CMSC 858F: Assignment 2

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Please TYPE in your solutions after each problem and put your homework in my mailbox in the first floor of AV Williams. For definitions, please see slides, handwritten notes, and other course materials (or even Wikipedia).

Question 1 : A coalition game G is **convex (supermodular)** if for all $A, B \subseteq N$ we have $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$. Prove that a game G is convex if and only if for all $i \in N$ and for all $S \subseteq T \subseteq N - \{i\}$, we have $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$

Proof : We have to show

$$\forall\, A, B\; v(A \cup B) + v(A \cap B) \geq v(A) + v(B) \Leftrightarrow \forall i \in N, \forall\, S \subseteq T \subseteq N - \{i\} \quad v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$$

First, we show that the left side implies the right side. Let $i \in N$ be an arbitrary element and S, T be two arbitrary subsets s.t $S \subseteq T \subseteq N\{i\}$. Let $A = S \cup \{i\}, B = T$. Therefore, $A \cup B = S \cup T \cup \{i\} = T \cup \{i\}$, and $A \cap B = (S \cup \{i\}) \cap T = S \cap T = S$. Hence, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B) \Rightarrow v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$

Now, we show that the right side implies the left side. Let $A, B \subseteq N$ be any two given sets. We know that

$$\forall i \in N, \forall\, S \subseteq T \subseteq N - \{i\} \quad v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S) \tag{1}$$

This can be easily extended to the following inequality:

$$\forall\, X, \forall\, S \subseteq T \subseteq N \setminus X \text{ we have } v(T \cup X) - v(T) \geq v(S \cup X) - v(S) \tag{2}$$

First we prove Equation 2. Let $X = \{x_1, x_2, \ldots, x_k\}$ and for each $i \in [k]$ let $X_i = \{x_1, \ldots, x_i\}$. Setting $i = x_1$ in Equation 1, we get

$$v(T \cup X_1) - v(T) \geq v(S \cup X_1) - v(S) \tag{3}$$

Setting $i = x_2$ and using the sets $S \cup X_1$ and $T \cup X_1$ in Equation 1, we get

$$v(T \cup X_2) - v(T \cup X_1) \geq v(S \cup X_2) - v(S \cup X_1) \tag{4}$$

Similarly, the final equation we obtain (by setting $i = x_k$ and using the sets $S \cup X_k$ and $T \cup X_k$ in Equation 1) is

$$v(T \cup X_k) - v(T \cup X_{k-1}) \geq v(S \cup X_k) - v(S \cup X_{k-1}) \tag{5}$$

Adding Equations 3-Equation 5 gives a telescoping sum and proves Equation 2. Now choose $S = A \cap B, T = B$ and $X = A \setminus (A \cap B)$ in Equation 2. Note that both $S \subseteq T$ and $T \subseteq N \setminus X$ are satisfied. Moreover, $T \cup X = A \cup B$ and $S \cup X = A \cap B$. Hence, the right side implies the left side.

Question 2 : Prove that Shapley values satisfy Symmetry, Dummy player, and Additivity axioms.

Proof : The payoff for player i is its Shapley value which is the following.

$$\Phi_i(v) = \sum_{S \subseteq N-\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(v(S+\{i\}) - v(S)\right) \tag{6}$$

Symmetry: Let i, j be two players such that

$$\forall S \subseteq N - \{i, j\} \quad v(S+\{i\}) = v(S+\{j\}) \tag{7}$$

We need to prove that

$$\sum_{S \subseteq N-\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(v(S+\{i\}) - v(S)\right) = \sum_{S' \subseteq N-\{j\}} \frac{|S'|!(n-|S'|-1)!}{n!} \left(v(S'+\{j\}) - v(S')\right) \tag{8}$$

