Algorithmic Game Theory

Spring 2014

Assignment 2

Instructor: Mohammad T. Hajiaghayi

Due date: **Friday, April 4, 2014 before 4pm**

Please TYPE in your solutions after each problem and put your homework in my mailbox in the first floor of AV Williams.

Please see slides, handwritten notes, and other course materials (or even Wikipedia) for definitions.

1. A coalition game G is **convex (supermodular)** if for all $S,T \subseteq N$,

$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$

Prove that a game G is convex if and only if for all i in N and for all $S \subseteq T \subseteq N\text{-}\{i\}$,

$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$

2. Prove that Shapley values satisfy Symmetry, Dummy player, and Additively axioms.

3. Prove formally that if f^{eq} is an equilibrium multicommodity flow, then for any other multicommodity flow f which satisfies the same demands, $< c(f^{eq}), f^{eq} - f > \leq 0$

4. Prove that a Valid Utility game with a non-decreasing objective function is (1,1)-smooth and thus its PoA is at mos ½.

5- Prove that in the LP for the load balancing problem presented in the class (see the slides) when each client i has an (unsplittable) integer demand $D_i \geq 0$, in any (optimum) basic feasible solution the number of $x_{i,j}$ assignment variables which are integral is at least the number of clients minus the number of access points (this means in a practical scenario when the number of clients is many more that the number of access points, all clients will be served unsplittably).

6- The employer of a weatherman is determined that he should provide a good prediction of the weather for the following day. The weatherman's instruments are good, and he can, with sufficient effort, tune them to obtain the correct value for the probability of rain on the next day. There are many days, and on the ith day the true probability of rain is called p_i. On the evening of the $(i-1)$th day, the weatherman submits his estimate q_i for the probability of rain on the following day, the ith one. Which payment scheme should the employer adopt to reward or penalize the weatherman for his predictions, so that he is motivated to correctly determine p_i (that is, to declare $q_i = p_i$)? Note that the employer does not know what the correct p_i is because he has no access to technical equipment, but he does know the q_i values that the weatherman provides, and he sees whether or not it is raining on each day.

7- Suppose we have *n* men and *n* women. Every man has a preference order over the *n* women, while every woman also has a preference order over the *n* men. A **matching** is a one-to-one mapping between the men and women, and it is perfect if all men and women are matched. A matching M is **unstable** if there exists a man and a woman who are not matched to each other in M, but prefer each other to their partners in M. Otherwise, the matching is called **stable**.
The following algorithm which is called the **men-proposing** algorithm was introduced by Gale and Shapley.

(i) Initially each woman is not tentatively matched.
(ii) Each man proposes to his most preferred woman.
(iii) Each woman evaluates her proposers, including the man she is tentatively matched to, if there is one, and rejects all but the most preferred one. She becomes tentatively matched to this latter man.
(iv) Each rejected man proposes to his next preferred woman.
(v) Repeat step (ii) and (iii) until each woman has a tentative match. At that point the tentative matches become final.

(Similarly, we could define a women-proposing algorithm.)
Part A) prove that the men-proposing algorithm yields a stable matching.

Part B) We say a woman a is **attainable** for a man x if there exists a stable matching M with M(x) = a. Let M be the stable matching produced by Gale-Shapley men-proposing algorithm. Then, prove

(i) For every man i, M(i) is the most preferred attainable woman for i.

(ii) For every woman j, $M^{-1}(j)$ is the least preferred attainable man for j.

8) Costco charges 55$ as a membership fee. Lots of other discount stores such as Walmart do not charge any membership fees. Model these two approaches with simple but realistic formula and analyze these two games. Especially via your formulation mention the situations that each of these approaches makes a better sense **(for this problem the main goal is to measure your abilities to model a real-world game).**