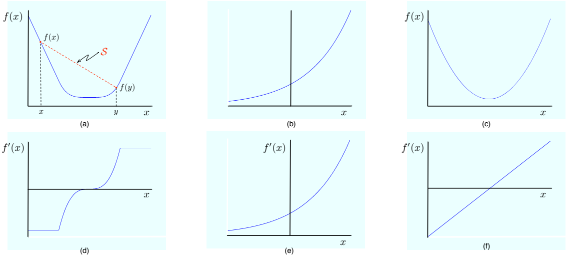


(a) (b) (c)

Monotonicity of Gradient and Convexity

- If $\mathbf{F} = \nabla f$, the monotonicity properties can be related to the convexity properties of f

a) f convex $\Leftrightarrow$ ∇f monotone $\Leftrightarrow$ $\nabla^2 f \succeq \mathbf{0}$
b) f strictly convex $\Leftrightarrow$ ∇f strictly monotone $\Leftarrow$ $\nabla^2 f \succ \mathbf{0}$
c) f strongly convex $\Leftrightarrow$ ∇f strongly monotone $\Leftrightarrow$ $\nabla^2 f - c\mathbf{I} \succeq \mathbf{0}$



Why are Monotone Mappings Important

- Arise from important classes of optimization/game-theoretic problems.
- Can articulate existence/uniqueness statements for such problems and VIs.
- Convergence properties of algorithms may sometimes (but not always) be restricted to such monotone problems.

Projection Algorithm

- If F were gradient of a convex function, it would be the same as gradient descent.

Algorithm 1: Projection algorithm with constant step-size

(S.0) : Choose any $\mathbf{x}^{(0)} \in \mathcal{K}$, and the step size $\tau > 0$; set $n = 0$.

(S.1) : If $\mathbf{x}^{(n)} = \prod_{\mathcal{K}} \left(\mathbf{x}^{(n)} - \mathbf{F}(\mathbf{x}^{(n)})\right)$, then: STOP.

(S.2) : Compute

$$\mathbf{x}^{(n+1)} = \prod_{\mathcal{K}} \left(\mathbf{x}^{(n)} - \tau\, \mathbf{F}(\mathbf{x}^{(n)})\right).$$

(S.3) : Set $n \leftarrow n + 1$; go to (S.1).

- In order to ensure the convergence of the sequence $\left\{\mathbf{x}^{(n+1)}\right\}_{n=0}^{\infty}$ (or a subsequence) to a fixed point of $\mathbf{\Phi}$, one needs some conditions of the mapping $\mathbf{F}$ and the step size $\tau > 0$. (Note that instead of a scalar step size, one can also use a positive definite matrix.)

Convergence

- **Theorem.** *Let* $\mathbf{F} : \mathcal{K} \to \mathbb{R}^n$*, where* $\mathcal{K} \subseteq \mathbb{R}^n$ *is closed and convex. Suppose* $\mathbf{F}$ *is strongly monotone and Lipschitz continuous on* $\mathcal{K}$*:* $\forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$*,*

$$(\mathbf{x}-\mathbf{y})^T(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})) \geq c_{\mathbf{F}}\|\mathbf{x}-\mathbf{y}\|^2, \quad \text{and} \quad \|\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})\| \leq L_{\mathbf{F}}\|\mathbf{x}-\mathbf{y}\|$$

and let

$$0 < \tau < \frac{2c_{\mathbf{F}}}{L_{\mathbf{F}}^2}.$$

Then, the mapping $\prod_{\mathcal{K}}\left(\mathbf{x}^{(n)} - \tau\mathbf{F}(\mathbf{x}^{(n)})\right)$ *is a contraction in the Euclidean norm with contraction factor*

$$\eta = 1 - L_{\mathbf{F}}^2\,\tau\left(\frac{2c_{\mathbf{F}}}{L_{\mathbf{F}}^2} - \tau\right).$$

Therefore, any sequence $\left\{\mathbf{x}^{(n)}\right\}_{n=0}^{\infty}$ *generated by Algorithm 1 converges linearly to the unique solution of the VI*$(\mathcal{K}, \mathbf{F})$*.*

Classical Model of Cournot Competition

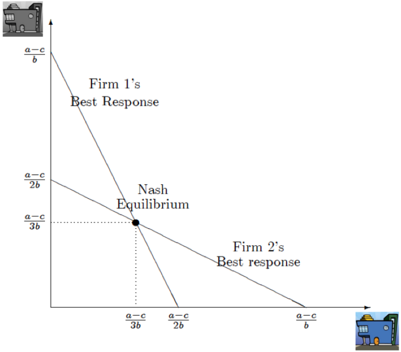
- Introduced by Antoine Cournot in 1838.
- All firms produce a homogeneous product.
- All the production is sold in the market.
- The market price is a function of total supply and is fixed for all firms.
- Firms have a cost function for the quantity they produce.
- Quantity is the strategic variable.

