Network Bargaining Games and Cooperative Game Theory

Hamid Mahini
University of Maryland

Bargaining

Bargaining

- Common wisdom has it that the whole is more than the sum of the parts.

- Two cooperative agents are often capable of generating a surplus that neither could achieve alone.
  - Trade creates value
  - Music studio, Music band - sell an album
  - Publishing house, author - print and sell a book
  - Job position
  - Partnership formation

Example

- Bargaining over a division of a cake

- Take-it-or-leave-it rule
  - I offer you a piece.
  - If you accept, we trade.
  - If you reject, no one eats.

- What is the equilibrium?
  - Power to the proposer.

Example

- Bargaining over a division of a cake

- Take-it-or-counteroffer rule
  - I offer you a piece.
  - If you accept, we trade.
  - If you reject, you may counteroffer (and δ of the cake remains, the rest melt)

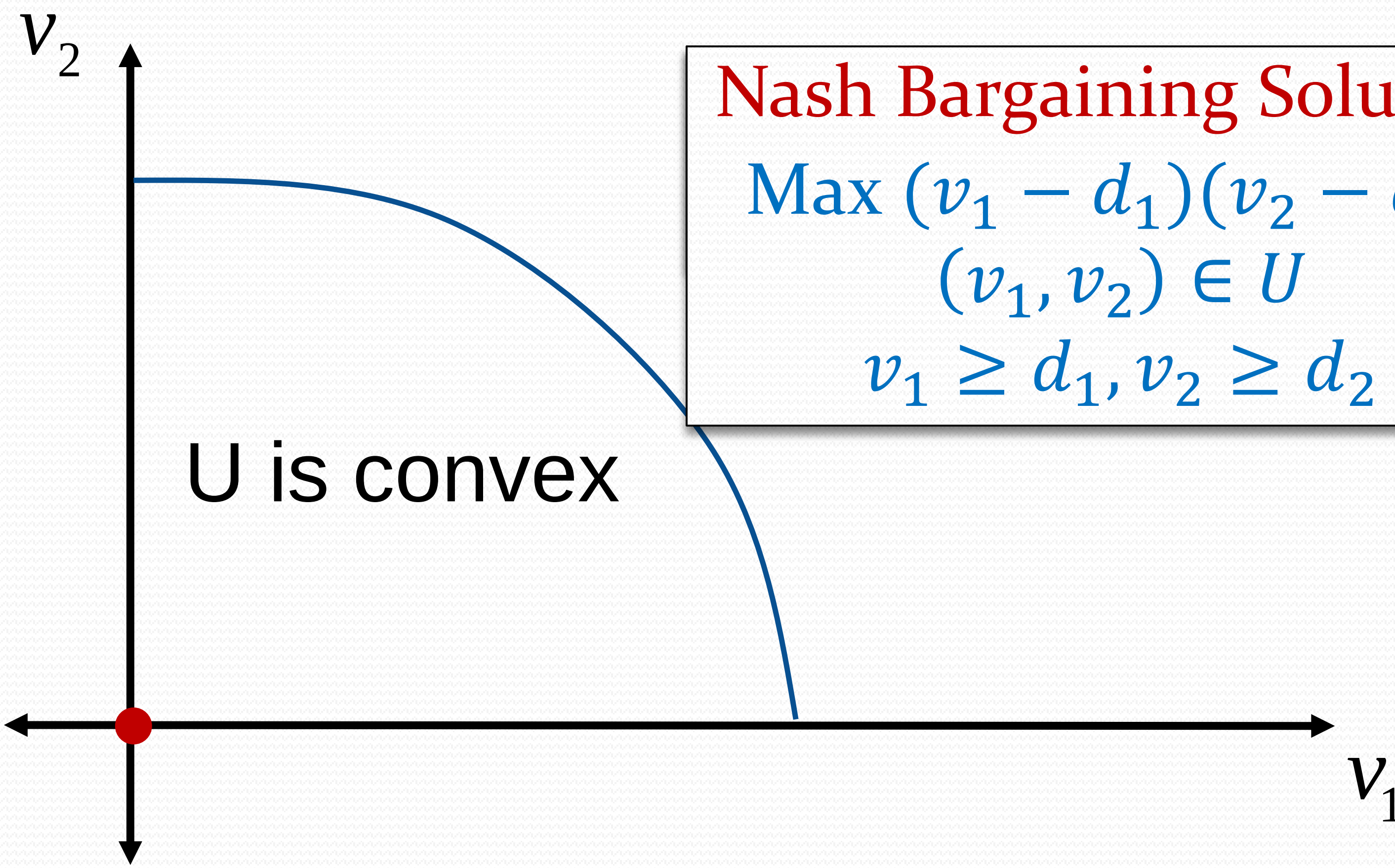
- What is the equilibrium?

Bargaining

What would be the outcome?

What is the right solution?

Nash Bargaining Solution



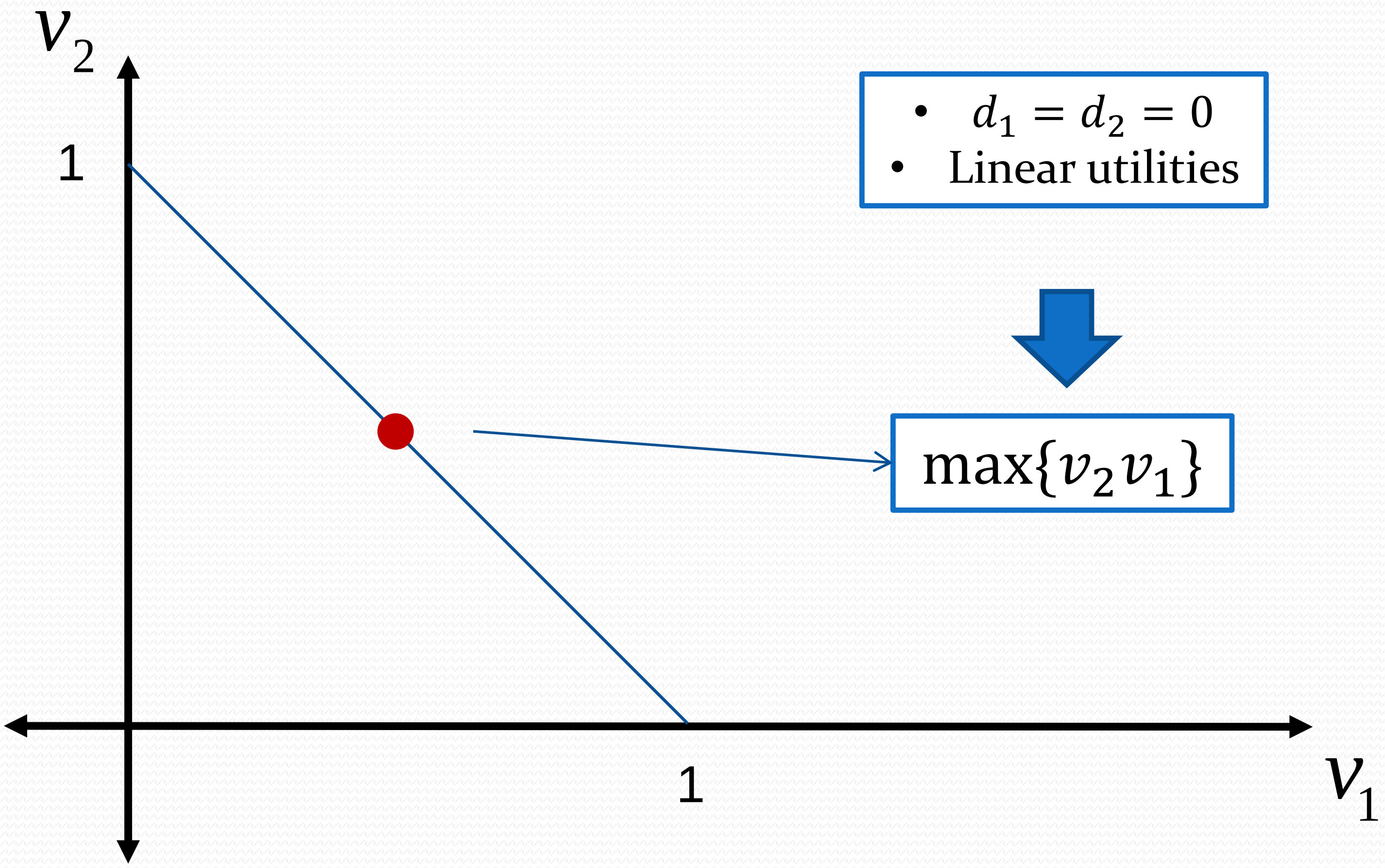
Nash Bargaining Solution

$$\text{Max } (v_1 - d_1)(v_2 - d_2)$$
$$(v_1, v_2) \in U$$
$$v_1 \geq d_1, v_2 \geq d_2$$

