CMSC 858F: Algorithmic Game Theory
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Multicast and Network Formation Games

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1 Introduction

We define the overarching goal of Network Formation Games to analyze the way (efficient) networks form under the existence of selfish agents, excluding a central authority. Generally, selfih behavior is defined as having two components: players want to *minimize the expenses* they incur for building the network and they also seek to obtain a *high quality of service* from the network. The strategy of each player is choosing a particular set of edges(generally path) according to such selfish behavior.

Out interest is to study the social cot of such a network, particularly, to answer questions like "'Does the game have a Nash equilibrium?"' and if it does, "'How much worse is it than the optimum?"'. The main tool used in answering such questions is the potential function method.

Within this class of game, there are several variations. One refers to the costs incurred by the agents. There are ways of varying the cost of building an edge, depending on latency and congestion. Another important variation is the cost sharing mechanism that decides the way agent pay for their strategies and, in particular, for an edge.

2 Potential Functions and Games

Definition 1 *For any finite game, an* **exact potential function** *is a function that maps evert strategy vector* S *to some real value and satisfies the following condition:*
if $S = (S_1, S_2, ..., S_k), S_i{}' \neq S_i$ *is an alternate strategy for some player* i*, and* $S' = (S_{-i}, S_i{}')$*,then*

$$\Phi(S) - \Phi(S') = u_i(S') - u_i(S).$$

A game that possesses an exact potential function is called an **exact potential game**.

Theorem 1 *Every potential game has at leat one pure Nash equilibrium, namely the strategy S that minimizes $\Phi(S)$.*

Proof: S is stable when no player can increase his/her utility by choosing a different strategy, i.e. when $\Phi(S)$ is at a local minimum. ■

Theorem 2 *In any finite potential game,* **best response dynamics** *always converge to a Nash equilibrium.*

Proof: Best response dynamics is the strategy which produces the most favorable outcome for a player, taking other player's strategies as given. Therefore, it simulates local search on Φ.(improving moves for players decrease Φ) ■

Theorem 3 *Assume that, for any outcome S,*

$$\frac{\mathsf{cost}(S)}{A} \leq \Phi(S) \leq B \cdot \mathsf{cost}(S)$$

for some constants $A, B > 0$. Then the price of stability is at most AB.

Proof: Best response dynamics is the strategy which produces the most favorable outcome for a player, taking other player's strategies as given. Therefore, it simulates local search on Φ.(improving moves for players decrease Φ) ■

Now let's consider some games that are relevant to network formation games.

3 Global Connection Game

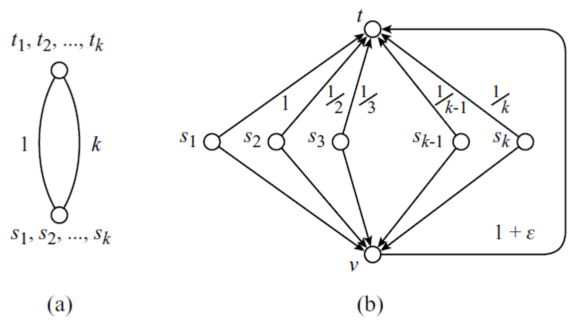
3.1 The Model

- a directed graph $G = (V, E)$
- nonnegative edge costs c_e for all edges $e \in E$. We will consider these costs to be fixed, though there are games in which that is not the case.
- k players, each with a specified source node s_i and sink node t_i

Player i's goal is to build a network in which t_i and s_i are connected, while minimizing construction costs. A strategy for player i is therefore a path from t_i to s_i. Apart from these parameters, we also need to set up the cost sharing mechanism, the way in which agents will contribute to building a network and, in particular, the set of edges in their strategy. A natural choice is the **Shapley cost-sharing mechanism**, also known as the fair mechanism, fair cost allocation, egalitarian cost sharing etc. That just means that all the agents using a particular edge share its cost. Formally, if k_e is the number of players whose path contains e, then e assigns a cost share of c_e/k_e to each of them. Also, the social objective for this game is simply the cost of the constructed network.(sum of the cost played by all players)

3.2 Price of Stability

Let us first look at some examples, taken from the AGT textbook:



We can see that:

- In figure a:
  - 2 equilibria, one of value 1(also OPT) and the other one of value k
  - price of stability is 1
  - price of anarchy is k (in fact, the PoA cannot exceed k on any network)
- In figure b:
  - 1 equilibrium, of value $H_k = \Sigma_{j=1}^{k} \frac{1}{j}$
  - OPT is at $1 + \epsilon$
  - price of stability is roughly H_k

Theorem 4 *The price of stability in the global connection game with* k *players is at most* H_k.

