# Algorithmic Game Theory

Spring 2014

Solutions to Assignment 1

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1. Find all Nash equilibria and correlated equilibria of the following game. Explain why there is no other Nash equilibria or correlated equilibria.

For the second player, strategy C strictly dominates strategies A and B. Thus, A and B are neither Nash equilibrium nor correlated Nash equilibrium and we can remove these two actions from the second player's strategy set. In the remaining game strategy A with value 2 strictly dominates strategies B and C for the first player. Thus, the only Nash equilibrium and the only correlated nash equilibrium is (A,C).

| | A | B | C |
|---|---|---|---|
| A | 2,2 | 3,1 | 2,3 |
| B | 1,3 | 1,1 | -1,4 |
| C | 3,2 | 2,-1 | 1,3 |

2. In number-distance game each player selects a number from set {-1,0,2}. The utility of the first player is |A-B| and the utility of the second player is A-|A-B|, where A is the number that first player selects and B is the number that the second player selects. Present a Nash equilibrium of the number-distance game.

The payoff matrix of the number-distance game is as follow. We know that each player is invariant to the different actions in her support. First we guessed that all of the actions are in supports for both players. Let x,y,z be the probability that the first players plays -1,0,2 respectively and Let p,q,r be the probability that the second players plays -1,0,2 respectively.

| | -1 | 0 | 2 |
|---|---|---|---|
| -1 | 0,-1 | 1,-2 | 3,-4 |
| 0 | 1,-1 | 0,0 | 2,-2 |
| 2 | 3,-1 | 2,0 | 0,2 |

For the first player we have:

0*p+1*q+3*r = 1*p+0*q+2*r = 3*p+2*q+0*r

p+q+r=1

and for the second player we have:

-1*x-1*y-1*z = -2*x+0*y+0*z = -4*x-2*y+2*z

x+y+z=1

One can easily see that **(**(0.5, 0, 0.5) **,** (0.5, 0, 0.5)**)** is a mixed Nash equilibrium.

3. Is there any game with at least 2 correlated equilibria s.t. in one of them both of the players make more profits?

Yes. Here, if we just select AA it is a correlated nash equilibrium. Also, if we just select BB it is a correlated nash equilibrium. However, in the second one bout of the players make more profit.

| | A | B |
|---|---|---|
| A | 1,1 | 0,0 |
| B | 0,0 | 2,2 |

4. Consider a first price auction with three players. Assume that the value of the good for each player is drawn uniformly at random from [0,1], and this is a common knowledge of the players know this fact. What is the best bid for each player?

Consider that each player knows her own valuation and this valuation is private. However, all of the players have the common knowledge that the values are drawn uniformly at random from [0,1]. The game is symmetric and all of the players are rational so they bid the same amounts if they see the same valuation. However, since the valuations may be different, they may have different bids. Assume the optimal bid is $a*v$, where v is the value of the bidder. By symmetricity, we can assume other players bid $b*V$ where V is the random variable that indicates the value of the good for the player. Since the bid of each of the two other players is selected uniformly at random from 0 to b, the probability that the player wins is $(v*b/a)^2$ (which is equal to the probability that this player has the maximum value). In the case she wins, her revenue is $v-b*v = (1-b)*v$; otherwise it is zero. Thus, her expected revenue is $(v*b/a)^2*(1-b)*v$ which maximize at $b = \frac{2}{3}$.

5. Consider an English auction and assume that the owner can pretends to be one of the bidders and places bids on the item. If the owner wins, she do not pay anything and the item remains her property. There is K bidder (other than the owner) and the value of the good for each bidder is drawn uniformly at random from [0,d]. Also the value of the good for the owner is v. The auction increases by e which is vanishingly small compare to d. What is the owner's optimal bidding strategy?

