CMSC 858F
Introduction to Game Theory*

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*Some of these slides are originally prepared by Professor Dana Nau.
What is Game Theory?

- Game Theory is about interactions among self-interested agents (players)
- Different agents have different preferences (i.e. like some outcomes more than others)
- Note that game theory is not a tool; it is a set of concepts.
- Goals of this course:
  - Formal definitions and technicality of the algorithms
  - Better understanding of real-world games
Algorithmic Game Theory

- Algorithm Game Theory is often viewed as “incentive-aware algorithm design”
- Algorithm design often deals with dumb objects though Algorithmic Game Theory often deals with smart (self-interested) objects
- Combines Algorithm Design and Game Theory
- Also known as Mechanism Design
- Goal of Mechanism Design
  - Encourage selfish agents to act socially by designing rewarding rules such that when agents optimize their own objective, a social objective is met
Some Fields where Game Theory is Used

- Economics, business
  - Markets, auctions
  - Economic predictions
  - Bargaining, fair division
Some Fields where Game Theory is Used

- Government, politics, military
  - Negotiations
  - Voting systems
  - International relations
  - War
  - …

World War 1
army trench
Some Fields where Game Theory is Used

- Biology, psychology, sociology
  - Population ratios, territoriality
  - Social behavior
  - ...
Some Fields where Game Theory is Used

- Engineering, computer science
  - Game programs
  - Computer and communication networks
  - Road networks
  - …
Games in Normal Form

- A (finite, \(n\)-person) **normal-form game** includes the following:
  1. An ordered set \(N = (1, 2, 3, \ldots, n)\) of agents or **players**:
  2. Each agent \(i\) has a finite set \(A_i\) of possible actions
     - An **action profile** is an \(n\)-tuple \(a = (a_1, a_2, \ldots, a_n)\), where \(a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\)
     - The set of all possible action profiles is \(A = A_1 \times \cdots \times A_n\)
  3. Each agent \(i\) has a real-valued **utility** (or **payoff**) function \(u_i(a_1, \ldots, a_n) = i\)'s payoff if the action profile is \((a_1, \ldots, a_n)\)

- Most other game representations can be reduced to normal form

- Usually represented by an \(n\)-dimensional **payoff** (or **utility**) **matrix**
  - for each action profile, shows the utilities of all the agents

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The Prisoner’s Dilemma

- Scenario: The police are holding two prisoners as suspects for committing a crime
  - For each prisoner, the police have enough evidence for a 1 year prison sentence
  - They want to get enough evidence for a 4 year prison sentence
  - They tell each prisoner,
    - “If you testify against the other prisoner, we’ll reduce your prison sentence by 1 year”
  - $C =$ Cooperate (with the other prisoner): refuse to testify against him/her
  - $D =$ Defect: testify against the other prisoner
  - Both prisoners cooperate $\Rightarrow$ both go to prison for 1 year
  - Both prisoners defect $\Rightarrow$ both go to prison for $4 - 1 = 3$ years
  - One defects, other cooperates $\Rightarrow$ cooperator goes to prison for 4 years; defector goes free

\[
\begin{array}{c|cc}
  & C & D \\
  C & -1, -1 & -4, 0 \\
  D & 0, -4 & -3, -3 \\
\end{array}
\]
Prisoner’s Dilemma

General form:

\[ c > a > d > b \]
\[ 2a > b + c \]

We used this:

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Equivalent:

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Game theorists usually use this:

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Utility Functions

- Idea: the preferences of a rational agent must obey some constraints
- Agent’s choices are based on rational preferences
  ⇒ agent’s behavior is describable as maximization of expected utility
- Constraints:
  - **Orderability** (sometimes called **Completeness**):
    \[(A > B) \lor (B > A) \lor (A \sim B)\]
  - **Transitivity**:
    \[(A > B) \land (B > C) \Rightarrow (A > C)\]

- **Theorem** (Ramsey, 1931; von Neumann and Morgenstern, 1944).
- Given preferences satisfying the constraints above, there exists a real-valued function \( u \) such that
  \[
u(A) \geq u(B) \iff A \succeq B\]  
  \( (*) \)

  \( u \) is called a **utility function**
Utility Scales for Games

- Suppose that all the agents have rational preferences, and that this is common knowledge* to all of them
- Then games are insensitive to positive affine transformations of one or more agents’ payoffs
  - Let $c$ and $d$ be constants, $c > 0$
  - For one or more agents $i$, replace every payoff $x_{ij}$ with $cx_{ij} + d$
  - The game still models the same sets of rational preferences

*Common knowledge is a complicated topic; I’ll discuss it later
Common-payoff Games

- **Common-payoff game:**
  - For every action profile, all agents have the same payoff
  - Also called a **pure coordination** game or a **team game**
    - Need to coordinate on an action that is maximally beneficial to all

- **Which side of the road?**
  - 2 people driving toward each other in a country with no traffic rules
  - Each driver independently decides whether to stay on the left or the right
  - Need to coordinate your action with the action of the other driver

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A Brief Digression

- **Mechanism design**: set up the rules of the game, to give each agent an incentive to choose a desired outcome
- E.g., the law says what side of the road to drive on
  - Sweden on September 3, 1967:
Zero-sum Games

- These games are purely competitive

- **Constant-sum** game:
  - For every action profile, the sum of the payoffs is the same, i.e.,
  - there is a constant $c$ such for every action profile $\mathbf{a} = (a_1, \ldots, a_n)$,
    - $u_1(\mathbf{a}) + \ldots + u_n(\mathbf{a}) = c$

- Any constant-sum game can be transformed into an equivalent game in which the sum of the payoffs is always 0
  - Positive affine transformation: subtract $c/n$ from every payoff

- Thus constant-sum games are usually called **zero-sum** games
Examples

- **Matching Pennies**
  - Two agents, each has a penny
  - Each independently chooses to display Heads or Tails
    - If same, agent 1 gets both pennies
    - Otherwise agent 2 gets both pennies

- **Penalty kicks in soccer**
  - A kicker and a goalie
  - Kicker can kick left or right
  - Goalie can jump to left or right
  - Kicker scores if he/she kicks to one side and goalie jumps to the other

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<th>Heads</th>
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<td>-1, 1</td>
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<td>Tails</td>
<td>-1, 1</td>
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Another Example: Rock-Paper-Scissors

- **Two players.** Each simultaneously picks an action: *Rock, Paper, or Scissors.*

- **The rewards:**
  - *Rock* beats *Scissors*
  - *Scissors* beats *Paper*
  - *Paper* beats *Rock*

- **The matrices:**

  $$R_1 = \begin{pmatrix} R & P & S \\ R & 0 & -1 & 1 \\ P & 1 & 0 & -1 \\ S & -1 & 1 & 0 \end{pmatrix} \quad R_2 = \begin{pmatrix} R & P & S \\ R & 0 & 1 & -1 \\ P & -1 & 0 & 1 \\ S & 1 & -1 & 0 \end{pmatrix}$$
A game is **nonconstant-sum** (usually called **nonzero-sum**) if there are action profiles \( \mathbf{a} \) and \( \mathbf{b} \) such that

\[
u_1(\mathbf{a}) + \ldots + u_n(\mathbf{a}) \neq u_1(\mathbf{b}) + \ldots + u_n(\mathbf{b})\]

- e.g., the Prisoner’s Dilemma

**Battle of the Sexes**

- Two agents need to coordinate their actions, but they have different preferences
- Original scenario:
  - husband prefers football, wife prefers opera
- Another scenario:
  - Two nations must act together to deal with an international crisis, and they prefer different solutions
Symmetric Games

- In a **symmetric** game, every agent has the same actions and payoffs
  - If we change which agent is which, the payoff matrix will stay the same

- For a 2x2 symmetric game, it doesn’t matter whether agent 1 is the row player or the column player
  - The payoff matrix looks like this:

- In the payoff matrix of a symmetric game, we only need to display \( u_1 \)
  - If you want to know another agent’s payoff, just interchange the agent with agent 1

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<td>( a_2 )</td>
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Strategies in Normal-Form Games

- **Pure strategy**: select a single action and play it
  - Each row or column of a payoff matrix represents both an action and a pure strategy

- **Mixed strategy**: randomize over the set of available actions according to some probability distribution
  - $s_i(a_j) = \text{probability that action } a_j \text{ will be played in mixed strategy } s_i$

- The **support** of $s_i = \{\text{actions that have probability } > 0 \text{ in } s_i\}$

- A pure strategy is a special case of a mixed strategy
  - support consists of a single action

- A strategy $s_i$ is **fully mixed** if its support is $A_i$
  - i.e., nonzero probability for every action available to agent $i$

- **Strategy profile**: an $n$-tuple $s = (s_1, \ldots, s_n)$ of strategies, one for each agent
Expected Utility

- A payoff matrix only gives payoffs for pure-strategy profiles
- Generalization to mixed strategies uses expected utility
  - First calculate probability of each outcome, given the strategy profile (involves all agents)
  - Then calculate average payoff for agent i, weighted by the probabilities
  - Given strategy profile $s = (s_1, \ldots, s_n)$
    - expected utility is the sum, over all action profiles, of the profile’s utility times its probability:

$$u_i(s) = \sum_{a \in A} u_i(a) P[a \mid s]$$

i.e.,

$$u_i(s_1, \ldots, s_n) = \sum_{(a_1, \ldots, a_n) \in A} u_i(a_1, \ldots, a_n) \prod_{j=1}^{n} s_j(a_j)$$
Some Comments about Normal-Form Games

- Only two kinds of strategies in the normal-form game representation:
  - **Pure strategy**: just a single action
  - **Mixed strategy**: probability distribution over pure strategies
    - i.e., choose an action at random from the probability distribution
- The normal-form game representation may see very restricted
  - No such thing as a conditional strategy
    (e.g., cross the bay if the temperature is above 70)
  - No temperature or anything else to observe
- However much more complicated games can be mapped into normal-form games
  - Each pure strategy is a description of what you’ll do in every situation you might ever encounter in the game
- In later sessions, we see more examples
How to reason about games?