We prove the above equality by giving a bijection ($b : \mathcal{S} \leftrightarrow \mathcal{S}'$) between sets $\mathcal{S} = \{S | S \subseteq N - \{i\}\}$ and $\mathcal{S}' = \{S' | S' \subseteq N - \{j\}\}$. For each $S \in \mathcal{S}$ its corresponding set $b(S)$ in $\mathcal{S}'$ is equal to S but i is replaced with j if $i \in S$. In order to prove 8, it is sufficient to show that

$$\frac{|S|!(n-|S|-1)!}{n!} \left(v(S+\{i\}) - v(S)\right) = \frac{|b(S)'|!(n-|b(S)'|-1)!}{n!} \left(v(b(S)'+\{j\}) - v(b(S)')\right) \tag{9}$$

When S and $b(S)$ neither have i nor j then 9 holds since $S = b(S)$. When $S \neq b(S)$ then by definition of b the only difference is that the member i is replaced by j. In this case, 9 holds by 7.

Dummy player: We have that i is a dummy player for which $\forall$ S such that $i \notin S$ we have $v(S+\{i\}) = v(S) + v(i)$. The payment of player i can be calculated by 6 as follows:

$$\begin{aligned}
\Phi_i(v) &= \sum_{S \subseteq N-\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(v(S+\{i\}) - v(S)\right) \\
&= \sum_{S \subseteq N-\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(v(i)\right) \\
&= v(i) \cdot \sum_{S \subseteq N-\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \\
&= v(i) \cdot \sum_{i=0}^{n-1} \binom{n-1}{s} \frac{s!(n-s-1)!}{n!} \\
&= v(i) \cdot \sum_{i=0}^{n-1} \binom{n-1}{s} \frac{s!(n-s-1)!}{n!} \\
&= v(i) \cdot \sum_{i=0}^{n-1} \frac{1}{n} \\
&= v(i)
\end{aligned}$$

Additivity: Let v, w be the two gain functions of two arbitrary coalition games.

$$\begin{aligned}
\Phi_i(v+w) &= \sum_{S \subseteq N-\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(v(S+\{i\}) + w(S+\{i\}) - v(S) - W(S)\right) \\
&= \sum_{S \subseteq N-\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(v(S+\{i\}) - v(S)\right) + \sum_{S \subseteq N-\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(w(S+\{i\}) - W(S)\right) \\
&= \Phi_i(v) + \Phi_i(w)
\end{aligned}$$

Question 3 : Prove formally that if f^{eq} is an equilibrium multicommodity flow, then for any other multicommodity flow f which satisfies the same demands, $\langle\ c(f^{\mathrm{eq}}), f^{\mathrm{eq}} - f\ \rangle \leq 0$.

Proof : We prove this problem by contradiction. If we assume otherwise then we have

$$\langle\ c(f^{eq}), f^{eq} - f\ \rangle > 0 \Rightarrow c(f^{eq}) \cdot f^{eq} > c(f^{eq}) \cdot f. \tag{10}$$

As all the costs and flows between all the source and destination pairs are non-negative then from Equation 10, we can conclude that there exist an index i such that we have

$$c(f^{eq}) \cdot f^{i,eq} > c(f^{eq}) \cdot f^{i}. \tag{11}$$

Expanding the inequality over the set $\mathcal{P}_i$ of all $s_i - d_i$ paths, we have

$$\sum_{p \in \mathcal{P}_i} c_p(f^{eq}) f_p^{i,eq} > \sum_{p \in \mathcal{P}_i} c_p(f^{eq}) f_p^{i}. \tag{12}$$

Since f^{eq} is an equilibrium flow, if $f_P^{eq} > 0$, then $c_P(f^{eq}) \leq c_{P'}(f^{eq})$ for all $P' \in \mathcal{P}_i$. That is, all paths $P \in \mathcal{P}_i$ used by the equilibrium flow f^{eq} have a common cost, say L. Moreover, $c_P(f) \geq L$ for every path $P \in \mathcal{P}_i$. Let $A = \{p \in \mathcal{P}_i \mid f_p^{i,eq} > 0\}$ and $B = \mathcal{P}_i \setminus A$. Let the total flow between $s_i - d_i$ be F_i. Then, we have the following two equations:

$$\sum_{p \in \mathcal{P}_i} c_p(f^{eq}) f_p^{i,eq} = \sum_{p \in A} c_p(f^{eq}) f_p^{i,eq} + \sum_{p \in B} c_p(f^{eq}) f_p^{i,eq} = \sum_{p \in A} c_p(f^{eq}) f_p^{i,eq} = L \cdot \sum_{p \in A} f_p^{i,eq} = L \cdot F_i. \tag{13}$$

$$\sum_{p \in \mathcal{P}_i} c_p(f^{eq}) f_p^{i} \geq L \cdot \sum_{p \in \mathcal{P}_i} f_p^{i} = L \cdot F_i. \tag{14}$$