In an English auction there is no intention for a bidders to bid more than her valuation. All of the players bid to their value unless there is no other player. Thus, only the winner of the auction (the person with the highest value) stops bidding before reaching her value. The winner pays as much as the second highest value. Thus, there is an incentive for the owner to pretends the second highest value to be higher. The owner can continue bidding with the last player to increase her benefit. However, there is a risk that the other bidder stops bidding and the owner wins the auction (which may not be good).
The auction increases by e which is vanishingly smaller than d. Thus, the probability that

two bidder stop bidding in the same stage is close to zero. There is no risk for the owner to bid unless there just remains one other bidder. So, she bids until all-except-one bidder stop. After that, she may continue bidding to maximize her expected benefit.
Let the second highest value to be s. If the owner continues bidding up to b; with probability (b-s)/(d-s) the owner wins the auction and her utility will remain v .With probability 1-(b-s)/(d-s), the other bidder wins the auction and pays b+e to the owner . Thus, the expected utility of the owner is ((b-s)/(d-s))v+(1-(b-s)/(d-s))(b+e) ≃ ((b-s)/(d-s))v+(1-(b-s)/(d-s))b. We set the first derivative to zero to find b which maximize the function. v/(d-s)+1-2b/(d-s)+s/(d-s)=0 → b=(v+d)/2.

6. Find a Minmax strategy and a Maxmin strategy of the first player in the following game.

| | A | B |
|---|---|---|
| A | 0,0 | 0.2,1 |
| B | 1,0.5 | -20,-10 |

**In the minmax strategy:**

Probability that player 1 plays A [10.5/11.5]

Probability that player 1 plays B [ 1/11.5 ]

**In the maxmin strategy:**

Probability that player 1 plays A [ 21/21.2 ]

Probability that player 1 plays B [ 1/21.2 ]

If the first player plays A with probability q and plays B with probability 1-q, the utility of the second player is at most max(q*0+(1-q)*0.5,q*1+(1-q)*(-10)). This minimize at q*0+(1-q)*0.5 = q*1+(1-q)*(-10). So we have

0.5-0.5q=q-10+10q → 10.5==11.5q → q=10.5/11.5

If the first player plays A with probability p and plays B with probability 1-p, her utility is at least min(p*0+(1-p)*1,p*(0.2)+(1-p)*(-20)). This maximize at p*0+(1-p)*1 = p*(0.2)+(1-p)*(-20) so we have:

1-p=0.2p-20+20p → 21=21.2p → p=21/21.2

7. Find all Minmax strategies and Maxmin strategies of the first player in the Rock-Paper-Scissors game mentioned in the class.

Recall that Rock-Paper-Scissors game is a zero sum game. Thus, Minmax strategy and Maxmin strategy and nash equilibrium are the same. By symmetry the nash equilibrium is (⅓, ⅓, ⅓ ) i.e. If each of the players selects each of Rock, Paper, Scissors, with probability ⅓ no one prefers to change her action.

**8.** Recall that a game *G* is ***simple*** if for every coalition *S*, either *v*(*S*) = 1 or *v*(*S*) = 0 and in addition $v(N) = 1$, where $N$ is the grand coalition. A player $i$ is a **veto player** if $v(S) = 0$ for any $S$ □ $N - \{i\}$.

**Part A)** Prove that in a simple game, the core is empty if there is no veto player.

If there is no veto player, the core is empty. If player i receives a positive payoff, xi, there must be a winning coalition with a subset of the remaining players. They could form a coalition and increase their payoffs by dividing xi (and the payoffs of the other players not in the winning coalition) amongst each other. Thus, any payoff profile in which anyone receives a positive payoff can be excluded from the core.

**Part B)** Prove that in a simple game in which there are veto players, the core is exactly the set all payoff vectors in which non-veto players get 0.

If there is at least one veto player, then the core is exactly the set of nonnegative feasible payoffs that gives zero to all of the non-veto players. Part (a) shows that anything that does not give everything to the veto players cannot be in the core. To show that any way of allocating the payoff to the veto players is in the core, notice that any coalition S that improves upon the feasible payoff x must include all of the veto players. But then the total amount given to the veto players must exceed 1, which is impossible.