Proof: Define the function $\Phi(S) = \Sigma_e \Phi_e(S)$, where, for each edge, $\Phi_e(S) = c_e \cdot H_{k_e}$. Φ is a potential function! Moreover, $\mathsf{cost}(S) \leq \Phi(S) \leq H_k \cdot \mathsf{cost}(S)$. ■

The proof extends to the following cases:

- each edge has a **nondecreasing concave cost function** $c_e(x)$, where x is the number of players using edge e.
- c_e is monotone increasing and concave and we add delays d_i
- add capacities

A nice observation is that the proof doesn't depend on the topology of the network, which allows us to extend it to a game in which players attempt to share a set of resources. However, there is a big difference between the directed and the undirected case. The same happens when we consider weighted players.

4 The Non-Cooperative Multicast Game

4.1 The Model

The mode for the Non-Cooperative Multicast Game is basically similar to the Global Connection Game, except:

- the graph is undirected
- all players are interested in connecting to the same sink

4.2 Price of Anarchy

While Figure a discouraged us from from studying the Price of Anarchy, [2] notice that:

- the expensive solution cannot be reached if we initially start with an "empty" configuration and let users join one-by-one
- this leads to an *online* version of the game, introduced by [2]

Furthermore, under this assumption, the following results are obtained:

- upper bound of $O(\sqrt{n}\log^2 n)$
- lower bound of $\Omega(\frac{\log n}{\log\log n})$

The upper bound is obtained by considering a two round game:

- first, all players join one-by-one
  - forms a *greedy online Steiner tree* which is only $O(\log n)$ away from the cost of an optimal Steiner tree
  - that, in turn, is $O(\sqrt{n})$ away from OPT
- players take turns in choosing their strategy by best-response
  - in this round, we lose at most another factor of $O(\log n)$ with respect to the cost of the solution obtained from the first round

However, it is NP-hard to find a Nash equilibrium that minimizes the potential function. But there is hope!
We introduce the **Fractional Multicast Game**, in which:

- each user is allowed to split its connection to the source into several paths

- a potential function exists, even for the weighted case

Moreover, a Nash equilibrium that minimizes the potential function can be computed in polynomial time using linear programming. **Proof:**

- split every edge into n copies of it, copy e_i having price c_e/i
- write an LP minimizing the potential function
- characterize an optimal solution(canonical flow)
- rearrange the output flow of the LP into a canonical flow that is not larger than the potential of the original flow f

■

We also obtain that

Theorem 5 *The price of anarchy for the Fractional Multicast Game is* $O(\log n)$.

and

Theorem 6 *A Nash equilibrium exists in the Weighted Fractional Multicast Game.*

[1] improve the bound on the PoA for the integral case by showing that, in Phase 1, the greedy algorithm has competitive ratio $O(\log^2 n)$. We therefore get that:

Theorem 7 *The Nash equilibrium reached by the two-phase Multicast Cost Sharing game with best response dynamics has cost* $O(\log^3 n)\mathrm{OPT}$.

As a sidenote, there is an interesting new game that has just been introduced by [3].

