Parameterizing the Winner Determination Problem for Combinatorial Auctions

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ABSTRACT

Combinatorial auctions have been studied by the multi-agent systems community for some time, since these auctions are an effective mechanism for resource allocation when agents are self-interested. One challenge, however, is that the winner-determination problem (WDP) for combinatorial auctions is NP-hard in the general case. However, there are ways to leverage meaningful structure in the auction so as to achieve a polynomial-time algorithm for the WDP. In this paper, using the formal scope of parameterized complexity theory, we systematically investigate alternative parameterizations of the bids made by the agents (i.e. the input to the WDP for combinatorial auctions) and are able to determine when a parameterization reduces the complexity of the WDP (fixed-parameter tractable), and when a particular parameterization results in the WDP remaining hard (fixed-parameter intractable). Our results are relevant to auction designers since they provide information as to what types of bidding-restrictions are effective for simplifying the winner determination problem, and which would simply limit the expressiveness of the agents while not providing any additional computational gains.

Categories and Subject Descriptors

F.2 [**Analysis of Algorithms and Problem Complexity**]: Miscellaneous—*combinatorial auctions, winner determination problem*

General Terms

Algorithms and Theory

Keywords

Auction and mechanism design, combinatorial auctions, parameterized complexity

1. INTRODUCTION

Auctions are used throughout today's economy to allocate goods, resources, and services to agents, and there are numerous investigations into auction design [11, 5, 15, 12, 13, 9]. The process of determining the allocation of items is known as the *winner determination problem* (WDP) [5]. The efficiency of the WDP greatly affects its real-world applicability, as waiting a long time to determine a winner is not realistic.

We will consider a type of auction where multiple items are sold. Typically, when there are many items to auction the seller will hold multiple single-item auctions or parallel auctions. Unfortunately, these types of mechanisms do not allow items to be sold in bundles where the agents have preferences over those bundles. For instance, an agent's value for a bundle may not always be the sum of the agent's value for each individual item in that bundle. Situations arise where obtaining two items together is worth more to an agent than the sum of the values of the individual items. It follows that we want a mechanism that allows agents to group items together into a single bid; *combinatorial auctions* (CAs) allow us to model this behaviour [5].

A CA consists of a set of n agents, m items to simultaneously auction, and the communication of bids to the seller. An *atomic bid* consists of a set of items and the price the agent is willing to pay for those items. A *bid graph* is a graph in which each vertex represents a single atomic bid and is labeled with the highest bid for its bundle of items. Two vertices are adjacent in a bid graph if and only if they have an item in common. Agents must communicate bids for every possible combination of items, and we assume that this is done using a language that enables the construction of a bid graph. For details on possible languages we direct the reader to Nisan [9]. Solving the WDP for the general CA is NP-complete [11], and this result is reflected in the fact that finding the maximum weighted independent set of a bid graph is also NP-complete [7]. Even if a problem is NP-hard, we must often research its potential solutions because in practice, some solution is required. There are a number of different approaches to analyzing NP-hard problems, one of which is parameterized complexity theory. We examine the graph representation of the CA through the formal theory of parameterized complexity.

Parameterized complexity theory, introduced by Downey and Fellows [6], relaxes the requirement that an algorithm runs in polynomial time; instead allowing the running time to be large in terms of one or more *parameters*, provided that it is polynomial with respect to the input size. The goal is to design algorithms that run efficiently if the parameters are sufficiently small, regardless of the size of the input. If such an algorithm exists, the problem it solves is called fixed-parameter tractable (FPT).

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In a CA we view the numbers of agents, items, and the sizes of the bids as parameters; hence, placing a bound on some or all of these parameters may help restrict the problem space and allow us to derive more efficient solutions. Alternatively, some parameterizations may not reduce the complexity from the NP-hard general problem. Such a parameterization is said to be fixed-parameter intractable. Negative results such as this can be very important in helping us understand what makes the problem difficult to solve. Further, by showing certain parameterizations to be hard, we provide cases to the research community where we know efficient algorithms cannot be found. This allows others to focus on other parameterizations that may yeild positive results.