From Equation 13 and Equation 14, we get

$$\sum_{p \in \mathcal{P}_i} c_p(f^{eq}) f_p^{i,eq} = L \cdot F_i \leq \sum_{p \in \mathcal{P}_i} c_p(f^{eq}) f_p^{i}. \tag{15}$$

which contradicts Equation 12.

Question 4 : Prove that a Valid Utility game with a non-decreasing objective function is $(1,1)$-smooth and thus its PoA is at least $\frac{1}{2}$.

Proof : In a Valid Utility Game, we are given a ground set E. For each player $i \in [k]$, there is strategy set $S_i \subseteq 2^E$ and a payoff function π_i. In addition, we are given a submodular value function $V : 2^E \Rightarrow \mathcal{R}$, i.e., $V(X \cap Y) + V(X \cup Y) \leq V(X) + V(Y)$ for every $X, Y \subseteq E$. Given an outcome s, we use $U(s) \subseteq E$ to denote $\bigcup_{i=1}^{k} s_i$, which is the union of all the strategies of players in s. Given an outcome s, its objective value function is given by $W(s) = V(U(s))$. In a Valid Utility Game, the following two conditions hold:

$$\text{For every player } i \text{ and every outcome } s, \text{ we have } \pi_i(s) \geq W(s) - W(0, s_{-i}) \tag{16}$$

$$\text{For every outcome } s, \text{ we have } W(s) \geq \sum_{i=1}^{k} \pi_i(s) \tag{17}$$

First we show that a Valid Utility game with a non-decreasing objective function is $(1,1)$-smooth. Note that a game is (λ, μ)-smooth if for every pair s, s^* of outcomes we have $\sum_{i=1}^{k} \pi_i(s_i^*, s_{-i}) \geq \lambda \cdot W(s^*) - \mu \cdot W(s)$. Fix any two outcomes s, s^* of the game. For each $i \in [k]$, let $Z_i = U(S) \bigcup (\cup_{j=1}^{i} s_i^*)$. Then we have

$$\begin{aligned}
\sum_{i=1}^{k} \pi_i(s_i^*, s_{-i}) &\geq \sum_{i=1}^{k} \Big(V(U(s_i^*, s_{-i})) - V(U(\emptyset, s_{-i})) \Big) && \text{From Equation 16} \\
&\geq \sum_{i=1}^{k} \Big(V(Z_i) - V(Z_{i-1}) \Big) && \text{From Submodularity with } X = U(s_i^*, s_{-i}) \text{ and } Y = Z_{i-1} \\
&= V(U(s) \cup U(s^*)) - V(U(s)) && \text{Telescoping sum} \\
&\geq V(U(s^*)) - V(U(s)) && \text{Since } V \text{ is non-decreasing} \\
&= W(s^*) - W(s) && \text{By Definition of } W
\end{aligned}$$

Hence, the game is $(1,1)$-smooth. From Equation 17, we have $\sum_{i=1}^{k} \pi_i(s_i^*, s_{-i}) \leq W(s)$. Combining with above derivation, we get $W(s) \geq \sum_{i=1}^{k} \pi_i(s_i^*, s_{-i}) \geq W(s^*) - W(s)$. Therefore, $\dfrac{W(s)}{W(s^*)} \geq \dfrac{1}{2}$, i.e., the PoA is at least $\frac{1}{2}$.

Question 5 : Prove that in the LP for the load balancing problem presented in the class (see the slides) when each client i has an (unsplittable) integer demand $D_i \geq 0$, in any basic feasible solution the number of $x_{i,j}$ assignment variables which are integral is at least the number of clients minus the number of access points (this means in a practical scenario when the number of clients is many more that the number of access points, all clients will be served unsplittably).