- In single-agent decision theory, look at an **optimal strategy**
  - Maximize the agent’s expected payoff in its environment

- With multiple agents, the best strategy depends on others’ choices
- Deal with this by identifying certain subsets of outcomes called **solution concepts**

- First we discuss two solution concepts:
  - Pareto optimality
  - Nash equilibrium

- Later we will discuss several others
Pareto Optimality

- A strategy profile $s$ **Pareto dominates** a strategy profile $s'$ if
  - no agent gets a worse payoff with $s$ than with $s'$, i.e., $u_i(s) \geq u_i(s')$ for all $i$,
  - at least one agent gets a better payoff with $s$ than with $s'$, i.e., $u_i(s) > u_i(s')$ for at least one $i$

- A strategy profile $s$ is **Pareto optimal** (or **Pareto efficient**) if there’s no strategy profile $s'$ that Pareto dominates $s$
  - Every game has at least one Pareto optimal profile
  - Always at least one Pareto optimal profile in which the strategies are pure
Examples

The Prisoner’s Dilemma

- \((D,C)\) is Pareto optimal: no profile gives player 1 a higher payoff
- \((C, D)\) is Pareto optimal: no profile gives player 2 a higher payoff
- \((C,C)\) is Pareto optimal: no profile gives both players a higher payoff
- \((D,D)\) isn’t Pareto optimal: \((C,C)\) Pareto dominates it

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<td>5, 0</td>
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Which Side of the Road

- \((\text{Left,Left})\) and \((\text{Right,Right})\) are Pareto optimal
- In common-payoff games, all Pareto optimal strategy profiles have the same payoffs
  - If \((\text{Left,Left})\) had payoffs \((2,2)\), then \((\text{Right,Right})\) wouldn’t be Pareto optimal

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Best Response

- Suppose agent $i$ knows how the others are going to play
  - Then $i$ has an ordinary optimization problem:
    maximize expected utility

- We’ll use $s_{-i}$ to mean a strategy profile for all of the agents except $i$
  
  $$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

- Let $s_i$ be any strategy for agent $i$. Then
  
  $$\left(s_i, s_{-i}\right) = (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n)$$

- $s_i$ is a **best response** to $s_{-i}$ if for every strategy $s_i'$ available to agent $i$,
  
  $$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

- There is always at least one best response

- A best response $s_i$ is **unique** if $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for every $s_i' \neq s_i$
Best Response

- Given $s_{-i}$, there are only two possibilities:
  1. $i$ has a pure strategy $s_i$ that is a unique best response to $s_{-i}$
  2. $i$ has infinitely many best responses to $s_{-i}$

**Proof.** Suppose (1) is false. Then there are two possibilities:

- **Case 1:** $s_i$ isn’t unique, i.e., $\geq 2$ strategies are best responses to $s_{-i}$
  - Then they all must have the same expected utility
  - Otherwise, they aren’t all “best”
  - Thus any mixture of them is also a best response
  - Thus (2) happens.

- **Case 2:** $s_i$ isn’t pure, i.e., it’s a mixture of $k > 2$ actions
  - The actions correspond to pure strategies, so this reduces to Case 1
  - Thus (2) happens.

- **Theorem:** Always there exists a pure best response $s_i$ to $s_{-i}$

**Proof.** In both (1) and (2) above, there should be one pure best response.
Example

- Suppose we modify the Prisoner’s Dilemma to give Agent 1 another possible action:
  - Suppose 2’s strategy is to play action $C$
  - What are 1’s best responses?
  - Suppose 2’s strategy is to play action $D$
  - What are 1’s best responses?

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Nash Equilibrium

- **Equilibrium**: it is simply a state of the world where economic forces are balanced and in the absence of external influence the equilibrium variables will not change.
  - More intuitively, a state in which no person involved in the game wants any change.

- **Famous economic equilibria**: Nash equilibrium, Correlated equilibrium, Market Clearance equilibrium

- \( s = (s_1, \ldots, s_n) \) is a **Nash equilibrium** if for every \( i \), \( s_i \) is a best response to \( s_{-i} \)
  - Every agent’s strategy is a best response to the other agents’ strategies
  - No agent can do better by *unilaterally* changing his strategy

- **Theorem (Nash, 1951)**: Every game with a finite number of agents and actions has at least one Nash equilibrium

- In Which Side of the Road, (Left,Left) and (Right,Right) are Nash equilibria
  
- In the Prisoner’s Dilemma, \((D,D)\) is a Nash equilibrium
  - Ironically, it’s the only pure-strategy profile that *isn’t* Pareto optimal
Strict Nash Equilibrium

• A Nash equilibrium $\mathbf{s} = (s_1, \ldots, s_n)$ is strict if for every $i$, $s_i$ is the only best response to $\mathbf{s}_{-i}$
  - i.e., any agent who unilaterally changes strategy will do worse

• Recall that if a best response is unique, it must be pure
  ➢ It follows that in a strict Nash equilibrium, all of the strategies are pure

• But if a Nash equilibrium is pure, it isn’t necessarily strict

Which of the following Nash equilibria are strict? Why?

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Weak Nash Equilibrium

- If a Nash equilibrium $s$ isn’t strict, then it is weak
  - At least one agent $i$ has more than one best response to $s_{-i}$
- If a Nash equilibrium includes a mixed strategy, then it is weak
  - If a mixture of $k = 2$ actions is a best response to $s_{-i}$, then any other mixture of the actions is also a best response
- If a Nash equilibrium consists only of pure strategies, it might still be weak

- Weak Nash equilibria are less stable than strict Nash equilibria
  - If a Nash equilibrium is weak, then at least one agent has infinitely many best responses, and only one of them is in $s$
Finding Mixed-Strategy Nash Equilibria

- In general, it’s tricky to compute mixed-strategy Nash equilibria
  - But easier if we can identify the support of the equilibrium strategies
- In 2x2 games, we can do this easily
- We especially use theorem below proved earlier
  **Theorem A:** Always there exists a pure best response $s_i$ to $s_{-i}$
  **Corollary B:** If $(s_1, s_2)$ is a pure Nash equilibrium only among pure strategies, it should be a Nash equilibrium among mixed strategies as well
- Now let $(s_1, s_2)$ be a Nash equilibrium
  - If both $s_1, s_2$ have supports of size one, it should be one of the cells of the normal-form matrix and we are done by Corollary B
  - Thus assume at least one of $s_1, s_2$ has a support of size two.
Finding Mixed-Strategy Nash Equilibria

- Now if the support of one of $s_1, s_2$, say $s_1$, is of size one, i.e., it is pure, then $s_2$ should be pure as well, unless both actions of player 2 have the same payoffs; in this case any mixed strategy of both actions can be Nash equilibrium.

Thus in the rest we assume both supports have size two.

- Thus to find $s_1$ assume agent 1 selects action $a_1$ with probability $p$ and action $a'_1$ with probability $1-p$.
- Now since $s_2$ has a support of size two, its support must include both of agent 2’s actions, and they must have the same expected utility
  - Otherwise agent 2’s best response would be just one of them and its support has size one.

- Hence find $p$ such that $u_2(s_1, a_2) = u_2(s_1, a'_2)$, i.e., solve the equation to find $p$ (and thus $s_2$)
- Similarly, find $s_2$ such that $u_1(a_1, s_2) = u_1(a'_1, s_2)$
Finding Mixed-Strategy Nash Equilibria

Example: Battle of the Sexes

- We already saw pure Nash equilibria.
- If there’s a mixed-strategy equilibrium,
  - both strategies must be mixtures of \{Opera, Football\}
  - each must be a best response to the other
- Suppose the husband’s strategy is \( s_h = \{(p, \text{Opera}), (1-p, \text{Football})\} \)
- Expected utilities of the wife’s actions:
  \[
  u_w(\text{Opera}, s_h) = 2p; \quad u_w(\text{Football}, s_h) = 1(1-p)
  \]
- If the wife mixes the two actions, they must have the same expected utility
  - Otherwise the best response would be to *always* use the action whose expected utility is higher
  - Thus \(2p = 1 - p\), so \(p = 1/3\)
- So the husband’s mixed strategy is \( s_h = \{(1/3, \text{Opera}), (2/3, \text{Football})\} \)

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<tr>
<th>Wife</th>
<th>Opera</th>
<th>Football</th>
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<tr>
<td>Football</td>
<td>0, 0</td>
<td>1, 2</td>
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Finding Mixed-Strategy Nash Equilibria

- Similarly, we can show the wife’s mixed strategy is
  \[ s_w = \{(2/3, \text{Opera}), (1/3, \text{Football})\} \]

- So the mixed-strategy Nash equilibrium is \( (s_w, s_h) \), where
  \[ s_w = \{(2/3, \text{Opera}), (1/3, \text{Football})\} \]
  \[ s_h = \{(1/3, \text{Opera}), (2/3, \text{Football})\} \]

- Questions:
  \[ \text{Like all mixed-strategy Nash equilibria, } (s_w, s_h) \text{ is weak} \]
  - Both players have infinitely many other best-response strategies
  - What are they?
  \[ \text{How do we know that } (s_w, s_h) \text{ really is a Nash equilibrium?} \]
  - Indeed the proof is by the way that we found Nash equilibria \( (s_w, s_h) \)

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Finding Mixed-Strategy Nash Equilibria

- \( s_w = \{(2/3, \text{Opera}), (1/3, \text{Football})\} \)
- \( s_h = \{(1/3, \text{Opera}), (2/3, \text{Football})\} \)

- Wife’s expected utility is
  - \( 2(2/9) + 1(2/9) + 0(5/9) = 2/3 \)

- Husband’s expected utility is also 2/3

- It’s “fair” in the sense that both players have the same expected payoff

- But it’s Pareto-dominated by both of the pure-strategy equilibria
  - In each of them, one agent gets 1 and the other gets 2

- Can you think of a fair way of choosing actions that produces a higher expected utility?