There are a number of previous applications of parameterized complexity to mutil-agent systems. Shrot, Aumann, and Kraus study parameterizations of coalition formation problems, showing, among other things, that the problems are FPT in the number of goals, but $W[1]$-hard in the size of the coalition [14]. Betzler and Uhlmann explore control problems for different voting systems, concentrating on the addition and removal of candidates [2]. In addition, Betzler, Guo, and Niedermeier explore the parameterized complexity of the Dodgson Score and Young Score problems, when the winner is close to being a Condorcet winner [1].

There are also numerous examples in the literature where restrictions are placed on CAs in order to produce solutions to the WDP that run in polynomial time [11, 15, 10, 13, 4]. These results do not make explicit use of parameterized complexity theory, nor do they require its more complex notions, such as allowing the running time to be arbitrarily bad in terms of one or more parameters. However, the results presented are similar in nature to our work in that they work at restricting the problem in order to find efficient solutions. Further, there are some results presented that show restrictions where the problem remains NP-hard. For example, Rothkopf *et al.* consider a number of restrictions to combinatorial auctions that result in a polynomial WDP [11]. For instance, they show that allowing bids on arbitrary doubletons allows us to solve WDP in polynomial time, while allowing bids on tripletons reverts back to NP-hardness. Our investigation complements this previous work by studying various parameterized versions of the WDP, and also consider its parameterization after restricting the graph class of its bid graph. Our results are relevant to auction designers since they provide information as to what types of bidding-restrictions are effective for simplifying the winner determination problem, and which would simply limit the expressiveness of the agents while not providing any additional computational gains.

We now give an outline of the paper. We begin with a brief introduction to combinatorial auctions and the winner determination problem. We then give an overview of parameterized complexity and introduce two problems that will be used throughout the remainder of the paper. In the end of the Preliminaries section, we introduce bid graphs, which we use to represent combinatorial auctions.

In Section 3.1, we look at various parameterizations of the WDP. We show that when the total number of distinct atomic bits are bounded and the solution satisfies a given lower bound, then solving the WDP is fixed-parameter tractable. In contrast, when we restrict each agent to a bounded number of atomic bids, and require that the solution be greater than a lower bound, then solving the WDP is as hard as any solution in the general-case the WDP. This shows that when agents are "single-minded" [8], the WDP is still hard.

In Section 3.2, we restrict the structure of the graphical representation (bid graph) of each agent's bids. We then consider the WDP as it applies to these restricted combinatorial auctions, and require that the solution satisfies a given lower bound. When the bid graphs are interval graphs, the problem becomes fixed-parameter tractable. However, if the bid graphs are chordal graphs, the problem is still hard. For our main result, we show a class of parameterizations of the WDP that do not reduce the complexity of the problem. That is, we show that the problem remains hard when restricting the bid graphs to a wide range of graph classes. We conclude our results by showing that if the bid graph is a disconnected graph with at most C components, each having at most ℓ vertices, then finding a solution to the WDP that has value greater than some lower bound is fixed-parameter tractable. Finally, we present our conclusions.

2. PRELIMINARIES

2.1 Combinatorial Auctions

A CA consists of a set of n agents, and m items to be auctioned. The set of agents will be denoted by $N = \{1, 2, \ldots, n\}$ and the set of items by M. For any subset $S \subseteq M$, any agent i can place a bid $b_i(S) \in \mathbb{Z}$ on S. We assume that $b_i(S) \geq 0$ for all $S \subseteq M$ and that the agents are self-interested.

Sandholm notes that only the highest bid for each bundle needs to be considered in order to solve winner determination [12]. Thus, define $b^*(S)$ as follows:

$$b^*(S) = \max_{i \in N} b_i(S)$$

An atomic bid, denoted (S, p), includes a set of items $S \subseteq M$ and its bid value, $p \geq 0$. We assume that for each agent i, bids are submitted as a set of atomic bids, $\{(S_{i1}, p_{i1}), \ldots, (S_{ir_i}, p_{ir_i})\}$.