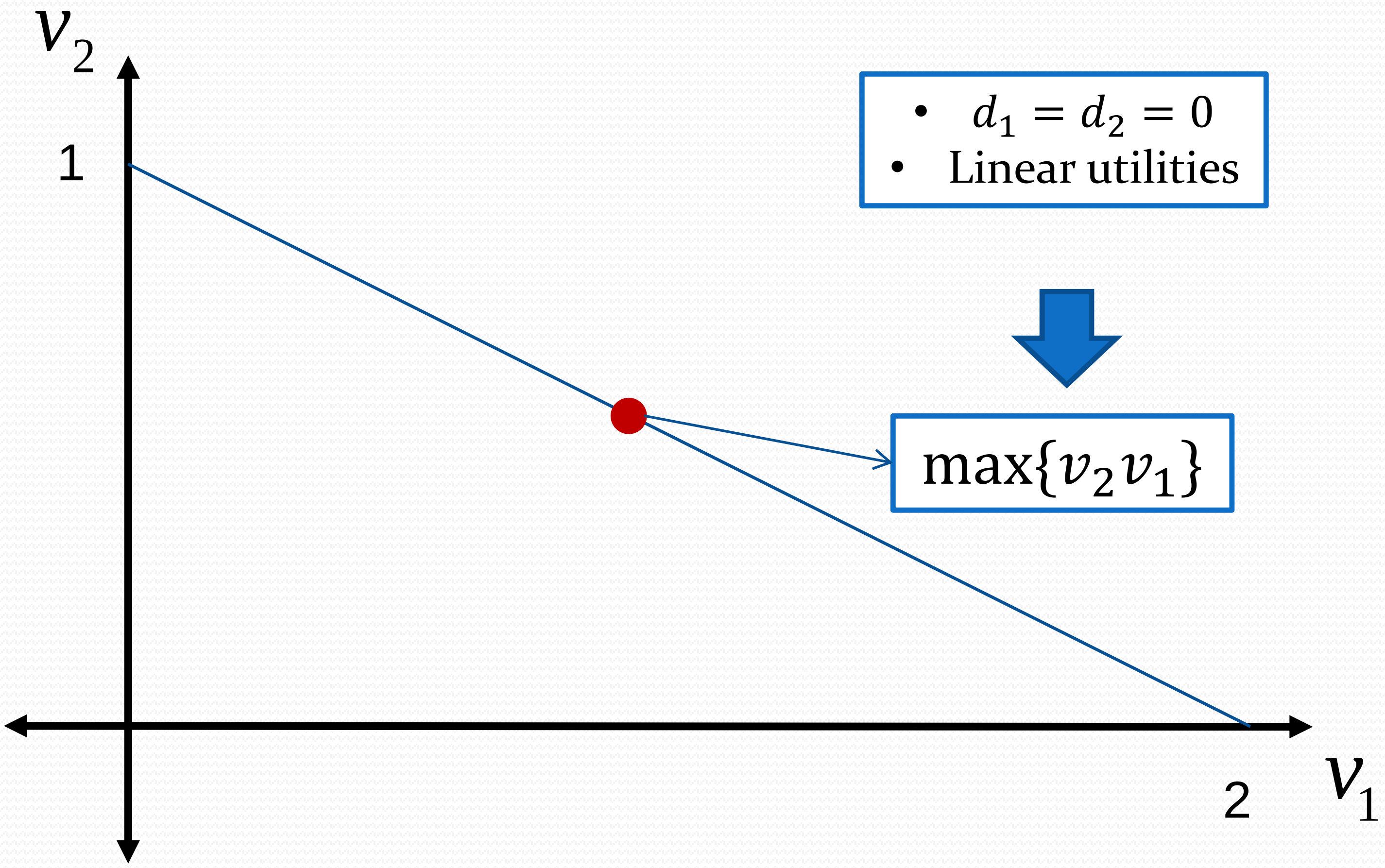
Nash Bargaining Solution

- Pareto Efficiency
  - $\nexists (v_1, v_2) \in U \; s.t. \; v \geq f(U, d) \; and \; \exists i \; s.t. \; v_i > f_i(U, d)$
- Symmetry
  - If U is symmetric and $d_1 = d_2$ then $f_1(U, d) = f_2(U, d)$
- Invariant to Equivalent Payoff Representation
  - Invariant to affine transform
- Independence of Irrelevant Alternatives
  - If $U' \subseteq U$ and $f(U, d) \in U'$ then $f(U', d) = f(U, d)$.