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Finding Mixed-Strategy Nash Equilibria

Matching Pennies

- Easy to see that in this game, no pure strategy could be part of a Nash equilibrium
  - For each combination of pure strategies, one of the agents can do better by changing his/her strategy
- Thus there isn’t a strict Nash equilibrium since it would be pure.

- But again there’s a mixed-strategy equilibrium
  - Can be derived the same way as in the Battle of the Sexes
    - Result is \((s, s)\), where \(s = \{ (\frac{1}{2}, \text{Heads}), (\frac{1}{2}, \text{Tails}) \}\)
Another Interpretation of Mixed Strategies

- Suppose agent $i$ has a deterministic method for picking a strategy, but it depends on factors that aren’t part of the game itself
  - If $i$ plays a game several times, $i$ may pick different strategies
- If the other players don’t know how $i$ picks a strategy, they’ll be uncertain what $i$’s strategy will be
  - Agent $i$’s mixed strategy is everyone else’s assessment of how likely $i$ is to play each pure strategy
- Example:
  - In a series of soccer penalty kicks, the kicker could kick left or right in a deterministic pattern that the goalie thinks is random
Complexity of Finding Nash Equilibria

- We’ve discussed how to find Nash equilibria in some special cases
  - Step 1: look for pure-strategy equilibria
    - Examine each cell of the matrix
    - If no cell in the same row is better for agent 1, and no cell in the same column is better for agent 2
      then the cell is a Nash equilibrium
  - Step 2: look for mixed-strategy equilibria
    - Write agent 2’s strategy as \{((q, b), (1–q, b'))\}; look for \(q\) such that \(a\) and \(a'\) have the same expected utility
    - Write agent 1’s strategy as \{((p, a), (1–p, a'))\}; look for \(p\) such that \(b\) and \(b'\) have the same expected utility
- More generally for two-player games with any number of actions for each player, if we know support of each, we can find a mixed-Nash equilibrium in polynomial-time by solving linear equations (via linear program).
- What about the general case?
Complexity of Finding Nash Equilibria

- General case: \( n \) players, \( m \) actions per player, payoff matrix has \( m^n \) cells
  (not in the book)
- Brute-force approach:
  - Step 1: Look for pure-strategy equilibria
    - At each cell of the matrix,
      - For each player, can that player do better by choosing a different action?
    - Polynomial time
  - Step 2: Look for mixed-strategy equilibria
    - For every possible combination of supports for \( s_1, \ldots, s_n \)
      - Solve sets of simultaneous equations
    - Exponentially many combinations of supports
    - Can it be done more quickly?
Complexity of Finding Nash Equilibria

- Two-player games
  - Lemke & Howson (1964): solve a set of simultaneous equations that includes all possible support sets for $s_1, \ldots, s_n$
    - Some of the equations are quadratic => worst-case exponential time
    - AI methods (constraint programming)
  - Sandholm, Gilpin, & Conitzer (2005)
    - Mixed Integer Programming (MIP) problem

- $n$-player games
  - van der Laan, Talma, & van der Heyden (1987)
  - Worst-case running time still is exponential in the size of the payoff matrix
Complexity of Finding Nash Equilibria

- There are special cases that can be done in polynomial time in the size of the payoff matrix
  - Finding pure-strategy Nash equilibria
    - Check each square of the payoff matrix
  - Finding Nash equilibria in zero-sum games (see later in this)
    - Linear programming
- For the general case,
  - It’s unknown whether there are polynomial-time algorithms to do it
  - It’s unknown whether there are polynomial-time algorithms to compute approximations
  - But we know both questions are PPAD-complete (but not NP-complete) even for two-player games (with some definition of PPAD introduced by Christos Papadimitriou in 1994)
- This is still one of the most important open problems in computational complexity theory
ε-Nash Equilibrium

- Reflects the idea that agents might not change strategies if the gain would be very small
- Let ε > 0. A strategy profile \( s = (s_1, \ldots, s_n) \) is an ε-Nash equilibrium if for every agent \( i \) and for every strategy \( s_i' \neq s_i \),
  \[
  u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) - \varepsilon
  \]
- ε-Nash equilibria exist for every \( \varepsilon > 0 \)
  - Every Nash equilibrium is an ε-Nash equilibrium, and is surrounded by a region of ε-Nash equilibria
- This concept can be computationally useful
  - Algorithms to identify ε-Nash equilibria need consider only a finite set of mixed-strategy profiles (not the whole continuous space)
  - Because of finite precision, computers generally find only ε-Nash equilibria, where \( \varepsilon \) is roughly the machine precision
- Finding an ε-Nash equilibrium is still PPAD-complete (but not NP-complete) even for two-player games
The Price of Anarchy (PoA)

- In the Chocolate Game, recall that

  - \((T_3, T_3)\) is the action profile that provides the best outcome for everyone

  - If we assume each payer acts to maximize his/her utility without regard to the other, we get \((T_1, T_1)\)

  - By choosing \((T_3, T_3)\), each player could have gotten 3 times as much

- Let’s generalize “best outcome for everyone”
The Price of Anarchy

- **Social welfare function**: a function $w(s)$ that measures the players’ welfare, given a strategy profile $s$, e.g.,
  - Utilitarian function: $w(s) = \text{average expected utility}$
  - Egalitarian function: $w(s) = \text{minimum expected utility}$

- **Social optimum**: benevolent dictator chooses $s^*$ that optimizes $w$
  - $s^* = \arg \max_s w(s)$

- **Anarchy**: no dictator; every player selfishly tries to optimize his/her own expected utility, disregarding the welfare of the other players
  - Get a strategy profile $s$ (e.g., a Nash equilibrium)
  - In general, $w(s) \leq w(s^*)$

**Price of Anarchy (PoA)** $= \max_{s \text{ is Nash equilibrium}} \frac{w(s^*)}{w(s)}$

- PoA is the most popular measure of inefficiency of equilibria.
- We are generally interested in PoA which is closer to 1, i.e., all equilibria are good approximations of an optimal solution.
The Price of Anarchy

- Example: the Chocolate Game
  - Utilitarian welfare function: 
    \[ w(s) = \text{average expected utility} \]
  - Social optimum: \( s^* = (T3,T3) \)
    - \( w(s^*) = 3 \)
  - Anarchy: \( s = (T1,T1) \)
    - \( w(s) = 1 \)
  - Price of anarchy
    \[ = \frac{w(s^*)}{w(s)} = \frac{3}{1} = 3 \]

- What would the answer be if we used the egalitarian welfare function?
The Price of Anarchy

- Sometimes instead of *maximizing* a welfare function $w$, we want to *minimize* a cost function $c$ (e.g. in Prisoner’s Dilemma)
  - Utilitarian function: $c(s) = \text{avg. expected cost}$
  - Egalitarian function: $c(s) = \text{max. expected cost}$
- Need to adjust the definitions
  - *Social optimum*: $s^* = \arg \min_s c(s)$
  - *Anarchy*: every player selfishly tries to minimize his/her own cost, disregarding the costs of the other players
    - Get a strategy profile $s$ (e.g., a Nash equilibrium)
    - In general, $c(s) \geq c(s^*)$
  - *Price of Anarchy (PoA)* = $\max_{s \text{ is Nash equilibrium}} \frac{c(s)}{c(s^*)}$
    - i.e., the reciprocal of what we had before
    - E.g. in Prisoner’s dilemma, $\text{PoA} = 3$

\[
\begin{array}{|c|c|c|}
\hline
& C & D \\
\hline
C & 3, 3 & 0, 5 \\
\hline
D & 5, 0 & 1, 1 \\
\hline
\end{array}
\]
Rationalizability

- A strategy is **rationalizable** if a *perfectly rational agent* could justifiably play it against *perfectly rational opponents*
  - The formal definition complicated
- Informally:
  - A strategy for agent $i$ is rationalizable if it’s a best response to strategies that $i$ could *reasonably* believe the other agents have
  - To be reasonable, $i$’s beliefs must take into account
    - the other agents’ knowledge of $i$’s rationality,
    - their knowledge of $i$’s knowledge of *their* rationality,
    - and so on so forth recursively
- A **rationalizable strategy profile** is a strategy profile that consists only of rationalizable strategies
Rationalizability

- Every Nash equilibrium is composed of rationalizable strategies.
- Thus the set of rationalizable strategies (and strategy profiles) is always nonempty.

**Example: Which Side of the Road**

- For Agent 1, the pure strategy \( s_1 = \text{Left} \) is rationalizable because
  - \( s_1 = \text{Left} \) is 1’s best response if 2 uses \( s_2 = \text{Left} \),
  - and 1 can reasonably believe 2 would rationally use \( s_2 = \text{Left} \), because
    - \( s_2 = \text{Left} \) is 2’s best response if 1 uses \( s_1 = \text{Left} \),
    - and 2 can reasonably believe 1 would rationally use \( s_1 = \text{Left} \), because
      - \( s_1 = \text{Left} \) is 1’s best response if 2 uses \( s_2 = \text{Left} \),
      - and 1 can reasonably believe 2 would rationally use \( s_2 = \text{Left} \), because
        - … and so on so forth…
Some rationalizable strategies are not part of any Nash equilibrium

Example: Matching Pennies

For Agent 1, the pure strategy \( s_1 = Heads \) is rationalizable because

- \( s_1 = Heads \) is 1’s best response if 2 uses \( s_2 = Heads \),
- and 1 can reasonably believe 2 would rationally use \( s_2 = Heads \), because
  - \( s_2 = Heads \) is 2’s best response if 1 uses \( s_1 = Tails \),
  - and 2 can reasonably believe 1 would rationally use \( s_1 = Tails \), because
    - \( s_1 = Tails \) is 1’s best response if 2 uses \( s_2 = Tails \),
    - and 1 can reasonably believe 2 would rationally use \( s_2 = Tails \), because
      - … and so on so forth…
Common Knowledge

- The definition of common knowledge is recursive analogous to the definition of rationalizability
- A property $p$ is *common knowledge* if
  - Everyone knows $p$
  - Everyone knows that everyone knows $p$
  - Everyone knows that everyone knows that everyone knows $p$
  - ...