Proof : See Lemma 1 (page 8) from the following paper - http://www-math.mit.edu/ hajiagha/cell-breath.pdf

Question 6 : The employer of a weatherman is determined that he should provide a good prediction of the weather for the following day. The weathermans instruments are good, and he can, with sufficient effort, tune them to obtain the correct value for the probability of rain on the next day. There are many days, and on the i^{th} day the true probability of rain is called p_i. On the evening of the $(i1)^{th}$ day, the weatherman submits his estimate q_i for the probability of rain on the following day, the i^{th} one. Which payment scheme should the employer adopt to reward or penalize the weatherman for his predictions, so that he is motivated to correctly determine p_i (that is, to declare $q_i = p_i$)? Note that the employer does not know what the correct p_i is because he has no access to technical equipment, but he does know the q_i values that the weatherman provides, and he sees whether or not it is raining on each day.

Proof : For a prediction x, suppose that we pay $f(x)$ if it rains and $g(x)$ if it does not rain. Then the expected payment is $E(x) = p \cdot f(x) + (1 - p) \cdot g(x)$. We want $E(x)$ to satisfy $E(p) > E(x)$ for every $x \in [0, 1] \setminus \{p\}$. We want the first derivative to be equal to zero at $x = p$. Now, $E'(x) = p \cdot f'(x) + (1 - p) \cdot g'(x)$. One simple choice for f and g so that $E'(p) = 0$ is $f(x) = \log x$ and $g(x) = \log(1 - x)$ since $E'(p) = p \cdot f'(p) + (1 - p) \cdot g'(p) + p\frac{1}{p} + (1 - p)\frac{-1}{1-p} = 0$. Also, note that $E'(x) = p \cdot f'(x) + (1 - p) \cdot g'(x) = \frac{p}{x} - \frac{1-p}{1-x} = \frac{p-x}{x(1-x)}$. Hence, $E'(x) > 0$ for $x \in (0, p)$ and $E'(x) < 0$ for $x \in (p, 1)$. This shows that $E(x)$ indeed achieves its maxima at $x = p$.

Question 7 : Suppose we have n men and n women. Every man has a preference order over the n women, while every woman also has a preference order over the n men. A **matching** is a one-to-one mapping between the men and women, and it is perfect if all men and women are matched. A matching M is **unstable** if there exists a man and a woman who are not matched to each other in M, but prefer each other to their partners in M. Otherwise, the matching is called **stable**. The following algorithm which is called the **men-proposing** algorithm was introduced by Gale and Shapley:

1. Initially each woman is not tentatively matched.
2. Each man proposes to his most preferred woman.
3. Each woman evaluates her proposers, including the man she is tentatively matched to, if there is one, and rejects all but the most preferred one. She becomes tentatively matched to this latter man.
4. Each rejected man proposes to his next preferred woman.
5. Repeat Steps 2 and 3 until each woman has a tentative match. At that point the tentative matches become final.

Similarly, we could define a **women-proposing** algorithm.
Part (A): Prove that the men-proposing algorithm yields a stable matching.

Proof : It is easy to see that the algorithm terminates in $O(n^2)$ steps: at each step, the number of proposals increases by at least one. Now we claim that we get a perfect matching at the end. Suppose there is a man m who is not matched to any woman after the men-proposing algorithm ends. The man m must have proposed to every woman at this stage. However, once a woman receives her first proposal then she is forever matched. This is a contradiction since n women are matched, but at most $n-1$ men are matched. Hence, all men (and all women) are matched at end of men-proposing algorithm.

Now suppose that the matching is not stable. Then there exist two pairs (m, w) and (m', w') such that m prefers w' to w and w' prefers m to m'. Because m prefers w' over w, he would have proposed w' before w. From a woman's point of view, her matches only get better (if she gets a new prospective match who is lower on her list, then she rejects him). Since w' is finally matched to m', this means that she prefers m' to m. Contradiction.

