CMSC 858F: Algorithmic Game Theory
Spring 2014
Dueling Games

Instructor: Mohammad T. Hajiaghayi
Scribe: Sina Dehghani, Milad Gholami, and Saeed Seddighin

May 6, 2014

1 Overview

The problem of finding Nash equilibria is one of the fundamental problems in Algorithmic Game Theory, and has made a nice connection between Game Theory and Computer Science [9]. While Nash [10] introduced the concept of Nash equilibrium and proved every game has a mixed Nash equilibrium, finding a Nash equilibrium in reasonable time seems to be essential. The Nash equilibrium can be used as a basic equilibrium concept, only if it is efficiently computable such that it can be used for predicting the outcome of a real-world game. This highlights the role of computer scientists in this area which aim to design efficient algorithms for finding Nash equilibria of various games (see, e.g., [1, 2, 3, 4, 5, 6, 7]). In this study, we followed this line of research and to understand the polynomial-time algorithms for finding Nash equilibria of a broad range of zero-sum games.

In a **dueling game**, which was initiated by [8] (STOC 2011), two competitors try to design an algorithm for an optimization problem with an element of uncertainty, and the winner is the player who better solves the optimization problem.

2 Problem Definition

Formally, dueling games are two player zero-sum games with a set of strategies X, a set of possible situations Ω, a probability distribution p over Ω, and a cost function $\mathsf{c} : \mathsf{X} \times \Omega \to \mathbb{R}$ that defines the cost measure for each player based on her strategy and the element of uncertainty. The payoff of each player is defined as the probability that she beats her opponent minus the the probability that she is beaten. More precisely the utility function is defined as

$$h^A(x, y) = -h^B(x, y) = \Pr_{\omega \sim p}[c(x, \omega) < c(y, \omega)] - \Pr_{\omega \sim p}[c(x, \omega) > c(y, \omega)]$$

where x and y are strategies for player A and B respectively.

In the following there are two dueling games mentioned in [8].

Binary search tree duel. In the Binary search tree duel, there is a set of elements Ω and a probability distribution p over Ω. Each player is going to construct a binary search tree containing the elements of Ω. Strategy x beats strategy y for element $\omega \in \Omega$ if and only if the path from ω to the root in x is shorter than the path from ω to the root in y. Thus, the set of strategies X is the set of all binary search trees with elements of Ω, and $c(x, \omega)$ is defined to be the depth of element ω in strategy x.

Ranking duel. In the Ranking duel, there is a set of m pages Ω, and a probability distribution p over Ω, notifying the probability that each page is going to be searched. In the Ranking duel, two search engines compete against each other. Each search engine has to provide a permutation of these pages, and a player beats the other if page ω comes earlier in her permutation. Hence, set of strategies X is all $m!$ permutations of the pages and for permutation $x = (x_1, x_2, \dots, x_m)$ and page ω, $c(x, \omega) = i$ iff $\omega = x_i$.

3 Finding a Nash equilibrium

They first introduce a set of dueling games called *bilinear dueling games*, then provide an approach to find a Nash equilibrium in such games. Afterwards, by reducing some dueling games to bilinear dueling games they find the Nash equilibrium in those games.

3.1 Bilinear Dueling Games

Bilinear dueling games are the games in which each strategy $\hat{x}$ of Player A is a $n(A)$ dimensional point in Euclidean space. Let S_A be the convex hull of these strategy points and each point in S_A is a mixed strategy for player A. Similarly define strategy $\hat{y}$, $n(B)$, and S_B for player B. Moreover, utility function $h^A(\hat{x}, \hat{y})$ is of the form $\hat{x}^t M \hat{y}$ where M is a $n(A) \times n(B)$ matrix. Again for player B, $h^B(\hat{x}, \hat{y}) = -h^A(\hat{x}, \hat{y})$.

Immorlica et al.[8] provided a method for finding an equilibrium of a class of bilinear games which is defined as follows:

Definition 1 *Polynomially-representable bilinear dueling games: A bilinear dueling game is polynomially representable if one can present the convex hull of strategies* S_A *and* S_B *with* m *polynomial linear constraints, i.e. there are* m *vectors* $\{v_1, v_2, \dots, v_m\}$ *and* m *real numbers* $\{b_1, b_2, \dots, b_m\}$ *such that* $S_A = \{\hat{x} \in R^{n(A)} | \forall i \in \{1, 2, \dots, m\}, v_i.\hat{x} \geq b_i\}$. *Similarly* $S_B = \{\hat{y} \in R^{n(B)} | \forall i \in \{1, 2, \dots, m'\}, v'_i.\hat{y} \geq b'_i\}$.

3.2 Reduction

For the reduction, they provide the following four step approach:

1. Find an efficient function which transfers each strategy x to a point $\hat{x}$ in $n(A)$ dimensional space
2. Find a payoff matrix M such that the payoff of the game where players have strategies $\hat{x}$ and $\hat{y}$ is $\hat{x}^t M \hat{y}$, denoting that payoff is a bilinear function.
3. Find a polynomial set of hyperplanes which represents the feasible space of strategy points.
4. A randomized rounding algorithm to transfer the strategy point back to a mixed strategy in the original space.

3.3 Finding an equilibrium of the bilinear game

This is an instance of a zero-sum game where players' strategy spaces are subject to further linear constraints (about the expectation). The standard method is to turn the max-min problem into a max-max linear program by taking the dual of the inner minimization.

References

[1] Adrian Vetta. Nash equilibria in competitive societies, with applications to facility location, traffic routing and auctions. In FOCS, pages 416425, 2002.

[2] Xi Chen and Xiaotie Deng. Settling the complexity of two-player nash equilibrium. In FOCS, pages 261272, 2006.

[3] Constantinos Daskalakis and Christos Papadimitriou. Computing equilibria in anonymous games. In FOCS, pages 8393, 2007.

[4] Baruch Awerbuch, Yossi Azar, Amir Epstein, Vahab Seyed Mirrokni, and Alexander Skopalik. Fast convergence to nearly optimal solutions in potential games. In EC, pages 264273, 2008.

[5] Dimitris Fotakis, Spyros Kontogiannis, Elias Koutsoupias, Marios Mavronicolas, and Paul Spirakis. The structure and complexity of nash equilibria for a selfish routing game. Theor. Comput. Sci., 410 (36):33053326, 2009.

[6] Constantinos Daskalakis, Grant Schoenebeck, Gregory Valiant, and Paul Valiant. On the complexity of nash equilibria of action-graph games. In SODA, pages 710719, 2009.

[7] Constantinos Daskalakis, Paul W Goldberg, and Christos H Papadimitriou. The complexity of com- puting a nash equilibrium. SIAM J. Comput., 39(1):195259, 2009.

[8] Nicole Immorlica, Adam Tauman Kalai, Brendan Lucier, Ankur Moitra, Andrew Postlewaite, and Moshe Tennenholtz. Dueling algorithms. pages 215224, 2011.

[9] Christos H Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. J. Comput. Syst. Sci., 48(3):498532, 1994.

[10] John F Nash Jr. The bargaining problem. Econometrica: Journal of the Econometric Society, pages 155162, 1950.