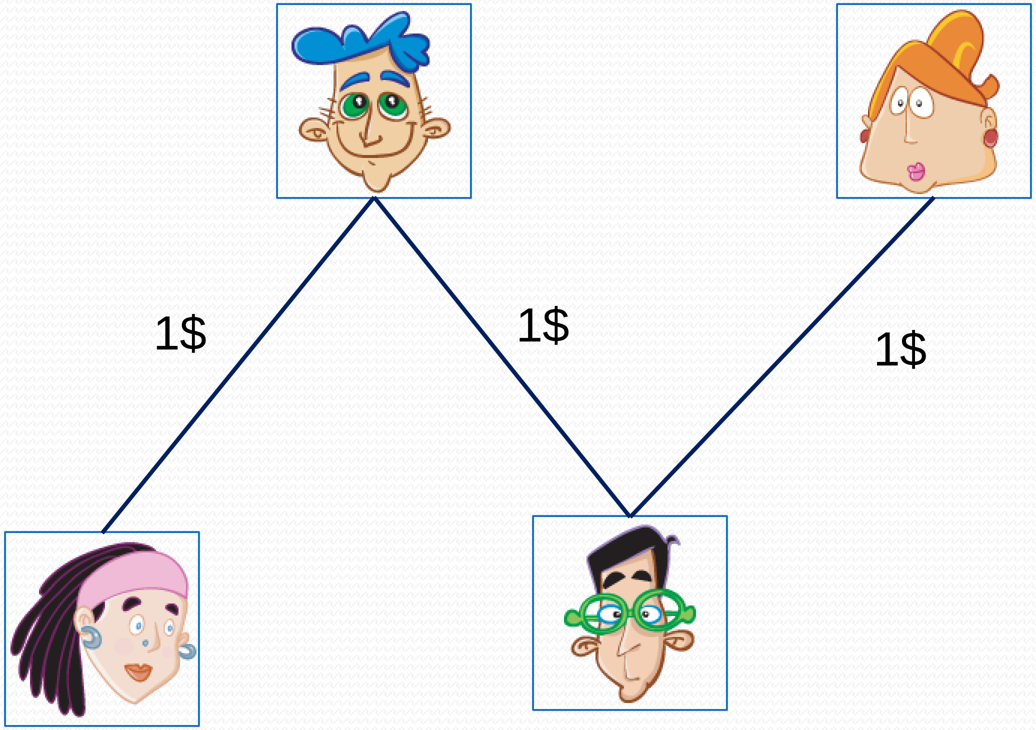
Nash Bargaining Solution



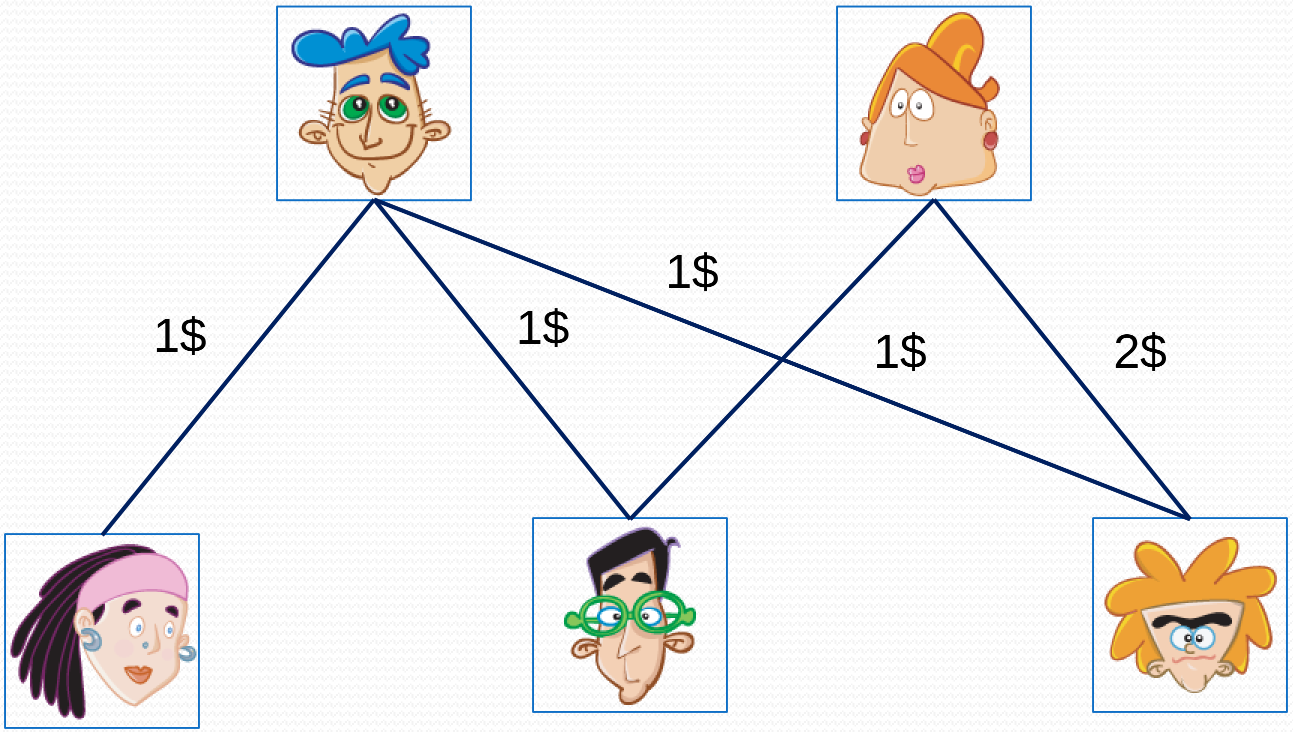
Nash Bargaining Solution



Bargaining Game



Bargaining Game



Bargaining Game

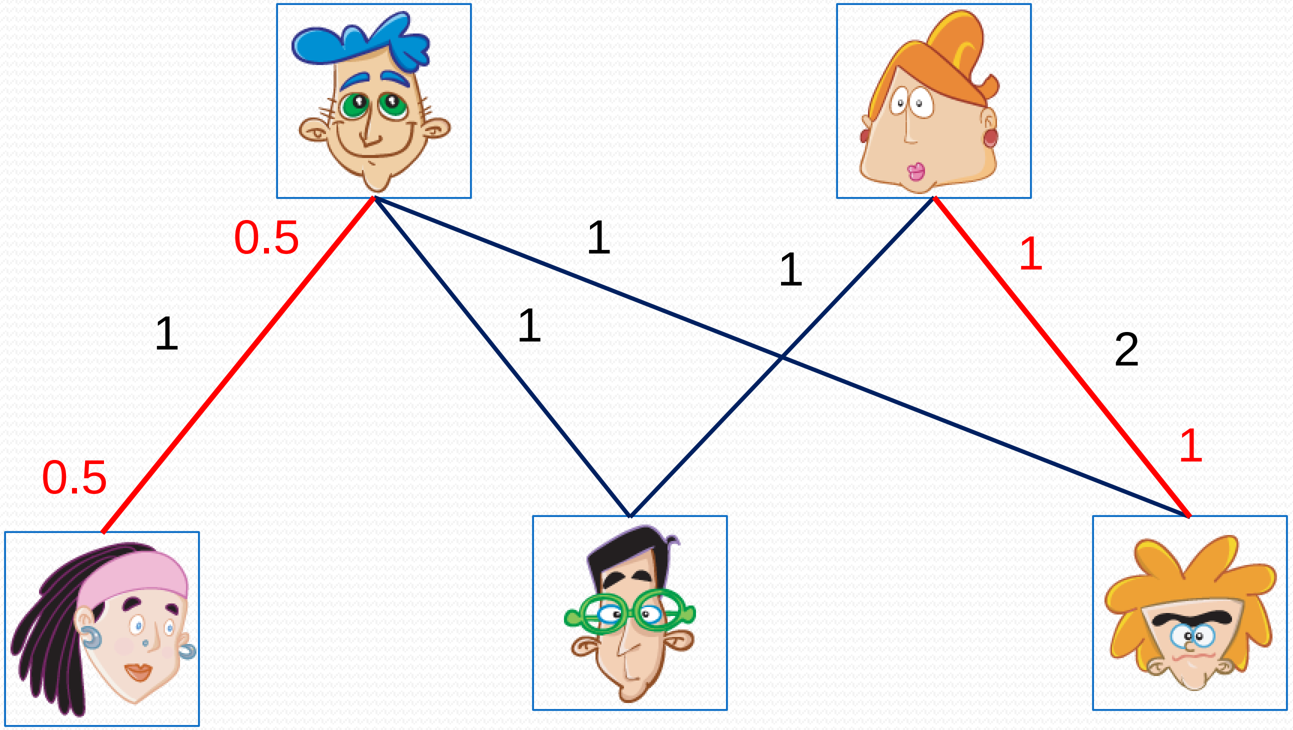
- They are n agents in the market.
- Each agent may participate in at most one contract.
- For each pair of agents i *and* j we are given weight $w_{i,j}$
  - Representing the surplus of a contract between i and j

Our main task is to predict the outcome of a network bargaining game.