We Aren’t Rational

- More evidence that we aren’t game-theoretically rational agents

- Why choose an “irrational” strategy?
  - Several possible reasons …
Reasons for Choosing “Irrational” Strategies

1. Limitations in reasoning ability
   - Didn’t calculate the Nash equilibrium correctly
   - Don’t know how to calculate it
   - Don’t even know the concept

2. Wrong payoff matrix - doesn’t encode agent’s actual preferences
   - It’s a common error to take an external measure (money, points, etc.) and assume it’s all that an agent cares about
   - Other things may be more important than winning
     - Being helpful
     - Curiosity
     - Creating mischief
     - Venting frustration

3. Beliefs about the other agents’ likely actions (next slide)
Beliefs about Other Agents’ Actions

- A Nash equilibrium strategy is best for you if the other agents also use their Nash equilibrium strategies.

- In many cases, the other agents won’t use Nash equilibrium strategies:
  - If you can guess what actions they’ll choose, then
    - You can compute your best response to those actions
      - maximize your expected payoff, given their actions
    - Good guess => you may do much better than the Nash equilibrium
    - Bad guess => you may do much worse
Worst-Case Expected Utility

- For agent $i$, the worst-case expected utility of a strategy $s_i$ is the minimum over all possible combinations of strategies for the other agents:

$$\min_{s_{-i}} u_i(s_i, s_{-i})$$

- Example: Battle of the Sexes

  - Wife’s strategy $s_w = \{(p, \text{Opera}), (1 - p, \text{Football})\}$
  - Husband’s strategy $s_h = \{(q, \text{Opera}), (1 - q, \text{Football})\}$
  - $u_w(p, q) = 2pq + (1 - p)(1 - q) = 3pq - p - q + 1$

  - For any fixed $p$, $u_w(p, q)$ is linear in $q$
    - e.g., if $p = \frac{1}{2}$, then $u_w(\frac{1}{2}, q) = \frac{1}{2} q + \frac{1}{2}$
  - $0 \leq q \leq 1$, so the min must be at $q = 0$ or $q = 1$
    - e.g., $\min_q \left( \frac{1}{2} q + \frac{1}{2} \right)$ is at $q = 0$
  - $\min_q u_w(p, q) = \min (u_w(p, 0), u_w(p, 1)) = \min (1 - p, 2p)$

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We can write $u_w(p, q)$ instead of $u_w(s_w, s_h)$
Maxmin Strategies

A maxmin strategy for agent $i$

- A strategy $s_1$ that makes $i$’s worst-case expected utility as high as possible:

$$\arg\max_{s_i} \min_{s_i} u_i(s_i, s_i)$$

- This isn’t necessarily unique
- Often it is mixed

Agent $i$’s maxmin value, or security level, is the maxmin strategy’s worst-case expected utility:

$$\max_{s_i} \min_{s_i} u_i(s_i, s_i)$$

For 2 players it simplifies to

$$\max_{s_1} \min_{s_2} u_1(s_1, s_2)$$

Also called maxmin
Example

- Wife’s and husband’s strategies
  - $s_w = \{(p, \text{Opera}), (1-p, \text{Football})\}$
  - $s_h = \{(q, \text{Opera}), (1-q, \text{Football})\}$

- Recall that wife’s worst-case expected utility is
  \[
  \min_q u_w(p,q) = \min (1-p, 2p)
  \]
  - Find $p$ that maximizes it

- Max is at $1-p = 2p$, i.e., $p = 1/3$
  - Wife’s maxmin value is $1-p = 2/3$
  - Wife’s maxmin strategy is
    \[
    \{(1/3, \text{Opera}), (2/3, \text{Football})\}
    \]

- Similarly,
  - Husband’s maxmin value is $2/3$
  - Husband’s maxmin strategy is
    \[
    \{(2/3, \text{Opera}), (1/3, \text{Football})\}
    \]
Minmax Strategies (in 2-Player Games)

- **Minmax strategy and minmax value**
  - Duals of their maxmin counterparts
- Suppose agent 1 wants to punish agent 2, regardless of how it affects agent 1’s own payoff
- Agent 1’s **minmax strategy** against agent 2
  - A strategy \( s_1 \) that minimizes the expected utility of 2’s best response to \( s_1 \)
    \[
    \arg \min_{s_1} \max_{s_2} u_2(s_1, s_2)
    \]
- Agent 2’s **minmax value** is 2’s maximum expected utility if agent 1 plays his/her minmax strategy:
  \[
  \min_{s_1} \max_{s_2} u_2(s_1, s_2)
  \]
- **Minmax strategy profile**: both players use their minmax strategies
Example

- Wife’s and husband’s strategies
  - \( s_w = \{(p, \text{Opera}), (1-p, \text{Football})\} \)
  - \( s_h = \{(q, \text{Opera}), (1-q, \text{Football})\} \)

- \( u_h(p,q) = pq + 2(1-p)(1-q) = 3pq - 2p - 2q + 2 \)

- Given wife’s strategy \( p \), husband’s expected utility is linear in \( q \)
  - e.g., if \( p = \frac{1}{2} \), then \( u_h(\frac{1}{2},q) = -\frac{1}{2} q + 1 \)

- Max is at \( q = 0 \) or \( q = 1 \)
  - \( \max_q u_h(p,q) = (2-2p, p) \)

- Find \( p \) that minimizes this

- Min is at \(-2p + 2 = p \) \( \rightarrow \) \( p = \frac{2}{3} \)

- Husband/s minmax value is \( \frac{2}{3} \)

- Wife’s minmax strategy is \( \{(\frac{2}{3}, \text{Opera}), (\frac{1}{3}, \text{Football})\} \)

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Minmax Strategies in $n$-Agent Games

- In $n$-agent games ($n > 2$), agent $i$ usually can’t minimize agent $j$’s payoff by acting unilaterally
- But suppose all the agents “gang up” on agent $j$
  - Let $s^*_{-j}$ be a mixed-strategy profile that minimizes $j$’s maximum payoff, i.e.,
    \[
    s^*_j = \arg\min_{s_j} \max_{s_j} u_j(s_j, s_{-j})
    \]
  - For every agent $i \neq j$, a **minmax strategy for $i$** is $i$’s component of $s_{-j}^*$

- Agent $j$’s **minmax value** is $j$’s maximum payoff against $s_{-j}^*$
  \[
  \max_{s_j} u_j(s_j, s^*_j) = \min_{s_j} \max_{s_j} u_j(s_j, s_j)
  \]
- We have equality since we just replaced $s_{-j}^*$ by its value above
Minimax Theorem (von Neumann, 1928)

**Theorem.** Let $G$ be any finite two-player zero-sum game. For each player $i$,

- $i$’s expected utility in any Nash equilibrium 
  $= i$’s maxmin value 
  $= i$’s minmax value 

- In other words, for every Nash equilibrium $(s_1^*, s_2^*)$,

$$u_1(s_1^*, s_2^*) = \min_{s_1} \max_{s_2} u_1(s_1, s_2) = \max_{s_1} \min_{s_2} u_1(s_1, s_2) = -u_2(s_1^*, s_2^*)$$

  - Note that since $-u_2 = u_1$ the third term does not mention $u_2$

**Corollary.** For two-player zero-sum games: \{Nash equilibria\} = \{maxmin strategy profiles\} = \{minmax strategy profiles\}

- Note that this is **not necessary true** for **non-zero-sum** games as we saw for Battle of Sexes in previous slides

- Terminology: the **value** (or **minmax value**) of $G$ is agent 1’s minmax value
Proof of Minimax Theorem

- Let \( u_2 = u_1 = u \) and let mixed strategies \( s_1 = x = (x_1, \ldots, x_k) \) and \( s_2 = y = (y_1, \ldots, y_r) \).
- Then \( u(x, y) = \sum i \sum j x_i y_j u_{i,j} = \sum j y_j \sum i x_i u_{i,j} \).
- We want to find \( x^* \) which optimizes \( v^1 = \max_x \min_y u(x,y) \).
- Since player 2 is doing his best response (in \( \min_y u(x,y) \)) he sets \( y_j > 0 \) only if \( \sum_i x_i u_{i,j} \) is minimized.

Thus \( v^1 = \sum_j \sum_i x_i y_j u_{i,j} = (\sum_j y_j) \min_j \sum_i x_i u_{i,j} = \min_j \sum_i x_i u_{i,j} \leq \sum_i x_i u_{i,j} \) for any \( j \).

Thus we have the following LP1 to find \( x^* \):

\[
\max v^1 \\
\text{such that } v^1 \leq \sum_i x_i u_{i,j} \text{ for all } j \\
\sum_i x_i = 1 \\
x_i \geq 0
\]
Proof of Minimax Theorem (continued)

- Similarly for \( v^2 = \min_y \max_x u(x,y) \) we have LP2
  \[
  \min v^2
  \] such that \( v^2 \geq \sum_j y_j u_{i,j} \) for all \( i \)
  \[
  \sum_j y_j = 1
  \]
  \[
  y_j \geq 0
  \]

- But LP1 And LP2 are duals of each other and by the (strong) duality theorem \( v^1 = v^2 \)

- Also note that if \((x,y)\) is a Nash equilibrium, \(x\) should satisfy LP1 (since we used only the fact that \(y\) is a best response to \(x\) in the proof) and \(y\) should satisfy LP2 (since we used only the fact that \(x\) is a best response to \(y\) in the proof) and thus \( u_1(x,y) = v^1 = v^2 \)
Dominant Strategies

- Let $s_i$ and $s_i'$ be two strategies for agent $i$
  - Intuitively, $s_i$ dominates $s_i'$ if agent $i$ does better with $s_i$ than with $s_i'$ for every strategy profile $s_{-i}$ of the remaining agents
- Mathematically, there are three gradations of dominance:
  - $s_i$ **strictly dominates** $s_i'$ if for every $s_{-i}$,
    $$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$
  - $s_i$ **weakly dominates** $s_i'$ if for every $s_{-i}$,
    $$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$
    and for at least one $s_{-i}$,
    $$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$
  - $s_i$ **very weakly dominates** $s_i'$ if for every $s_{-i}$,
    $$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$
Dominant Strategy Equilibria

- A strategy is **strictly** (resp., **weakly**, **very weakly**) dominant for an agent if it strictly (weakly, very weakly) dominates any other strategy for that agent.