Calculating the allocation of goods to agents can be done by solving a integer program, which is NP-hard. We define a *valid outcome* $\mathcal{X} = \{S_1, S_2, \ldots, S_l\}$ as a set of bundles of items, where $S_j \subseteq M$, $|S_j| \geq 1$ for all j and for every $S_j, S_k \in \mathcal{X}, j \neq k$, we have $S_j \cap S_k = \emptyset$. The solution to the winner determination problem is the following:

$$\max_{\mathcal{X}} \sum_{S \in \mathcal{X}} b^*(S)$$

where $\mathcal{X}$ is a valid outcome. It finds a set of disjoint subsets of M that maximizes the sum of the bids. Note that some items may not be included in any of the subsets $S \in \mathcal{X}$ and a valid outcome does not explicitly state who wins each of the subsets. However, this can be determined by giving every bundle $S \in \mathcal{X}$ to the agent who placed the highest bid for S.

Any items that are not included in a valid outcome we will consider to be in set S' with $b^*(S') = 0$. With this in mind, the summation can be maximized by a valid outcome $\mathcal{X}^*$ such that all items are included in the outcome.

Definition 2.1.1. *An* exhaustive valid outcome *is a valid outcome where every item is included in exactly one subset of the outcome.*

A solution to the WDP using the integer programming solution is an exhaustive valid outcome that maximizes the summation of bids on the subsets of that outcome.

2.1.1 Bid Graphs

Here we briefly define a bid graph and how such a graph is constructed. Put simply, each vertex in a bid graph is a bid, and an edge exists between two vertices if the two bids they represent share an item. Recall that N is our set of agents, and M our set of items. We denote the bid for agent i as V_i, which consists of atomic bids $(S_{i1}, p_{i1}), \ldots (S_{ir_i}, p_{ir_i})$ where $S_{ij} \subseteq M$ and $p_{ij} > 0$. For each agent i, we define r_i to be the number of atomic bids in V_i and let M_i represent the total number of items used over all atomic bids, counting each item exactly once. Lastly, we let R represent the number of distinct atomic bids across all V_i; we distinguish between two atomic bids by their item bundles. Equivalently, R represents the number of distinct subsets of items $S \subseteq M$ that have at least one atomic bid placed by an agent.

By definition, a bid graph is the intersection graph of the distinct atomic bids. We consider the vertices of the bid graph to each have label $\{i, S, p\}$, where $S \subseteq M$ is a bundle of items, and $p > 0$ denotes the highest bid value for that bundle by agent i. We note that $p = b^*(S)$ for bundle S. See Figure 1 for an example bid graph. The WDP is equivalent to finding a maximum weighted independent set on the constructed graph [12]. Sandholm *et al.* have shown that the bid graph can be constructed in polynomial-time, and in fact it can be constructed in time that is $O(n \cdot R^2 \cdot m^2)$ [12, 13]. It is also well known that we can translate to a CA from a bid graph in polyomial time. For the purposes of this paper, we will analyze the bid graph and consider the ramifications of parameterizing this graph and its associated CA.

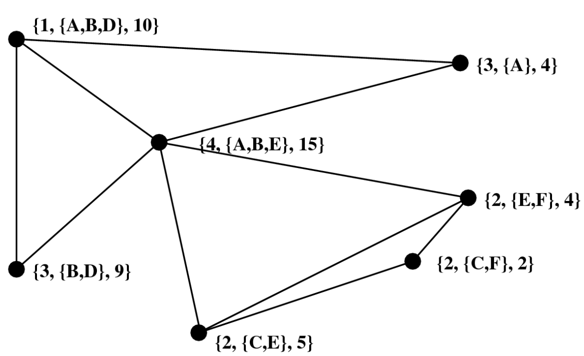

Figure 1: An example of a bid graph representing a CA. The CA consists of 4 agents, 5 items $\{A, B, C, D, E\}$, and 7 distinct atomic bids.

2.2 Parameterized Complexity

The winner determination problem for the general combinatorial auction is NP-complete [11]. Even though it is NP-complete, we must research its potential solutions because in practice, some solution is required. There are a number of different approaches to analyzing NP-hard problems, one of which is parameterized complexity theory.