Definition 2 *First, however, let's define the* ***Network Cutting Game****:*

- *players want to cut themselves from nodes in the network*
- *if the player does not meet the cut requirement, it pays a penalty cost*
- *does not, in general, have pure Nash equilibria*
- *for some special case, there exist approximate equilibria*

Definition 3 *The* ***Network Multicut Game*** *is a special instance:*

- *each player* i *wants to disconnect from some specific node* t_i
- *there always exists a 2-approximate Nash equilibrium as cheap as OPT*
- *proof is done by an algorithm that actually assigns edges of OPT to the players*
- *it can be shown that no player can reduce the cost by more than half by deviating from the state*

5 A new model

5.1 Related Games

First, let's look at the inspiration:

Definition 4 ***Network Creation Games***

- *each player can build a set of edges around him*
- *the objective is to be connected to all the other nodes in the graph*
- *the game comes in two flavors:* unilateral *and* bilateral*, depending on the cost-sharing scheme*
- unilateral *: at most one node pays for the edge*
- bilateral*: both of the nodes contribute to the cost of the edge*
- *constant bounds on the price of anarchy have been establihed for a variety of ranges of the cost of an edge*

This motivates our next game:

Definition 5 ***The Contribution Game***

- *introduced by [7]*
- *every player contributes to an edge(relationship) with a certain effort, within the limit of a particular budget*
- *there is a reward function for each edge*
- *the player's wellfare is the total sum of rewards he obtains for the relationships he establishes*
- *mixed results, depending on the nature of the reward function(ex: price of anarchy is at most 2 when the functions are concave)*
- *authors consider pairwise equilibrium, instead of Nash*

5.2 Our Game

This finally brings us to a new game:

Definition 6 ***The PeerWise Game***

- *players have a set of destinations they are trying to reacg*
- *each edge has a latency associated with it*

- *triangle inequality might not apply so it is often the case that a detour is faster*
- *connections(edges) are constructed based on "'mutual advantage"'*
- *the wellfare of the player is equal to the total sum of fastest(min latency) distances to its destinations*

Mutual advantage is defined according to a reward function depending on the node with which the current player wants to establish a connection. The reward function = difference between the player's wellfare when it makes the connection with the node - the player's wellfare when it doesn't connect to the node.

The model is inspired by the PeerWise latency-reducing overlay network introduced by [6]. In PeerWise, "'mutual advantage"' is a principle according to which to users establish a connection only if they can provide resources to each other

6 Further ideas

As possible questions, we have:

- can we use some of the methods in the games presented so far?
- how can we characterize a Nash equilibrium? Is there a potential function?
- it might be wiser to consider pairwise and approximate equilibrium
- can we extend the analysis to the case of multi-source case(i.e. Global Connection Game for an undirected graph)
- what about the case in which we allow for random replays and arrivals? ([2] show that in the case of a semi-random setting, the solution is within $\mathsf{O}(\mathsf{polylog}(n)\sqrt{n} \cdot \mathsf{OPT})$
- in case of the fractional multicast game, can we use an SDP instead of an LP?

References

[1] Moses Charikar, Howard Karloff, Claire Mathieu, Joseph Naor and Michael Saks, *Online multicast with egalitarian cost sharing*, **SPAA'08**

[2] Chandra Chekuri, Julia Chuzhoy, Liane Lewin-Eytan, Joseph Naor, Ariel Orda, *Non-cooperative multicast and facility location games*, **Proceedings of the 7th ACM conference on Electronic commerce**, 2006

[3] Elliot Anshelevich, Bugra Cakurlu, Ameya Hate, *Strategic Multiway Cut and Multicut Game*, manuscript 2010

[4] Martin Hoefer, *Strategic cooperation in cost sharing games*,**CoRR**,abs/1003.3131, 2010

[5] Elliot Anshelevich, Anirban Dasgupta, Jon Kleinberg, Eva Tardos, Tom Wexler and Tim Roughgarden, *The price of stability for network design with fair cost allocation*, **FOCS'04**

[6] Cristina Lumezanu, Randy Baden, Dave Levin, Neil Spring and Bobby Bhattacharjee, *Symbiotic relationships in Internet routing Overlays*, **NSDI**, 2009

[7] Elliot Anshelevich and Martin Hoefer, *Contribution games in social networks*,**CoRR**, abs/1004.1854, 2010

[8] Alex Fabrikant, Ankur Luthra, Elitza Maneva, Christos H. Papadimitriou and Scott Shenker, *On a newtork creation game*,**PODC'03**