- A strategy profile \((s_1, \ldots, s_n)\) in which every \(s_i\) is dominant for agent \(i\) (strictly, weakly, or very weakly) is a Nash equilibrium.
  - Why?
    - Such a strategy profile forms an equilibrium in strictly (weakly, very weakly) dominant strategies.
Examples

- Example: the Prisoner’s Dilemma
  
  - [http://www.youtube.com/watch?v=ED9gaAb2BEw](http://www.youtube.com/watch?v=ED9gaAb2BEw)

  - For agent 1, $D$ is strictly dominant
    
    - If agent 2 uses $C$, then
      - Agent 1’s payoff is higher with $D$ than with $C$
    
    - If agent 2 uses $D$, then
      - Agent 1’s payoff is higher with $D$ than with $C$

  - Similarly, $D$ is strictly dominant for agent 2

  - So $(D,D)$ is a Nash equilibrium in strictly dominant strategies

- How do strictly dominant strategies relate to strict Nash equilibria?
**Example: Matching Pennies**

- **Matching Pennies**
  - If agent 2 uses Heads, then
    - For agent 1, Heads is better than Tails
  - If agent 2 uses Tails, then
    - For agent 1, Tails is better than Heads
  - Agent 1 doesn’t have a dominant strategy
    => no Nash equilibrium in dominant strategies

- **Which Side of the Road**
  - Same kind of argument as above
  - No Nash equilibrium in dominant strategies
Elimination of Strictly Dominated Strategies

- A strategy $s_i$ is strictly (weakly, very weakly) dominated for an agent $i$ if some other strategy $s_i'$ strictly (weakly, very weakly) dominates $s_i$.

- A strictly dominated strategy can’t be a best response to any move, so we can eliminate it (remove it from the payoff matrix).
  - This gives a reduced game.
  - Other strategies may now be strictly dominated, even if they weren’t dominated before.

- **IESDS (Iterated Elimination of Strictly Dominated Strategies):**
  - Do elimination repeatedly until no more eliminations are possible.
  - When no more eliminations are possible, we have the maximal reduction of the original game.
If you eliminate a strictly dominated strategy, the reduced game has the same Nash equilibria as the original one.

Thus

\[ \{ \text{Nash equilibria of the original game} \} = \{ \text{Nash equilibria of the maximally reduced game} \} \]

Use this technique to simplify finding Nash equilibria.

- Look for Nash equilibria on the maximally reduced game.

In the example, we ended up with a single cell.

- The single cell must be a unique Nash equilibrium in all three of the games.

### IESDS

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### Reduced Game

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### Single Cell

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Even if $s_i$ isn’t strictly dominated by a pure strategy, it may be strictly dominated by a mixed strategy.

**Example:** the three games shown at right

1. **1st game:**
   - R is strictly dominated by L (and by C)
   - Eliminate it, get 2nd game

2. **2nd game:**
   - Neither $U$ nor $D$ dominates $M$
   - But $\{(1/2, U), (1/2, D)\}$ strictly dominates $M$
     - This wasn’t true before we removed $R$
   - Eliminate it, get 3rd game

3. **3rd game** is maximally reduced

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If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium.

----Roger Myerson
Correlated Equilibrium: Intuition

- Not every correlated equilibrium is a Nash equilibrium but every Nash equilibrium is a correlated equilibrium.
- We have a traffic light: a fair randomizing device that tells one of the agents to go and the other to wait.
- Benefits:
  - easier to compute than Nash, e.g., it is polynomial-time computable
  - fairness is achieved
  - the sum of social welfare exceeds that of any Nash equilibrium
Correlated Equilibrium

- Recall the mixed-strategy equilibrium for the Battle of the Sexes
  - \( s_w = \{(2/3, \text{Opera}), (1/3, \text{Football})\} \)
  - \( s_h = \{(1/3, \text{Opera}), (2/3, \text{Football})\} \)
- This is “fair”: each agent is equally likely to get his/her preferred activity
- But 5/9 of the time, they’ll choose different activities \( \Rightarrow \) utility 0 for both
  - Thus each agent’s expected utility is only 2/3
  - We’ve required them to make their choices independently
- Coordinate their choices (e.g., flip a coin) \( \Rightarrow \) eliminate cases where they choose different activities
  - Each agent’s payoff will always be 1 or 2; expected utility 1.5
- Solution concept: **correlated** equilibrium
  - Generalization of a Nash equilibrium
Correlated Equilibrium Definition

- Let $G$ be an 2-agent game (for now).
- Recall that in a (mixed) Nash Equilibrium at the end we compute a probability matrix (also known as joint probability distribution) $P = [p_{i,j}]$ where $\Sigma_{i,j} p_{i,j} = 1$ and in addition $p_{i,j} = q_i \cdot q_j'$ where $\Sigma_i q_i = 1$ and $\Sigma_j q_j' = 1$ (here $q$ and $q'$ are the mixed strategies of the first agent and the second agent).
- Now if we remove the constraint $p_{i,j} = q_i \cdot q_j'$ (and thus $\Sigma_i q_i = 1$ and $\Sigma_j q_j' = 1$) but still keep all other properties of Nash Equilibrium then we have a Correlated Equilibrium.
- Surely it is clear that by this definition of Correlated Equilibrium, every Nash Equilibrium is a Correlated Equilibrium as well but note vice versa.
- Even for a more general $n$-player game, we can compute a Correlated Equilibrium in polynomial time by a linear program (as we see in the next slide).
- Indeed the constraint $p_{i,j} = q_i \cdot q_j'$ is the one that makes computing Nash Equilibrium harder.
Computing CE

\[
\sum_{a \in A \mid a_i \in a} p(a)u_i(a) \geq \sum_{a \in A \mid a'_i \in a} p(a)u_i(a'_i, a_{-i}) \quad \forall i \in N, \forall a_i, a'_i \in A_f
\]

\[p(a) \geq 0\]

\[\sum_{a \in A} p(a) = 1\]

variables: \(p(a)\); constants: \(u_i(a)\)

we could find the social-welfare maximizing CE by adding an objective function

\[
\text{maximize: } \sum_{a \in A} p(a) \sum_{i \in N} u_i(a).
\]
Motivation of Correlated Equilibrium

- Let $G$ be an $n$-agent game
- Let “Nature” (e.g., a traffic light) choose action profile $a = (a_1, \ldots, a_n)$ randomly according to our computed joint probability distribution (Correlated Equilibrium) $p$.
- Then “Nature” tells each agent $i$ the value of $a_i$ (privately)
  - An agent can condition his/her action based on (private) value $a_i$
- However by the definition of best response in Nash Equilibrium (which also exists in Correlated Equilibrium), agent $i$ will not deviate from suggested action $a_i$
  - Note that here we implicitly assume because other agents are rational as well, they choose the suggested actions by the “Nature” which are given to them privately.
- Since there is no randomization in the actions, the correlated equilibrium might seem more natural.
Auctions

- An auction is a way (other than bargaining) to sell a fixed supply of a 
  commodity (an item to be sold) for which there is no well-established 
  ongoing market

- **Bidders** make **bids**
  - proposals to pay various amounts of money for the commodity

- Often the commodity is sold to the bidder who makes the largest bid

- Example applications
  - Real estate, art, oil leases, electromagnetic spectrum, electricity, eBay, google ads

- **Private-value auctions**
  - Each bidder may have a different bidder value or bidder valuation (BV), i.e., how much the commodity is worth to that bidder
  - A bidder’s BV is his/her private information, not known to others
  - E.g., flowers, art, antiques
Types of Auctions

- Classification according to the rules for bidding
  - English
  - Dutch
  - First price sealed bid
  - Vickrey
  - many others
  - On the following pages, I’ll describe several of these and will analyze their equilibria

- A possible problem is *collusion* (secret agreements for fraudulent purposes)
  - Groups of bidders who won’t bid against each other, to keep the price low
  - Bidders who place phony (phantom) bids to raise the price (hence the auctioneer’s profit)
- If there’s collusion, the equilibrium analysis is no longer valid
English Auction

- The name comes from oral auctions in English-speaking countries, but I think this kind of auction was also used in ancient Rome.

- Commodities:
  - antiques, artworks, cattle, horses, wholesale fruits and vegetables, old books, etc.

- Typical rules:
  - Auctioneer solicits an opening bid from the group.
  - Anyone who wants to bid should call out a new price at least $c$ higher than the previous high bid (e.g., $c = 1$ dollar).
  - The bidding continues until all bidders but one have dropped out.
  - The highest bidder gets the object being sold, for a price equal to his/her final bid.

- For each bidder $i$, let
  - $v_i = i$’s valuation of the commodity (private information).
  - $B_i = i$’s final bid.

- If $i$ wins, then $i$’s profit is $\pi_i = v_i - B_i$ and everyone else’s profit = 0.
English Auction (continued)

- Nash equilibrium:
  - Each bidder $i$ participates until the bidding reaches $v_i$, then drops out.
  - The highest bidder, $i$, gets the object, at price $B_i < v_i$, so $\pi_i = B_i - v_i > 0$
    - $B_i$ is close to the second highest bidder’s valuation.
  - For every bidder $j \neq i$, $\pi_j = 0$.