Parameterized complexity theory and the notion of fixed-parameter tractability were developed by Downey and Fellows to further classify intractable problems [6]. By relaxing the requirement that an algorithm runs in polynomial time, the theory allows the running time to be large in terms of one or more *parameters*, provided that it is polynomial with respect to the input size. The goal is to design algorithms that run efficiently if the parameters are sufficiently small, regardless of the size of the input. If such an algorithm exists, the problem it solves is called fixed-parameter tractable (FPT). We now give an overview of the fundamental concepts behind parameterized complexity.

DEFINITION 2.2.1 ([6]). *A* parameterized problem *is a language $L \subseteq \Sigma^* \times \Gamma^*$, where alphabets Σ and Γ are finite. For each $(x, y) \in L$, we call x the input and y the parameter.*

Many computation and optimization problems that are known to be NP-complete may be represented as parameterized problems in order to obtain new algorithms or complexity results. Often the parameter y is taken to be a positive integer and denoted instead as k. By separating the problem input into two parts, we hope to find an algorithm that has "good" running time as a function of $|x|$, while allowing for arbitrarily "bad" running time as a function of $|y|$. This leads to the formal notion of *fixed-parameter tractability*, which is central to parameterized complexity in the same way that *polynomial time* is central to classical computational complexity.

DEFINITION 2.2.2 ([6]). *A parameterized problem L is* fixed-parameter tractable *(FPT) if there exists an algorithm that, given input $(x, y) \in \Sigma^* \times \Gamma^*$, can correctly determine if $(x, y) \in L$ using time $f(|y|) \cdot p(|x|)$ for some computable function f and polynomial p.*

Fixed-parameter tractability generalizes polynomial time computability by admitting algorithms whose running time is exponential, but only with respect to the parameter. For some combinatorial problems, there are proofs of fixed-parameter intractability. However, as with the classical notion of intractability, to demonstrate fixed-parameter intractability for most interesting natural problems, we will need to make use of completeness theory. More precisely, we need to define what it means for one parameterized problem to be *reducible* to another. Conceptually, we wish to take a parameterized problem instance and find a mapping to the other parameterized problem instance such that an input is accepted in the first language if and only if the mapped input is in the second language. This mapping algorithm must be fixed-parameter tractable and the part of the mapping that calculates the second problem's parameter must be a function of only the first problem's parameter. Formally, we have the following definition:

DEFINITION 2.2.3 ([6]). *Let $L \subseteq \Sigma^* \times \Gamma^*$ and $L' \subseteq (\Sigma')^* \times (\Gamma')^*$ be parameterized problems. We say that L is* fixed-parameter reducible *to L' ($L \leq^{fpt} L'$) if there is a mapping $F : \Sigma^* \times \Gamma \mapsto (\Sigma')^* \times (\Gamma')^*$ such that*

1. *For all $(x, y) \in \Sigma^* \times \Gamma^*$, $(x, y) \in L$ if and only if $F(x, y) \in L'$,*
2. *F is computable in time $f(|y|) \cdot p(|x|)$, for some computable function f and polynomial p, and*
3. *For all $(x, y) \in \Sigma^* \times \Gamma^*$ where $F(x, y) = (x', y')$, we have $|y'| \leq g(|y|)$ for some computable function g.*

We now have the tools and definitions to perform fixed-parameter reductions. In Section 2.2.1, we will see two problems that are unlikely to be FPT. These problems are known as $W[1]$-hard, and are believed to be fixed-parameter intractable [6]. Later, we show that some parameterizations of the WDP are also fixed-parameter intractable by providing fixed-parameter reductions using a problem in Section 2.2.1. The notions of FPT and $W[1]$ are similar to that of P and NP, and for the purposes of this paper we will not delve further into the many other parameterized complexity classes, but instead refer the reader to Downey and Fellows for a complete description [6].

It is important to know if a parameterization of the WDP is $W[1]$-hard because, as we will see in later sections, $W[1]$-hardness implies that our parameterization does not provide a significant improvement in the running time of a solution to the WDP. Knowing which restrictions *not* to make is often just as important as making ones that lead to fixed-parameter tractability. For further details about the hierarchy and its definition, see Downey and Fellows [6]. For the classification of various parameterized problems see Cesati [3].