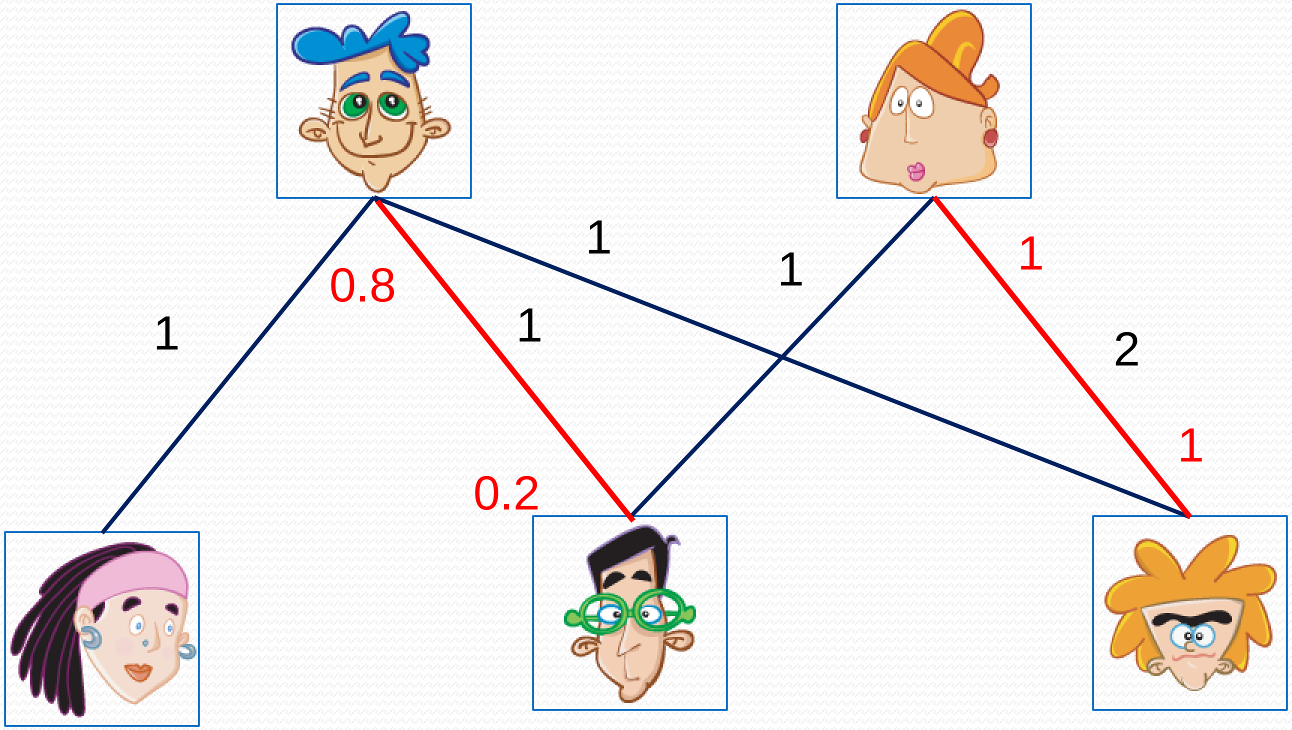
Bargaining Solution

- We call a set of contracts M feasible if:
  - Each agent i is in at most 1 contract.
- A *solution* $(\{z_{i,j}\}, M)$ *of a bargaining game is:*
  - A set of feasible contracts M.
  - For each (i, j) in M: $z_{i,j} + z_{j,i} = w_{i,j}$
  - $z_{i,j}$ is the amount of money i earns from the contract with j

Bargaining Solution - Example

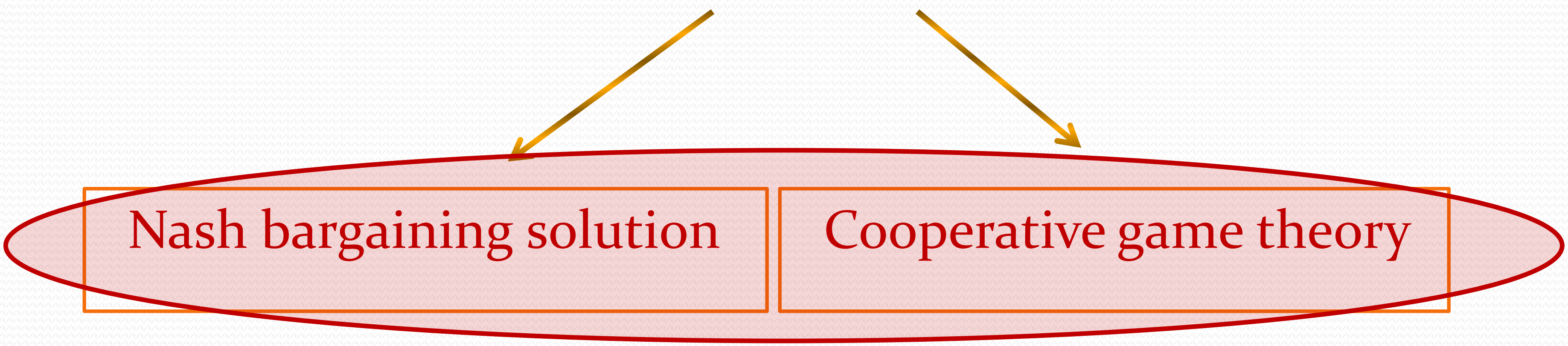


Bargaining Solution - Example



Bargaining Solution

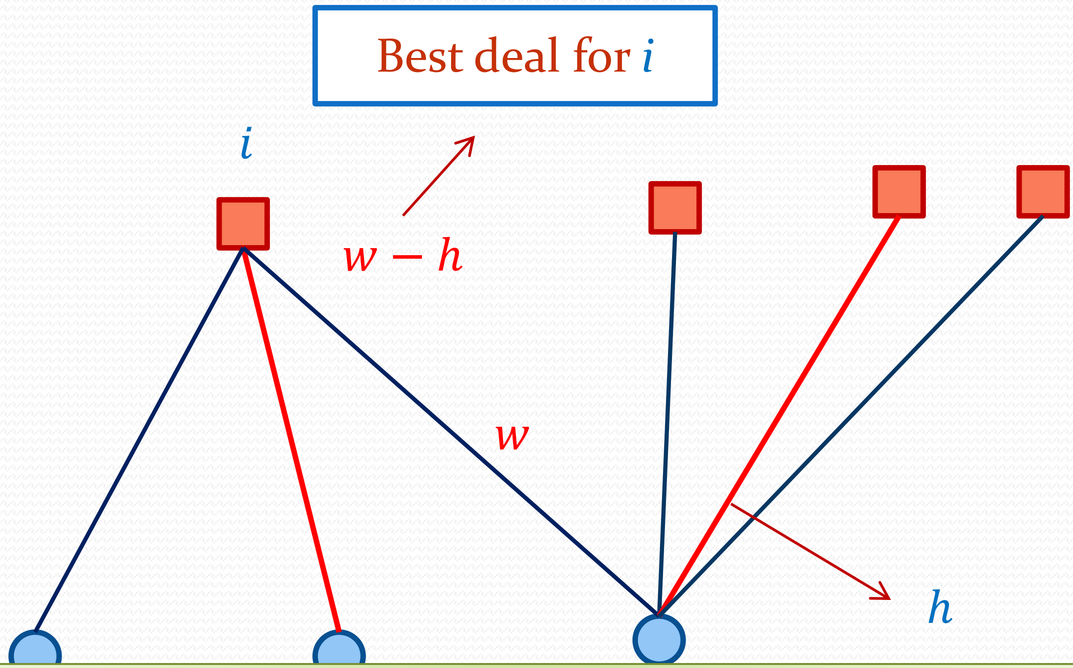
- The set of solution is quite large.
- Define a subset of solution as a result of the bargaining process.



Goal

- Nash bargaining solution.
  - Stable
  - Balanced
- Cooperative game theory solutions.
  - Core
  - Kernel
- Connection between these two views.