- Why is this an equilibrium?

- Suppose bidder $j$ deviates and none of the other bidders deviate
  - If $j$ deviates by dropping out earlier,
    - Then $j$’s profit will be 0, no better than before.
  - If $u$ deviates by bidding $B_i > v_j$, then
    - $j$ wins the auction but $j$’s profit is $v_j - B_j < 0$, worse than before.
If there is a large range of bidder valuations, then the difference between the highest and 2nd-highest valuations may be large

- Thus if there’s wide disagreement about the item’s value, the winner might be able to get it for much less than his/her valuation

Let \( n \) be the number of bidders

- The higher \( n \) is, the more likely it is that the highest and 2nd-highest valuations are close
  - Thus, the more likely it is that the winner pays close to his/her valuation
First-Price Sealed-Bid Auctions

- Examples:
  - construction contracts (lowest bidder)
  - real estate
  - art treasures

- Typical rules
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer
  - The auctioneer opens the bid and finds the highest bidder
  - The highest bidder gets the object being sold, for a price equal to his/her own bid
  - Winner’s profit = $BV - price paid
  - Everyone else’s profit = 0
First-Price Sealed-Bid (continued)

- Suppose that
  - There are \( n \) bidders
  - Each bidder has a private valuation, \( v_i \), which is private information
  - But a probability distribution for \( v_i \) is common knowledge
    - Let’s say \( v_i \) is uniformly distributed over \([0, 100]\)
  - Let \( B_i \) denote the bid of player \( i \)
  - Let \( \pi_i \) denote the profit of player \( i \)

- What is the Nash equilibrium bidding strategy for the players?
  - Need to find the optimal bidding strategies

- First we’ll look at the case where \( n = 2 \)
First-Price Sealed-Bid (continued)

• Finding the optimal bidding strategies
  ➢ Let $B_i$ be agent $i$’s bid, and $\pi_i$ be agent $i$’s profit
  ➢ If $B_i \geq v_i$, then $\pi_i \leq 0$
    • So, assuming rationality, $B_i < v_i$
  ➢ Thus
    • $\pi_i = 0$ if $B_i \neq \max_j \{B_j\}$
    • $\pi_i = v_i - B_i$ if $B_i = \max_j \{B_j\}$
  ➢ How much below $v_i$ should your bid be?
  ➢ The smaller $B_i$ is,
    • the less likely that $i$ will win the object
    • the more profit $i$ will make if $i$ wins the object
First-Price Sealed-Bid (continued)

- Case $n = 2$
  - Suppose your BV is $v$ and your bid is $B$
  - Let $x$ be the other bidder’s BV and $\alpha x$ be his/her bid, where $0 < \alpha < 1$
    - You don’t know the values of $x$ and $\alpha$
  - Your expected profit is
    - $E(\pi) = P(\text{your bid is higher}) \cdot (v - B) + P(\text{your bid is lower}) \cdot 0$
- If $x$ is uniformly distributed over $[0, 100]$, then the pdf is $f(x) = 1/100$, $0 \leq x \leq 100$
  - $P(\text{your bid is higher}) = P(\alpha x < B) = P(x < B/\alpha) = \int_0^{B/\alpha} (1/100) \, dx = B/100\alpha$
  - so $E(\pi) = B(v - B)/100\alpha$
- If you want to maximize your expected profit (hence your valuation of money is risk-neutral), then your maximum bid is
  - $\max_B B(v - B)/100\alpha = \max_B B(v - B) = \max_B Bv - B^2$
  - maximum occurs when $v - 2B = 0 \implies B = v/2$
- So, bid $\frac{1}{2}$ of what the item is worth to you!
First-Price Sealed-Bid (continued)

- With \( n \) bidders, if your bid is \( B \), then
  - \( P(\text{your bid is the highest}) = (B/100\alpha)^{n-1} \)

- Assuming risk neutrality, you choose your bid to be
  - \( \max_B B^{n-1}(v-B) = v(n-1)/n \)

- As \( n \) increases, \( B \rightarrow v \)
  - I.e., increased competition drives bids close to the valuations
Dutch Auctions

- **Examples**
  - flowers in the Netherlands, fish market in England and Israel, tobacco market in Canada

- **Typical rules**
  - Auctioneer starts with a high price
  - Auctioneer lowers the price gradually, until some buyer shouts “Mine!”
  - The first buyer to shout “Mine!” gets the object at the price the auctioneer just called
  - Winner’s profit = BV – price
  - Everyone else’s profit = 0

- **Dutch auctions are game-theoretically equivalent to first-price, sealed-bid auctions**
  - The object goes to the highest bidder at the highest price
  - A bidder must choose a bid without knowing the bids of any other bidders
  - The optimal bidding strategies are the same
Sealed-Bid, Second-Price Auctions

- Background: Vickrey (1961)
- Used for:
  - stamp collectors’ auctions
  - US Treasury’s long-term bonds
  - Airwaves auction in New Zealand
  - eBay and Amazon
- Typical rules
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer
  - The auctioneer opens the bid and finds the highest bidder
  - The highest bidder gets the object being sold, for a price equal to the second highest bid
- Winner’s profit = BV – price
- Everyone else’s profit = 0
Sealed-Bid, Second-Price (continued)

- Equilibrium bidding strategy:
  - It is a weakly dominant strategy to bid your true value: This property is also called **truthfulness** or **strategyproofness** of an auction.
  - To show this, need to show that overbidding or underbidding cannot increase your profit and might decrease it.
- Let $V$ be your valuation of the object, and $X$ be the highest bid made by anyone else
- Let $s_V$ be the strategy of bidding $V$, and $\pi_V$ be your profit when using $s_V$
- Let $s_B$ be a strategy that bids some $B \neq V$, and $\pi_B$ be your profit when using $s_B$
- There are $3! = 6$ possible numeric orderings of $B$, $V$, and $X$:
  - Case 1, $X > B > V$: You don’t get the commodity either way, so $\pi_B = \pi_V = 0$
  - Case 2, $B > X > V$: $\pi_B = V - X < 0$, but $\pi_V = 0$
  - Case 3, $B > V > X$: you pay $X$ rather than your bid, so $\pi_B = \pi_V = V - X > 0$
  - Case 4, $X < B < V$: you pay $X$ rather than your bid, so $\pi_B = \pi_V = V - X > 0$
  - Case 5, $B < X < V$: $\pi_B = 0$, but $\pi_V = V - X > 0$
  - Case 6, $B < V < X$: You don’t get the commodity either way, so $\pi_B = \pi_V = 0$
Sealed-Bid, Second-Price (continued)

- Sealed-bid, 2nd-price auctions are nearly equivalent to English auctions
  - The object goes to the highest bidder
  - Price is close to the second highest BV (close since the second highest bids just a bit below his actual BV)
Coalitional Games with Transferable Utility

- Given a set of agents, a coalitional game defines how well each group (or coalition) of agents can do for itself—its payoff
  - Not concerned with
    - how the agents make individual choices within a coalition,
    - how they coordinate, or
    - any other such detail

- **Transferable utility** assumption: the payoffs to a coalition may be freely redistributed among its members
  - Satisfied whenever there is a universal **currency** that is used for exchange in the system
  - Implies that each coalition can be assigned a single value as its payoff
Coalitional Games with Transferable Utility

- A **coalitional game with transferable utility** is a pair $G = (N, v)$, where
  - $N = \{1, 2, \ldots, n\}$ is a finite set of players
  - $(\text{nu}) \; v : 2^N \to \mathbb{R}$ associates with each coalition $S \subseteq N$ a real-valued payoff $v(S)$, that the coalition members can distribute among themselves

- $v$ is the **characteristic function**
  - We assume $v(\emptyset) = 0$ and that $v$ is non-negative.

- A coalition’s payoff is also called its **worth**

Coalitional game theory is normally used to answer two questions:

1. Which coalition will form?
2. How should that coalition divide its payoff among its members?

- The answer to (1) is often “the grand coalition” (all of the agents)
  - But this answer can depend on making the right choice about (2)
Example: A Voting Game

- Consider a parliament that contains 100 representatives from four political parties:
  - A (45 reps.), B (25 reps.), C (15 reps.), D (15 reps.)
- They’re going to vote on whether to pass a $100 million spending bill (and how much of it should be controlled by each party)
- Need a majority (≥ 51 votes) to pass legislation
  - If the bill doesn’t pass, then every party gets 0
- More generally, a voting game would include
  - a set of agents $N$
  - a set of winning coalitions $W \subseteq 2^N$
    - In the example, all coalitions that have enough votes to pass the bill
      - $\nu(S) = 1$ for each coalition $S \in W$
    - Or equivalently, we could use $\nu(S) = \$100$ million
      - $\nu(S) = 0$ for each coalition $S \not\in W$
Superadditive Games

- A coalitional game $G = (N, v)$ is **superadditive** if the union of two disjoint coalitions is worth at least the sum of its members’ worths
  - for all $S, T \subseteq N$, if $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$

- The voting-game example is superadditive
  - If $S \cap T = \emptyset$, $v(S) = 0$, and $v(T) = 0$, then $v(S \cup T) \geq 0$
  - If $S \cap T = \emptyset$ and $v(S) = 1$, then $v(T) = 0$ and $v(S \cup T) = 1$
  - Hence $v(S \cup T) \geq v(S) + v(T)$

- If $G$ is superadditive, the grand coalition always has the highest possible payoff
  - For any $S \neq N$, $v(N) \geq v(S) + v(N - S) \geq v(S)$

- $G = (N, v)$ is **additive** (or **inessential**), if
  - For $S, T \subseteq N$ and $S \cap T = \emptyset$, then $v(S \cup T) = v(S) + v(T)$
**Constant-Sum Games**

- \( G \) is **constant-sum** if the worth of the grand coalition equals the sum of the worths of any two coalitions that partition \( N \)
  - \( v(S) + v(N-S) = v(N) \), for every \( S \subseteq N \)