Cournot Oligopoly

- Single good produced by n firms.
- Cost for firm i for producing q_i units: $C_i(q_i)$, where C_i is nonnegative and increasing
- If firms' total output is Q then market price is $P(Q)$,
- P is nonincreasing
- Profit of firm i, as a function of all the firms' outputs: $\pi_i(q_1, ..., q_n) = q_i P(Q) - C_i(q_i)$

Cournot Oligopoly : Example

- Two firms.
- Inverse demand: $P(Q) = \max\{0, a - bQ\}$.
- constant unit cost: $C_i(q_i) = cq_i$.
- Utility function : $\pi_1(q_1, q_2) = q_1(a - bq_1 - bq_2) - cq_1$.

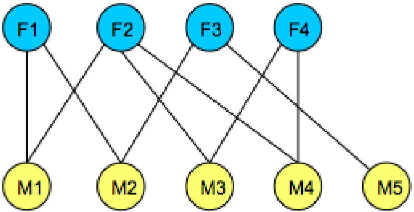


Utility Markets

- The distribution network fragments the market, e.g., natural gas, water and electricity.
- We can assume each firm has access to a subset of existing submarkets.
- Relations between suppliers and submarkets form a complex network.
- A market having access to multiple suppliers enjoys a lower price as a result of the competition.
- Multiple firms competing in multiple markets.

Notation

- n firms denoted by $\mathcal{F}$ that produce a homogeneous good.
- m markets denoted by $\mathcal{M}$.
- A bipartite graph $G = (\mathcal{F}, \mathcal{M}, \mathcal{E})$.
- An edge between vertices in the bipartite graph if firm j is able to produce the good in market i.



Notation

- Inverse demand (price) functions P_i for market i.
  - Function of total quantity produced in that market.
- Cost function c_j for for firm j.
  - Function of vector of quantities produced by the firm in each market.
- $N(j)$ is the set of neighbors of a node j in G.
- Revenue of firm j, denoted by R_j, is:

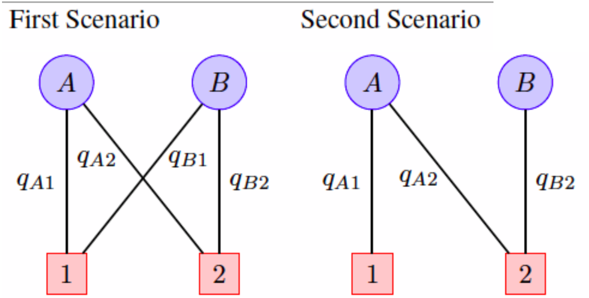
$$R_j = \sum_{i \in N(j)} P_i(D_i) q_{ij} \tag{1}$$

- Profit of firm j, denoted by π_j, is:

$$\pi_j = R_j - c_j(\vec{s_j}). \tag{2}$$

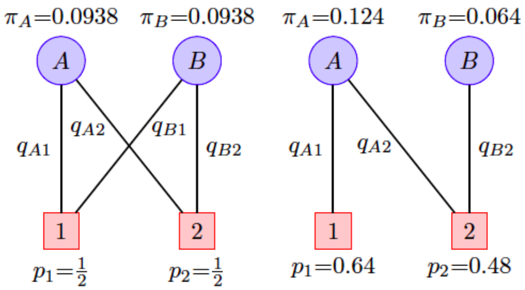
An Example

- Firm $i \in \{A, B\}$ produces quantity q_{ij} of the good in market $j \in \{1, 2\}$.



An Example

- Let $p_i(\mathbf{q}) = 1 - q_{Ai} - q_{Bi}$ be the market prices.
- Let $c_i(\mathbf{q}) = \frac{1}{2}(q_{i1} + q_{i2})^2$ be the cost of production.
- Profit of firm A in second scenario:
$\pi_A(\mathbf{q}) = q_{A1}(1 - q_{A1}) + q_{A2}(1 - q_{A2} - q_{B2}) - \frac{1}{2}(q_{A1} + q_{A2})^2$.



Cournot Nash Equilibrium

- Quantities produced by firms represent a Cournot-Nash equilibrium if none of the firms can increase their profits by unilaterally changing production quantities.

Example cont'd

Any Nash equilibrium of this game satisfies the set of equations:

$$\textbf{Either } q_{A1} = 0 \text{ and } \frac{\partial \pi_A}{\partial q_{A1}} \leq 0 \textbf{ Or } \frac{\partial \pi_A}{\partial q_{A1}} = 0$$

$$\textbf{Either } q_{A2} = 0 \text{ and } \frac{\partial \pi_A}{\partial q_{A2}} \leq 0 \textbf{ Or } \frac{\partial \pi_A}{\partial q_{A2}} = 0$$

$$\textbf{Either } q_{B1} = 0 \text{ and } \frac{\partial \pi_A}{\partial q_{B1}} \leq 0 \textbf{ Or } \frac{\partial \pi_A}{\partial q_{B1}} = 0$$

$$\textbf{Either } q_{B2} = 0 \text{ and } \frac{\partial \pi_A}{\partial q_{B2}} \leq 0 \textbf{ Or } \frac{\partial \pi_A}{\partial q_{B2}} = 0$$