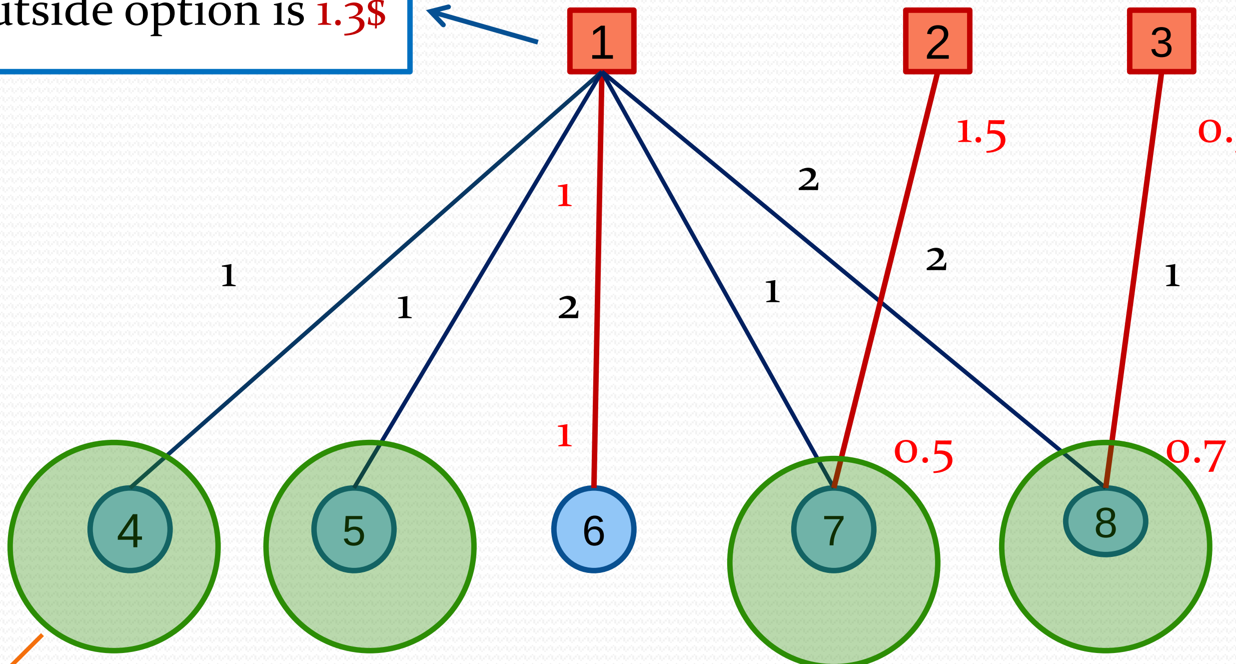
Outside Option



- The outside option of an agent i is the best deal she could make with someone outside the contracting set M.

Outside Option

The Outside option is 1.3$



The best deal is 1$

The best deal is 1$

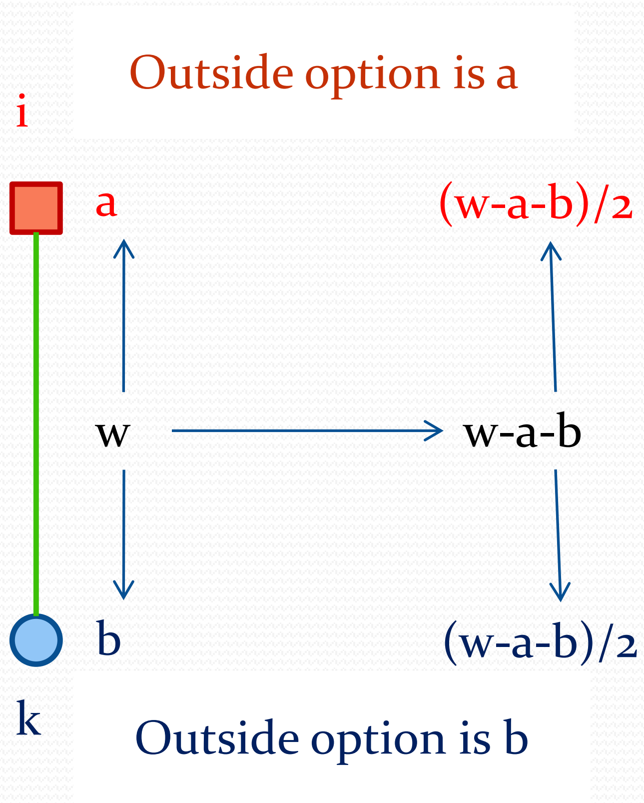
The best deal is 0.5$

The best deal is 1.3$

Stable and Balanced Solutions

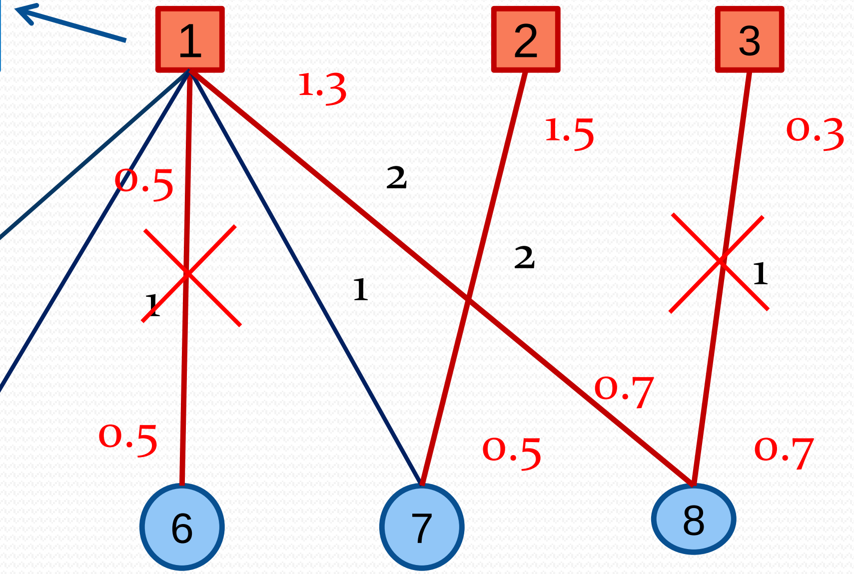
- *A* solution is stable if no agent has better outside option.
- Nash additionally argued that agents tend to split surplus equally.
- A solution is balanced if agents split the net surplus equally.
  - Each agent gets its outside option in a contract.
  - Then divide the money on the table equally.