2.2.1 Two Important Parameterized Problems

For our proofs of fixed-parameter intractability, we will need to reduce from known fixed-parameter intractable problems. For this reason, we introduce two problems that are both known to be $W[1]$-complete [6].

k-WEIGHTED INDEPENDENT SET

Input: A graph $G = (V, E)$, together with function $\omega : V \mapsto \mathbb{Z}$, $\omega(v) > 0$ for all $v \in V$.

Parameter: Positive integer k.

Question: Does there exist a set of vertices $V' \subseteq V$ such that for all $u, v \in V'$, $\{u, v\} \notin E$, and $\sum_{v \in V'} (\omega(v) \geq k$?

k-WEIGHTED SET PACKING

Input: A finite family of sets $S = S_1, \ldots, S_q$ along with their weights $W = W_1, \ldots, W_n$, $W_i > 0$ and $W_i \in \mathbb{Z}$ for all i.

Parameter: Positive integer k.

Question: Does S contain a sub-family F of mutually disjoint sets such that the sum of their corresponding weights is at least k?

LEMMA 2.2.4 ([6]). *k-WEIGHTED INDEPENDENT SET and k-WEIGHTED SET PACKING are both $W[1]$-complete.*

3. PARAMETERIZATION

The winner determination problem on the constructed bid graph is equivalent to finding its maximum weighted independent set. In order to parameterize the problem we need to consider what factors may affect the running time of a solution. Given the construction and nature of the WDP, we need to either restrict the number of vertices, the number of edges, or enforce some structure on the graph itself. There are many ways to accomplish this but only some restrictions will lead to fixed-parameter tractable algorithms.

For parameterizing the problem, we can restrict any combination of the problem's parameters. The number of items, number of agents, number of atomic bids, and the number times bids of different agents intersect are just some of the possible parameters. We will consider a few of these parameters and also look at restricting the graph class of our bid graph. We begin with a simple parameterization of the WDP and show it is $W[1]$-complete.

In the following example, the parameter acts as a lower bound on the sum of the winning bids. As we show below, this parameterization is $W[1]$-complete.

k-WINNER DETERMINATION (k-WD)

Input: A set of agents N, items M, and bids $V = V_1, V_2, \ldots V_n$.

Parameter: Positive integer k.

Question: Does there exist a set of mutually disjoint atomic bids $\{(S_1, p_1), (S_2, p_2), \ldots, (S_\ell, p_\ell)\}$ such that $\sum_{j=1}^{\ell} (p_j) \geq k$?

PROPOSITION 3.0.5. *k-WD is $W[1]$-complete.*

PROOF. We begin by providing a fixed-parameter reduction from k'-WEIGHTED INDEPENDENT SET to k-WD. Given an arbitrary graph as input to k'-WEIGHTED INDEPENDENT SET, we construct a combinatorial auction with one agent. We number the edges of the graph as $1, 2 \ldots m$. For each vertex v, we create an atomic bid (S_v, p_v), where $S_v = \{$ edge numbers incident with $v\}$, and p_v is the weight of vertex v in the k'-WEIGHTED INDEPENDENT SET problem. Further, we set the parameter $k = k'$.

Finding mutually disjoint atomic bids is then equivalent to finding an independent set, hence the sum of the bid prices is equivalent to the sum of the independent set's vertices' corresponding weights. Therefore, if we have an independent set of weight at least k' then we also have mutually disjoint atomic bids whose prices sum to at least k, by construction. The reduction is a fixed-parameter reduction and so we have that k-WD is $W[1]$-hard.

In Section 2.1.1, we have seen that k-WD can be used to construct a bid graph in polynomial time. As the WDP is equivalent to finding a maximum weighted independent set on the bid graph, it follows directly that by restricting our attention to k-WD we only need to find an independent set of size at least $k' = k$. Therefore, k-WD is in $W[1]$ and hence is $W[1]$-complete. ☐

3.1 Parameterizing the Auction

We will now demonstrate some of the ways one can parameterize the WDP and the effect, if any, each parameterization has on its complexity. There are many choices to be made and the ones we make are by no means exhaustive; there may be many other parameterizations for which the problem remains hard, and others that demonstrate fixed-parameter tractability. We begin our parameterizations by considering the number of distinct atomic bids in a CA.