Question 7 : Part (B): We say a woman a is attainable for a man x if there exists a stable matching M with $M(x) = a$. Let M be the stable matching produced by Gale-Shapley men-proposing algorithm. Then, prove the following:

1. For every man i, $M(i)$ is the most preferred attainable woman for i.
2. For every woman j, $M^{-1}(j)$ is the least preferred attainable man for j.

Proof :

1. Suppose not. Since we get a stable matching M at the end, it follows that some man was rejected by an attainable woman. Consider the first such rejection: suppose man m by an attainable woman w. Since men propose in decreasing order, it follows that w is most preferred attainable woman for m. At the time of rejection of m by w, it follows that w is engaged to a man m' whom she prefers to m. Since w is an attainable woman for m, there is a stable matching, say M^* where m and w are matched. Let w' be the woman matched to m' in M^*. In M, the first rejection of a man by an attainable woman was rejection of m by w. So, till that point m' was not rejected by an attainable partner. Since men propose in decreasing order of preference and w' is an attainable woman for m', it follows that m' prefers w to w'. Also w prefers m' to m, since she rejected m in favor of m' during formation of M. This contradicts the fact that M^* is a stable matching, since m' and w cause it to be unstable.

2. Suppose not. So there is a pair $(m, w) \in M$ such that m is not the least preferred attainable partner for w. So, there exists a stable matching M^* such that $(m', w) \in M^*$ and w prefers m to m'. Let w' be such that $(m, w') \in M^*$. By the first part, we know that w is most preferred attainable woman for m. Also $(m, w') \in M^*$ and M^* is stable implies m prefers w to w'. This contradicts the fact that M^* is a stable matching, since m and w cause it to be unstable.

Question 8 : Costco charges 55$ as a membership fee. Lots of other discount stores such as Walmart do not charge any membership fees. Model these two approaches with simple but realistic formula and analyze these two games. Especially via your formulation mention the situations that each of these approaches makes a better sense. **(For this problem the main goal is to measure your abilities to model a real-world game)**.

Solution by Melika: Costco sells food and essential needs. It sells products at low prices but in large amount and the wholesale price would be a lot lower. Let C be the number of customers of Costco and S be the average value of the total shopping that a customer does during a year. Costco gives a discount of almost 30% on the products that it sells. Hence utility of Costco is given by

$$U = 55 \cdot C + C \cdot S \cdot 0.7$$

The 55$ membership charge might seem to decrease the number of customers. However, since Costco sells essential things in bulk and with low price many people pay this fee. This prepaid charge makes 70% of the total income of Costco. Also this prepaid charge encourages people to shop at Costco more frequently. Most of people are satisfied with the discounts that Costco offers so it does not lose many customers because of this annual fee.

Walmart does not charge this annual fee. Therefore, if someone needs something immediately they would go to Walmart and not worry about the membership. This gives Walmart a great pricing power. Let C' be the number of customers of Walmart. Walmart does not give any discount to its customers and all of its income is from selling its products at their actual price. Let S' be the average value of the total shopping that a customer does during a year. Then utility of Walmart is

$$U' = C' \cdot S'$$

Solution by Reza: Costco charges 55$ annually. This means that the clients who pay this amount will visit the store frequently during the year. Therefore, Costco can claim that it has a group of customers who usually buy their needs from this store and would not go to other stores. Therefore, it gives bargaining power to Costco when he wants to make deals with producers. On the other hand since Costco charges clients annually it has to make incentive for them to pay that money and hence its prices must be competitive with Walmart considering the annual fee. Therefore, Costco will face a trade-off in this situation.

From the customer perspective, they want to minimize their payments. Hence they calculate the total discounts they get from Costco minus their annual fee and if it was large they go to Costco. It also convenient for the customers to go to a single store as opposed to try to find the best prices of the other stores.

Costco is good for usual non-expensive needs such as foods since customers do that frequently and tend to not search for the best quality and prices each time. Therefore they are looking for a convenient place that has good quality and price. Walmart is good for special needs and low-quality goods with very cheap prices.