Balanced solution

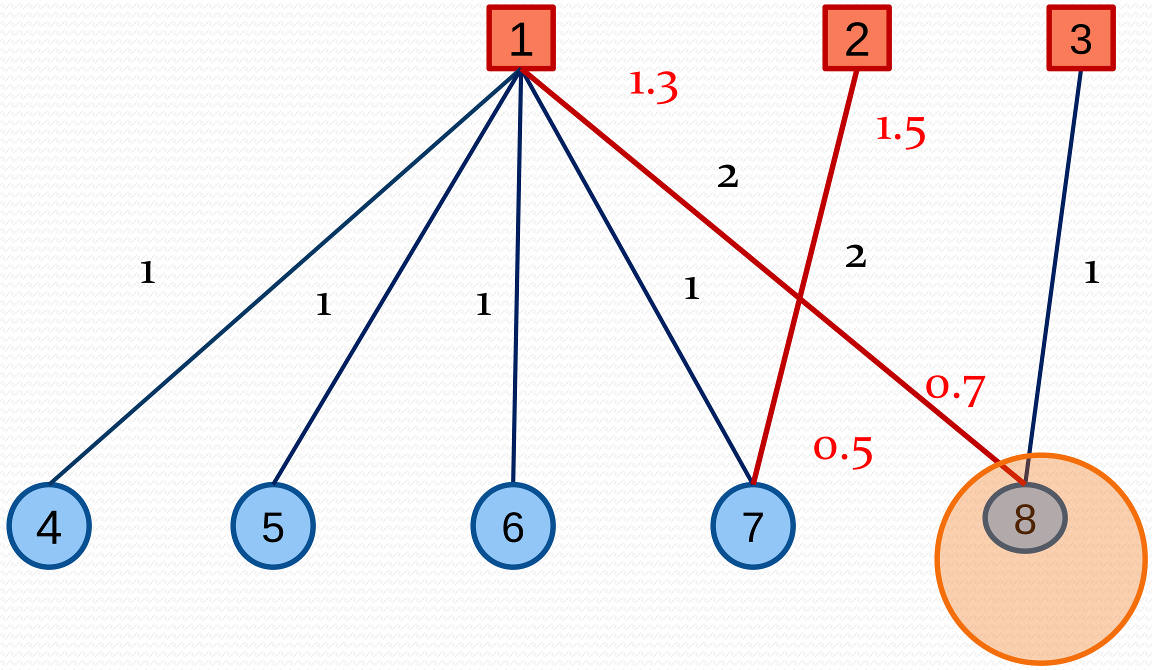


Stable Solution

The Outside option is 1.3$

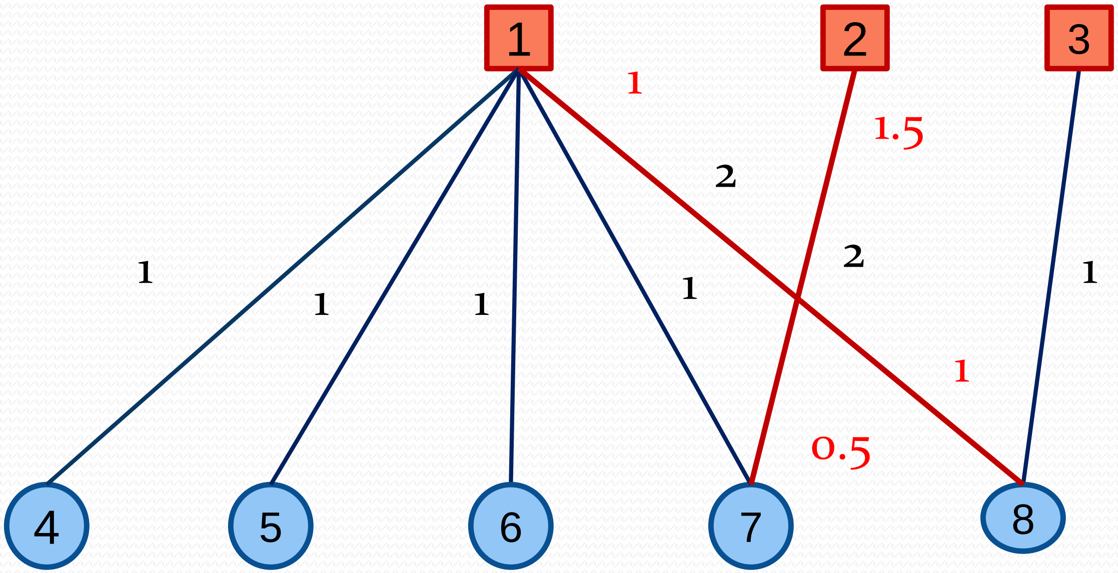


Stable Solution



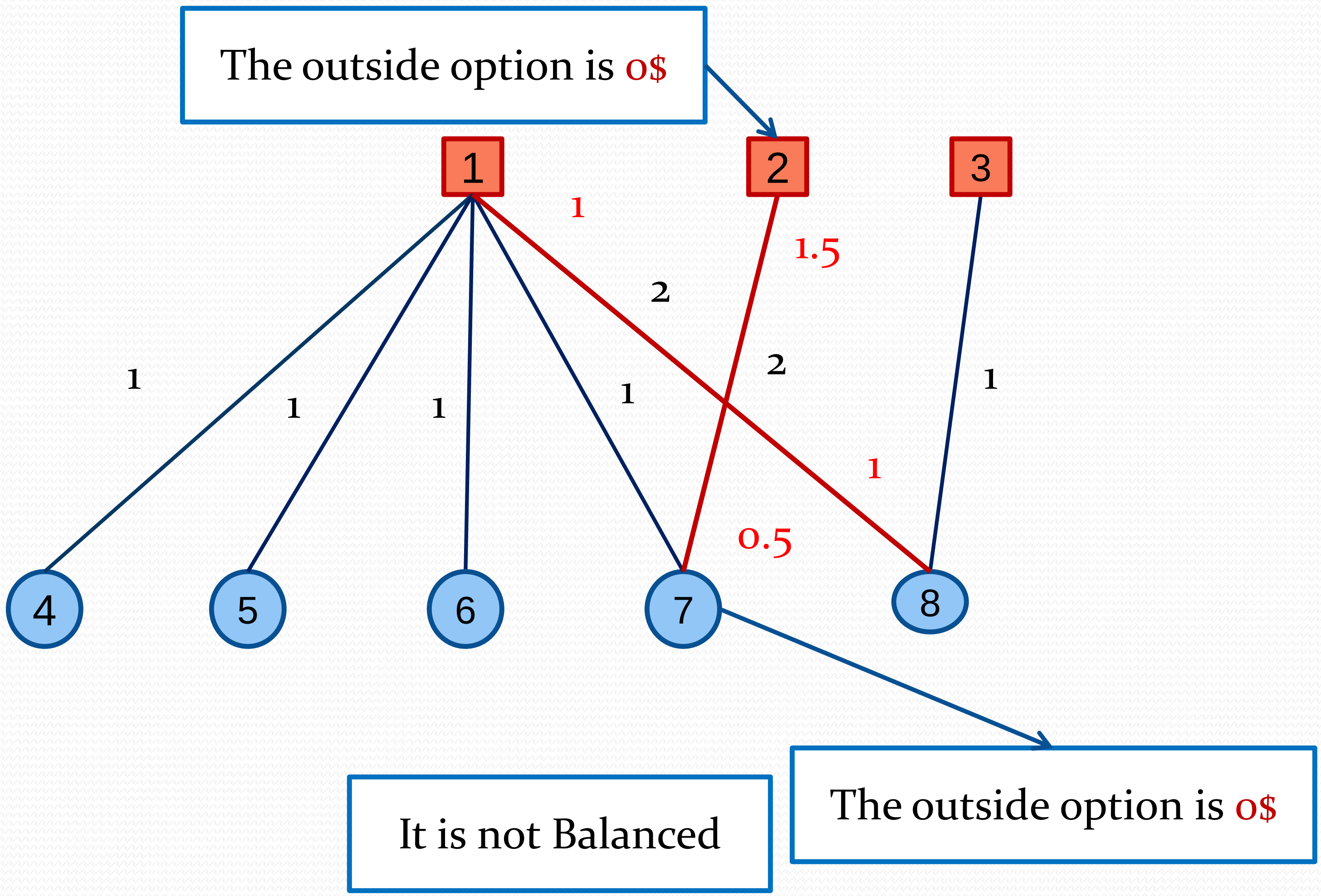
The Outside option is 1 $

Stable Solution

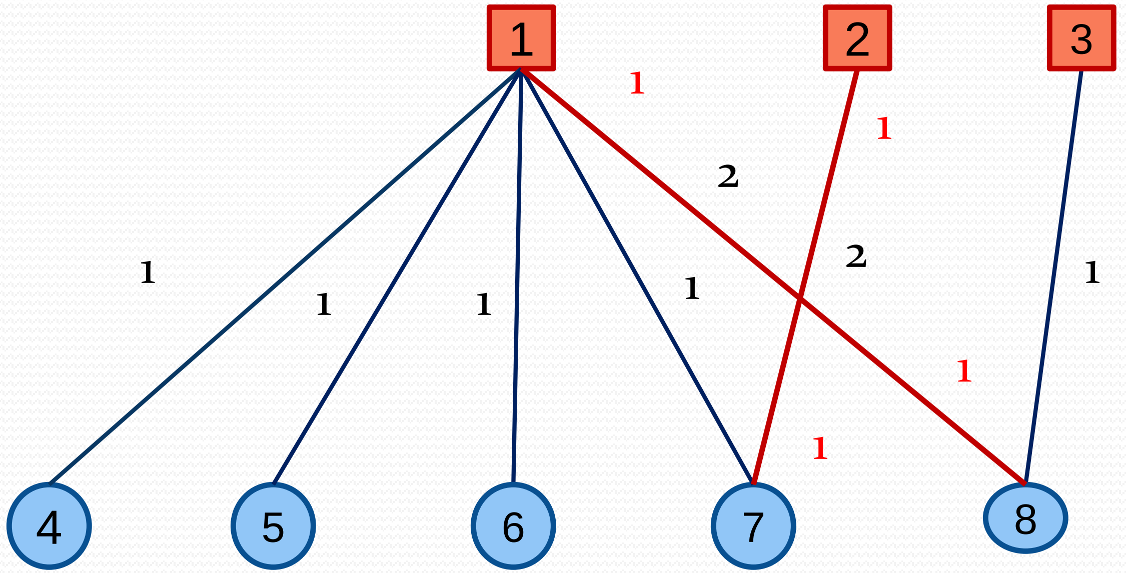


Stable Solution

Balanced Solution



Balanced Solution



Balanced Solution

Cooperative game theory

- A cooperative game is defined by a set of agents N.
- A value function $v: 2^N \rightarrow R^+ \cup \{0\}$
  - The value of a set of agents represents the surplus they can achieve.
- The goal is to define an outcome of the game $\{x_i\}$