- Every additive game is constant-sum
  - additive \( \Rightarrow v(S) + v(N-S) = v(S \cup (N-S)) = v(N) \)

- But not every constant-sum game is additive
  - Example is a good exercise
Convex Games

- $G$ is **convex (supermodular)** if for all $S, T \subseteq N$,
  
  \[ v(S \cup T) + v(S \cap T) \geq v(S) + v(T) \]

- It can be shown the above definition is equivalent to for all $i$ in $N$ and for all $S \subseteq T \subseteq N - \{i\}$,
  
  \[ v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S) \]

  - Prove it as an exercise

- Recall the definition of a superadditive game:
  
  \[ \text{for all } S, T \subseteq N, \text{ if } S \cap T = \emptyset, \text{ then } v(S \cup T) \geq v(S) + v(T) \]

- It follows immediately that every super-additive game is a convex game
Simple Coalitional Games

- A game \( G = (N, v) \) is simple for every coalition \( S \),
  - either \( v(S) = 1 \) (i.e., \( S \) wins) or \( v(S) = 0 \) (i.e., \( S \) loses)
  - Used to model voting situations (e.g., the example earlier)
- Often add a requirement that if \( S \) wins, all supersets of \( S \) would also win:
  - if \( v(S) = 1 \), then for all \( T \supseteq S \), \( v(T) = 1 \)
- This doesn’t quite imply superadditivity
  - Consider a voting game \( G \) in which 50% of the votes is sufficient to pass a bill
  - Two coalitions \( S \) and \( T \), each is exactly 50% \( N \)
  - \( v(S) = 1 \) and \( v(T) = 1 \)
  - But \( v(S \cup T) \neq 2 \)
Proper-Simple Games

- $G$ is a **proper simple game** if it is both simple and constant-sum
  - If $S$ is a winning coalition, then $N - S$ is a losing coalition
    - $v(S) + v(N - S) = 1$, so if $v(S) = 1$ then $v(N - S) = 0$

- Relations among the classes of games:

  $\{\text{Additive games}\} \subseteq \{\text{Super-additive games}\} \subseteq \{\text{Convex games}\}$
  $\{\text{Additive games}\} \subseteq \{\text{Constant-sum game}\}$
  $\{\text{Proper-simple games}\} \subseteq \{\text{Constant-sum games}\}$
  $\{\text{Proper-simple games}\} \subseteq \{\text{Simple game}\}$
Analyzing Coalitional Games

- Main question in coalitional game theory
  - How to divide the payoff to the grand coalition?
- Why focus on the grand coalition?
  - Many widely studied games are super-additive
    - Expect the grand coalition to form because it has the highest payoff
  - Agents may be required to join
    - E.g., public projects often legally bound to include all participants
- Given a coalitional game \( G = (N, v) \), where \( N = \{1, \ldots, n\} \)
  - We’ll want to look at the agents’ shares in the grand coalition’s payoff
    - The book writes this as \((\Psi) \psi(N,v) = x = (x_1, \ldots, x_n)\), where \(\psi_i(N,v) = x_i\) is the agent’s payoff
  - We won’t use the \(\psi\) notation much
    - Can be useful for talking about several different coalitional games at once, but we usually won’t be doing that
Terminology

- Feasible payoff set
  
  = \{ \text{all payoff profiles that don’t distribute more than the worth of the grand coalition} \}
  
  = \{(x_1, \ldots, x_n) \mid x_1 + x_2 + \ldots + x_n \leq v(N) \}

- Pre-imputation set

  \mathcal{P} = \{ \text{feasible payoff profiles that are efficient, i.e., distribute the entire worth of the grand coalition} \}
  
  = \{(x_1, \ldots, x_n) \mid x_1 + x_2 + \ldots + x_n \} = v(N)

- Imputation set

  \mathcal{C} = \{\text{payoffs in } \mathcal{P} \text{ in which each agent gets at least what he/she would get by going alone (i.e., forming a singleton coalition)}\}
  
  = \{(x_1, \ldots, x_n) \in \mathcal{P} : \forall i \in N, x_i \geq v(\{i\}) \}

\textbf{impute: verb [trans.] } \text{represent as being done, caused, or possessed by someone; attribute : the crimes imputed to Richard.}
Fairness, Symmetry

- What is a **fair** division of the payoffs?
  - Three axioms describing fairness
    - *Symmetry, dummy player, and additivity* axioms

- Definition: agents *i* and *j* are **interchangeable** if they always contribute the same amount to every coalition of the other agents
  - i.e., for every *S* that contains neither *i* nor *j*, \( v(S \cup \{i\}) = v(S \cup \{j\}) \)

- **Symmetry axiom**: in a fair division of the payoffs, interchangeable agents should receive the same payments, i.e.,
  - if *i* and *j* are interchangeable and \((x_1, \ldots, x_n)\) is the payoff profile, then \( x_i = x_j \)
Agent $i$ is a **dummy player** if $i$’s contributes to any coalition is exactly the amount $i$ can achieve alone

- i.e., for all $S$ s.t. $i \notin S$, $v(S \cup \{i\}) = v(S) + v(\{i\})$

**Dummy player axiom**: in a fair distribution of payoffs, dummy players should receive payment equal to the amount they achieve on their own

- i.e., if $i$ is a dummy player and $(x_1, \ldots, x_n)$ is the payoff profile, then $x_i = v(\{i\})$
Additivity

- Let $G_1 = (N, v_1)$ and $G_2 = (N, v_2)$ be two coalitional games with the same agents.
- Consider the combined game $G = (N, v_1 + v_2)$, where
  \[ (v_1 + v_2)(S) = v_1(S) + v_2(S) \]
- **Additivity axiom**: in a fair distribution of payoffs for $G$, the agents should get the sum of what they would get in the two separate games.
  \[ \text{i.e., for each player } i, \quad \psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2) \]
Shapley Values

- Recall that a pre-imputation is a payoff division that is both feasible and efficient

- **Theorem.** Given a coalitional game \((N,v)\), there’s a unique pre-imputation \(\varphi(N,v)\) that satisfies the Symmetry, Dummy player, and Additivity axioms. For each player \(i\), \(i\)’s share of \(\varphi(N,v)\) is

\[
\varphi_i(N,v) = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} |S|! \left( |N| - |S| - 1 \right)! (v(S \cup \{i\}) - v(S))
\]

- \(\varphi_i(N,v)\) is called \(i\)’s **Shapley value**
  - Lloyd Shapley introduced it in 1953

- It captures agent \(i\)’s **average marginal contribution**
  - The average contribution that \(i\) makes to the coalition, averaged over every possible sequence in which the grand coalition can be built up from the empty coalition
Shapley Values

- Suppose agents join the grand coalition one by one, all sequences equally likely
- Let $S = \{\text{agents that joined before } i\}$ and $T = \{\text{agents that joined after } i\}$
  - $i$’s marginal contribution is $v(S \cup \{i\}) - v(S)$
  - independent of how $S$ is ordered, independent of how $T$ is ordered
- $\Pr[S, \text{then } i, \text{then } T]$
  $= (\# \text{ of sequences that include } S \text{ then } i \text{ then } T) / (\text{total } \# \text{ of sequences})$
  $= \frac{|S|! \cdot |T|!}{|N|!}$
- Let $\varphi_{i,S} = \Pr[S, \text{then } i, \text{then } T] \times i$’s marginal contribution when it joins
- Then $i,S = \frac{|S|! \cdot (|N| - |S|) \cdot 1!}{|N|!} \cdot (v(S \cup \{i\}) - v(S))$
- Let $\varphi_i(N,v) = \text{expected contribution over all possible sequences}$
- Then $i(N,v) = \sum_{S \subseteq N \setminus \{i\}} i,S = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} |S|! \cdot (|N| - |S|) \cdot 1! \cdot (v(S \cup \{i\}) - v(S))$
Example

- The voting game again
  - Parties A, B, C, and D have 45, 25, 15, and 15 representatives
  - A simple majority (51 votes) is required to pass the $100M bill
- How much money is it fair for each party to demand?
  - Calculate the Shapley values of the game
- Every coalition with \( \geq 51 \) members has value 1; other coalitions have value 0
- Recall what it means for two agents \( i \) and \( j \) to be interchangeable:
  - for every \( S \) that contains neither \( i \) nor \( j \), \( v(S \cup \{i\}) = v(S \cup \{j\}) \)
- B and C are interchangeable
  - Each adds 0 to \( \emptyset \), 1 to \( \{A\} \), 0 to \( \{D\} \), and 0 to \( \{A,D\} \)
- Similarly, B and D are interchangeable, and so are C and D
- So the fairness axiom says that B, C, and D should each get the same amount
Recall that \( i_{i,S} = \frac{|S|!(|N| - |S| - 1)!(v(S \cup \{i\}) - v(S))}{|N|!} \)

\[
i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)!(v(S \cup \{i\}) - v(S))
\]

In the example, it will be useful to let \( \varphi'_{i,S} \) be the term inside the summation

Hence \( \varphi'_{i,S} = |N|! \varphi_{i,S} \)

Let's compute \( \varphi_A(N, v) \)

\( N = |\{A,B,C,D\}| = 4 \), so \( A_{i,S} = |S|!(3 - |S|)! (v(S \cup A) - v(S)) \)

\( S \) may be any of the following:

\( \emptyset, \{B\}, \{C\}, \{D\}, \{B,C\}, \{B,D\}, \{C,D\} \)

We need to sum over all of them:

\[
A(N, v) = \frac{1}{4!} (A_{i,S} + A_{\{B\}} + A_{\{C\}} + A_{\{D\}} + A_{\{B,C\}} + A_{\{B,D\}} + A_{\{C,D\}} + A_{\{B,C,D\}})
\]
\[ A_{N} = \frac{1}{4!(3 \cdot n(S))} ( (\nu(S \cap A) - \nu(S)) ) \]