$v(S) =$ Maximum value of $\sum_{(i,j)\in M} w_{i,j}$ over all feasible contract M

Core

- An outcome $\{x_i\}$ is in the core if and only if:
  - Each set of agents should earn in total at least as much as they can achieve alone: $\sum_{i \in S} x_i \geq v(S)$
  - Total surplus of all agents is exactly divided among the agents: $\sum_{i \in \mathrm{N}} x_i = v(N)$

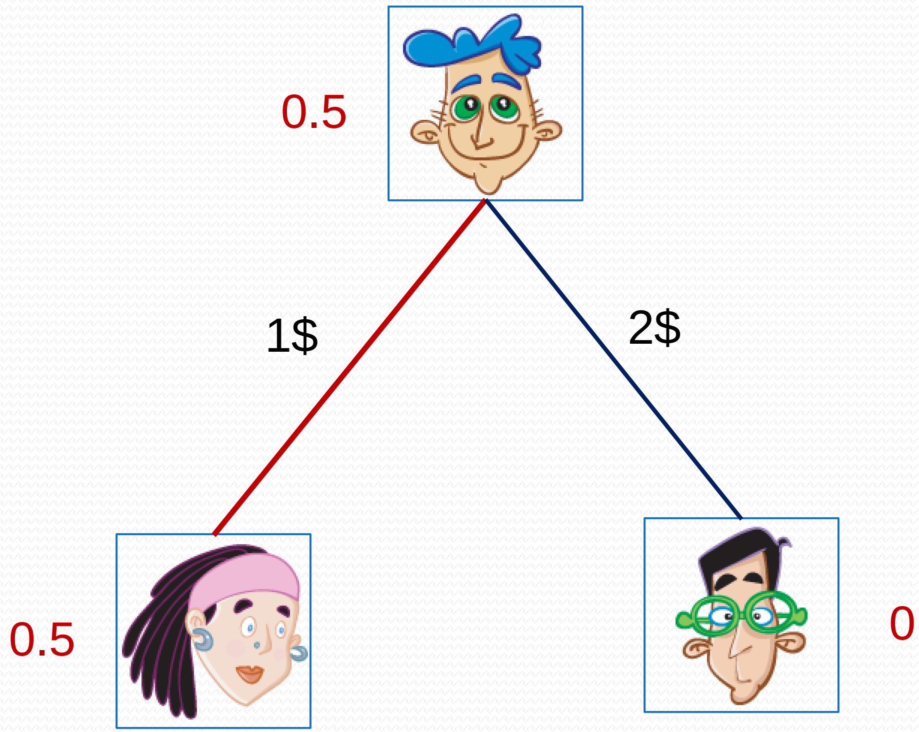
Prekernel

- The power of *i* over *j* is the maximum amount *i* can earn without cooperation with *j*.

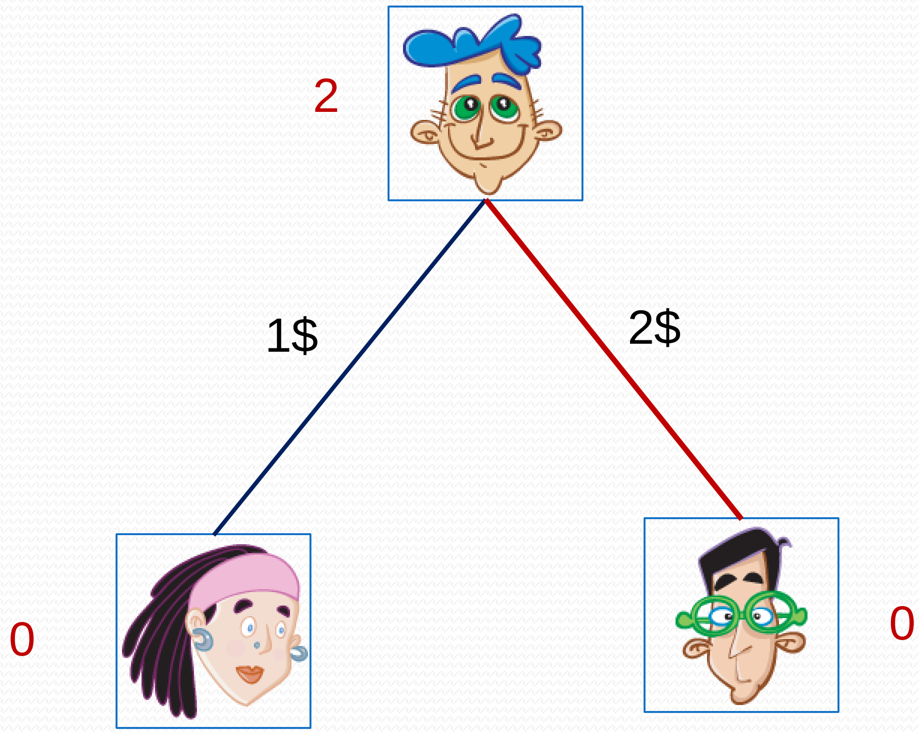
$$s_{ij}(x) = \max\left\{\nu(S) - \sum_{k \in S} x_k : S \subseteq N, S \ni i, S \not\ni j\right\}$$