A has 45 members
B has 25 members
C has 15 members
D has 15 members

\[ \nu(S) = 0 \rightarrow \nu(\{A\}) - \nu(\emptyset) = 0 \rightarrow \phi'_{A,\emptyset} = 0! \cdot 3! \cdot 0 = 0 \]

\[ \nu(S) = \{B\} \rightarrow \nu(\{A,B\}) - \nu(\{B\}) = 1 \rightarrow \phi'_{A,\{B\}} = 1! \cdot 2! \cdot 1 = 2 \]

\[ \nu(S) = \{C\} \rightarrow \text{same} \]

\[ \nu(S) = \{D\} \rightarrow \text{same} \]

\[ \nu(S) = \{B,C\} \rightarrow \nu(\{A,B,C\}) - \nu(\{B,C\}) = 1 \rightarrow \phi'_{A,\{B,C\}} = 2! \cdot 1! \cdot 1 = 2 \]

\[ \nu(S) = \{B,D\} \rightarrow \text{same} \]

\[ \nu(S) = \{C,D\} \rightarrow \text{same} \]

\[ \nu(S) = \{B,C,D\} \rightarrow \nu(\{A,B,C,D\}) - \nu(\{B,C,D\}) = 1 \rightarrow \phi'_{A,\{B,C,D\}} = 3! \cdot 0! \cdot 0 = 0 \]

\[ A(N, \nu) = \frac{1}{4!} ( \frac{1}{4!} ( \frac{1}{4!} ( \frac{1}{4!} ( \frac{1}{4!} ( \frac{1}{4!} ( \frac{1}{4!} ( \frac{1}{4!} ( \frac{1}{4!} ( A_{\emptyset} + A_{\{B\}} + A_{\{C\}} + A_{\{D\}} + A_{\{B,C\}} + A_{\{B,D\}} + A_{\{C,D\}} + A_{\{B,C,D\}} ) ) ) ) ) ) ) ) ) ) ) ) ) )

\[ = \frac{1}{24} (0 + 2 + 2 + 2 + 2 + 2 + 2 + 0) = 12 \div 24 = \frac{1}{2} \]
\[ \varphi_i(N, \nu) = \frac{1}{|N|!} \sum_{S \subseteq N - \{i\}} |S|! \left( |N| - |S| - 1 \right)! (\nu(S \cup \{i\}) - \nu(S)) \]

- Similarly, \( \varphi_B = \varphi_C = \varphi_D = 1/6 \)
  - The text calculates it using Shapley’s formula
- Here’s another way to get it:
  - If \( A \) gets \( \frac{1}{2} \), then the other \( \frac{1}{2} \) will be divided among \( B, C, \) and \( D \)
  - They are interchangeable, so a fair division will give them equal amounts: \( 1/6 \) each
- So distribute the money as follows:
  - \( A \) gets \( (1/2) \times 100M = 50M \)
  - \( B, C, D \) each get \( (1/6) \times 100M = 16\frac{2}{3}M \)
Stability of the Grand Coalition

- Agents have incentive to form the grand coalition iff there aren’t any smaller coalitions in which they could get higher payoffs.

- Sometimes a subset of the agents may prefer a smaller coalition.

- Recall the Shapley values for our voting example:
  - \( A \) gets $50M; \( B, C, D \) each get $16\frac{2}{3}M
  - \( A \) on its own can’t do better
  - But \( \{ A, B \} \) have incentive to defect and divide the $100M
    - e.g., $75M for \( A \) and $25M for \( B \)

- What payment divisions would make the agents want to join the grand coalition?
The Core

- The core of a coalitional game includes every payoff vector $\mathbf{x}$ that gives every sub-coalition $S$ at least as much in the grand coalition as $S$ could get by itself.
  - All feasible payoff vectors $\mathbf{x} = (x_1, \ldots, x_n)$ such that for every $S \subseteq N$,
    
    $$ x_i - v(S) \geq 0, \quad i \in S $$

- For every payoff vector $\mathbf{x}$ in the core, no $S$ has any incentive to deviate from the grand coalition.
  - i.e., form their own coalition, excluding the others

- It follows immediately that if $\mathbf{x}$ is in the core then $\mathbf{x}$ is efficient.
  - Why?
Analogy to Nash Equilibria

- The core is an analog of the set of all Nash equilibria in a noncooperative game
  - There, no agent can do better by deviating from the equilibrium
- But the core is stricter
  - No set of agents can do better by deviating from the grand coalition
- Analogous to the set of strong Nash equilibria
  - Equilibria in which no coalition of agents can do better by deviating
- Unlike the set of Nash equilibria, the core may sometimes be empty
  - In some cases, no matter what the payoff vector is, some agent or group of agents has incentive to deviate
Example of an Empty Core

- Consider the voting example again:
  - Shapley values are $50M to A, and $16.33M each to B, C, D
- The minimal coalitions that achieve 51 votes are
  - \{A,B\}, \{A,C\}, \{A,D\}, \{B,C,D\}
- If the sum of the payoffs to B, C, and D is < $100M, this set of agents has incentive to deviate from the grand coalition
  - Thus if \( x \) is in the core, \( x \) must allocate $100M to \{B, C, D\}
  - But if B, C, and D get the entire $100M, then A (getting $0) has incentive to join with whichever of B, C, and D got the least
    - e.g., form a coalition \{A,B\} without the others
  - So if \( x \) allocates the entire $100M to \{B,C,D\} then \( x \) cannot be in the core
- So the core is empty
Simple Games

- There are several situations in which the core is either guaranteed to exist, or guaranteed not to exist
  - The first one involves simple games
- Recall: $G$ is simple for every coalition $S$, either $v(S) = 1$ or $v(S) = 0$
- Player $i$ is a veto player if $v(S) = 0$ for any $S \subseteq N - \{i\}$
- Theorem. In a simple game, the core is empty iff there is no veto player
- Example: previous slide
Theorem. In a simple game in which there are veto players, the core is \{all payoff vectors in which non-veto players get 0\}

Example: consider a modified version of the voting game

- An 80\% majority is required to pass the bill

Recall that A, B, C, and D have 45, 25, 15, and 15 representatives

- The minimal winning coalitions are \{A, B, C\} and \{A, B, D\}
- All winning coalitions must include both A and B
- So A and B are veto players
  - The core includes all distributions of the $100M among A and B
  - Neither A nor B can do better by deviating
Non-Additive Constant-Sum Games

- Recall:
  - $G$ is constant-sum if for all $S$, $\nu(S) + \nu(N - S) = \nu(N)$
  - $G$ is additive if $\nu(S \cup T) = \nu(S) + \nu(T)$ whenever $S$ and $T$ are disjoint
- **Theorem.** Every non-additive constant-sum game has an empty core
- **Example:** consider a constant-sum game $G$ with 3 players a, b, c
  - Suppose $\nu(a) = 1$, $\nu(b) = 1$, $\nu(c) = 1$, $\nu(\{a,b,c\})=4$
  - Then $\nu(a) + \nu(\{b,c\}) = \nu(\{a,b\})+\nu(c) = \nu(\{a,c\}) + \nu(b) = 4$
  - Thus $\nu(\{b,c\}) = 4 - 1 = 3 \neq \nu(b) + \nu(c)$
  - So $G$ is not additive
- Consider $x = (1.333, 1.333, 1.333)$
  - $\nu(\{a,b\}) = 3$, so if $\{a,b\}$ deviate, they can allocate $(1.5,1.5)$
- To keep $\{a,b\}$ from deviating, suppose we use $x = (1.5, 1.5, 1)$
  - $\nu(\{a,c\}) = 3$, so if $\{a,c\}$ deviate, they can allocate $(1.667, 1.333)$
Convex Games

- Recall:
  - $G$ is **convex** if for all $S, T \subseteq N$, $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$

- **Theorem.** Every convex game has a nonempty core

- **Theorem.** In every convex game, the Shapley value is in the core
Modified Parliament Example

- 100 representatives from four political parties:
  - A (45 reps.), B (25 reps.), C (15 reps.), D (15 reps.)
- Any coalition of parties can approve a spending bill worth $1K times the number of representatives in the coalition:

\[
v(S) = \sum_{i \in S} \$1000 \times \text{size}(i)
\]

\[
v(A) = $45K, \quad v(B) = $25K, \quad v(C) = $15K, \quad v(D) = $15K,
\]

\[
v(\{A,B\}) = $70K, \quad v(\{A,C\}) = $60K, \quad v(\{A,D\}) = $60K,
\]

\[
v(\{B,C\}) = $40K, \quad v(\{B,D\}) = $40K, \quad v(\{C,D\}) = $30K, \ldots
\]

\[
v(\{A,B,C,D\}) = $100K
\]

- Is the game convex?
Modified Parliament Example

Let $S$ be the grand coalition

- What is each party’s Shapley value in $S$?

Each party’s Shapley value is the average value it adds to $S$, averaged over all 24 of the possible sequences in which $S$ might be formed:

$A, B, C, D;$ $A, B, D, C;$ $A, C, B, D;$ $A, C, D, B;$ etc

In every sequence, every party adds exactly $1K$ times its size

Thus every party’s Shapley value is $1K$ times its size:

- $\varphi_A = $45K,
- $\varphi_B = $25K,
- $\varphi_C = $15K,
- $\varphi_D = $15K
Modified Parliament Example

- Suppose we distribute $\nu(S)$ by giving each party its Shapley value.
- Does any party or group of parties have an incentive to leave and form a smaller coalition $T$?
  - $\nu(T) = $1K times the number of representatives in $T$
    - = the sum of the Shapley values of the parties in $T$
  - If each party in $T$ gets its Shapley value, it does no better in $T$ than in $S$
  - If some party in $T$ gets more than its Shapley value, then another party in $T$ will get less than its Shapley value
- No case in which every party in $T$ does better in $T$ than in $S$
- No case in which all of the parties in $T$ will have an incentive to leave $S$ and join $T$
- Thus the Shapley value is in the core