Prekernel: power of *i* over *j* = power of *j* over *i*

Core - example

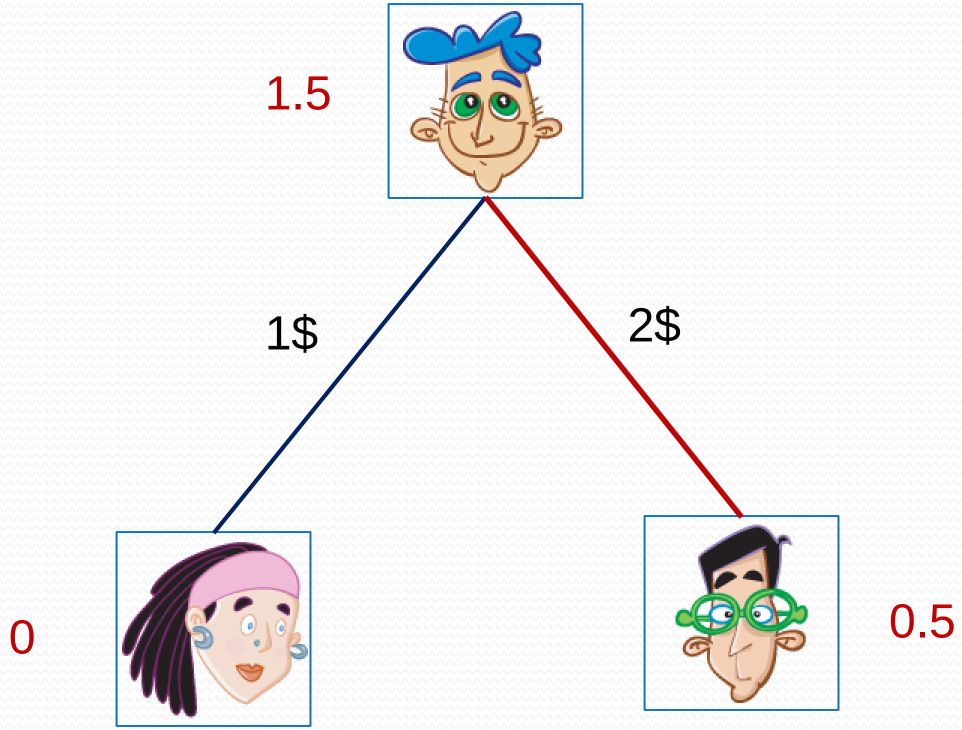


Prekernel - example



$$s_{ij}(x) = \max\left\{\nu(S) - \sum_{k\in S} x_k : S \subseteq N, S \ni i, S \not\ni j\right\}$$

Prekernel - example



$$s_{ij}(x) = \max\left\{\nu(S) - \sum_{k \in S} x_k : S \subseteq N, S \ni i, S \not\ni j\right\}$$

Characterizing Stable Solutions

Primal

Maximize $\sum_{ij} w_{ij} x_{ij}$

Subject to $\sum_j x_{ij} \leq 1, \quad \forall i$

$x_{ij} \geq 0, \forall i, j$

Dual

Minimize $\sum_i u_i$

Subject to $u_i + u_j \geq w_{ij}, \forall i, j$

$u_i \geq 0, \forall i$

A stable solution ≈ a pair of optimum solutions of the above linear programs

Characterizing Stable Solutions

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Stable to LP

- given $(\{z_{ij}\}, M)$
- $x_{ij} = 1$ iff $(i, j) \in M$
- $u_i = z_{ij}$ iff $(i, j) \in M$

Characterizing Stable Solutions

Primal

Maximize $\sum_{ij} w_{ij} x_{ij}$

Subject to $\sum_j x_{ij} \leq 1, \quad \forall i$

$x_{ij} \geq 0, \forall i, j$

Dual

Minimize $\sum_i u_i$

Subject to $u_i + u_j \geq w_{ij}, \forall i, j$

$u_i \geq 0, \forall i$

LP to Stable

- given $(\{x_{ij}\}, \{u_i\})$
- $(i, j) \in M$ iff $x_{ij} = 1$
- $z_{ij} = u_i$ for all $x_{ij} = 1$

Core = Stable

Stable ⊆ Core

- We use the characterization of stable solutions
- Consider $(\{x_{ij}\}, \{u_i\})$
- Define $x_i = u_i$
- We should prove:
  - $\sum_{i \in \mathrm{N}} x_i = v(N)$
  - $\sum_{i \in R} x_i \geq v(R)$

Core = Stable

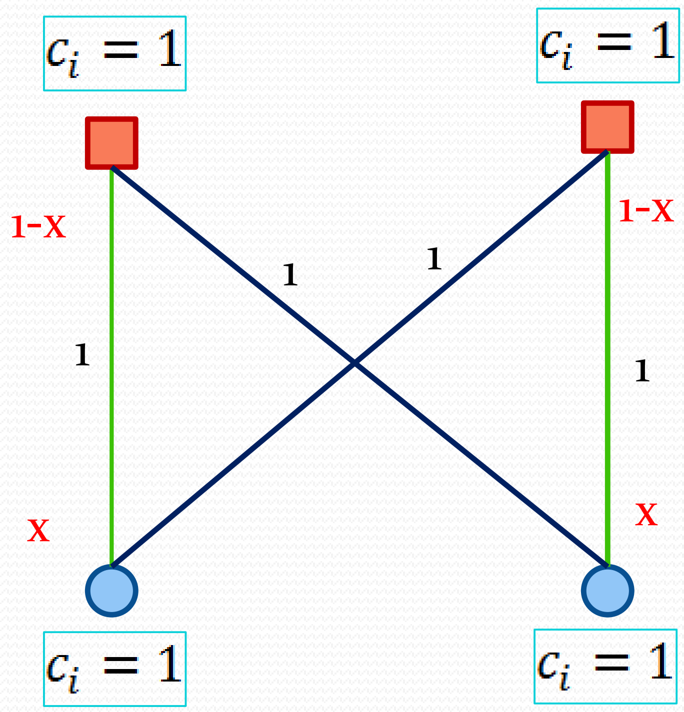
Core ⊆ Stable

- Assume $(\{x_i\})$ is in the core.
- Consider an optimal set of contracts M
- Set $z_{ij} = x_i$ and $z_{ji} = x_j$ for all $(i, j) \in M$
  - $\sum_i x_i = v(N)$ = maximum matching
- Set $u_i = x_i$
  - $(\{u_i\})$ is a feasible solution for the dual.

Core ∩ Kernel = Balanced

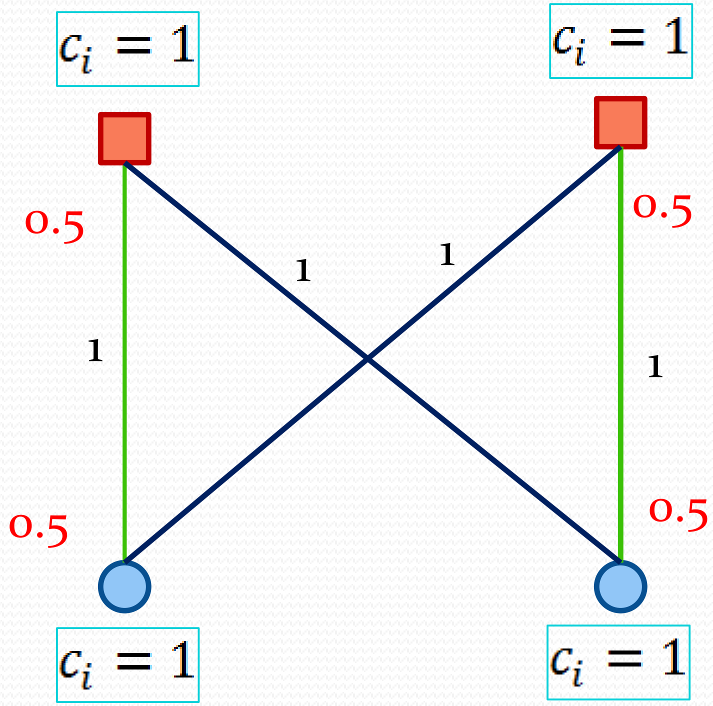
- Assume $(\{x_i\})$ is in the core ∩ kernel.
- Construct $(\{z_{ij}\}, M)$ based on the previous approach.
- Define $\hat{s}_{ij} = \alpha_i - z_{ij}$
- Prove $s_{ij} = \hat{s}_{ij}$

Stable, Balanced are too large



Stable and Balanced Solution

Unique outcome



The symmetric solution

Unique outcome, Nucleolus

- The excess of s set S is the extra earning of S in $\{x_i\}$.
  - $\epsilon(S) = \sum_{i \in S} x_i - \nu(S)$
- Let $\epsilon = (\epsilon_{S_1}, \ldots, \epsilon_{S_{2^N}})$ be the vector of excesses sorted in non-decreasing order.

- The nucleolus is unique.
- It is in the kernel.
- It is in the core if core is non-empty.
- Characterized by a set of simple, reasonable axiom.
  - Symmetry