The number of vertices in the bid graph is exactly the number of distinct atomic bids across all agents, R. A simplistic algorithm would search through every possible combination of vertices to try and find the maximum weighted independent set. The number of edges is bounded above by $\binom{R}{2}$, and so such a naive algorithm would require time of at most $O(R^2 \cdot 2^R)$. What is interesting here is that the number of items does not affect the result at all. Recalling that $O(n \cdot R^2 \cdot m^2)$ was our bid graph construction time, the total required time to find a solution to the WDP is $O(n \cdot R^2 \cdot m^2 + R^2 \cdot 2^R)$. This leads to the following parameterization:

R, k-WINNER DETERMINATION (R, k-WD)

Input:	Agents N, items M, and bids $V = V_1, V_2, \ldots V_n$, where the total number of distinct atomic bids across all V_i is at most R.
Parameter:	Positive integers R, k.
Question:	Does there exist a set of mutually disjoint atomic bids $\{(S_{i_1}, p_{i_1}), (S_{i_2}, p_{i_2}), \ldots, (S_{i_\ell}, p_{i_\ell})\}$ such that $\sum_{j=1}^{\ell}(p_{i_j}) \geq k$?

From the above, R, k-WD is fixed-parameter tractable, since we can find the maximum weighed independent set and then verify that its value is at least k. By noticing that we really only require checking all possible combinations of up to and including m sets of R atomic bids we can come up with a possibly better bound if $m \leq \frac{1}{2}R$. By checking $\binom{R}{1} + \binom{R}{2} + \ldots + \binom{R}{m}$ sets of atomic bids we have included every possible item and if a maximum choice exists we must have already seen it. For each set we need to check if any of at most $\binom{m}{2}$ edges exist among the chosen atomic bids. The algorithm then runs in time $O(n \cdot R^2 \cdot m^2 + m^2 \cdot (\binom{R}{1} + \binom{R}{2} + \ldots + \binom{R}{m})) \in O(n \cdot R^2 \cdot m^2 + m^3 \cdot (\frac{R}{m})^m)$, for $m \leq \frac{1}{2}R$. If $m > \frac{1}{2}R$, the new method is still faster than trying all 2^R possible combinations of atomic bids, but the order notation does not simplify as easily. Further, since we have the parameter k, it is possible to check even fewer combinations. However, we will not analyze the exact running time of the algorithm making use of the parameter.

A practical problem arises from this parameterization as the seller has no direct control over the total number of atomic bids. Further, without additional restrictions on n, the seller cannot possibly specify a restriction per agent based on knowledge of a bound on R.

Consider another view: parameterize the number of atomic bids per agent i as B. This parameterization results in a maximum value for R of $n \cdot B$, and thus the required running time for our naive algorithm becomes $O(n^3 \cdot B^2 \cdot m^2 + (n \cdot B)^2 \cdot 2^{n \cdot B})$. The naive algorithm is now exponential in n, which is not a parameter. However, it requires formal justification to show that this parameterization is fixed-parameter intractable.

B, k-WINNER DETERMINATION (B, k-WD)

Input:	Agents N, items M, and bids $V = V_1, V_2, \ldots V_n$, where the number of atomic bids in any V_i is at most B.
Parameter:	Positive integers B, k.
Question:	Does there exist a set of mutually disjoint atomic bids $\{(S_{i_1}, p_{i_1}), (S_{i_2}, p_{i_2}), \ldots, (S_{i_\ell}, p_{i_\ell})\}$ such that $\sum_{j=1}^{\ell}(p_{i_j}) \geq k$?

Proposition 3.1.1. *B, k-WD is $W[1]$-hard.*

Proof. We perform a fixed-parameter reduction from k'-WEIGHTED SET PACKING to B, k-WD. Our family of sets is $S = S_1, S_2, \ldots, S_q$ and from these we create a combinatorial auction with q agents. Subset S_i will be used to create an atomic bid (S_i, W_i) for agent i, $1 \leq i \leq q$. The result is a B, k-WD problem, with $B = 1$, and $k = k'$. Further, we have a set of mutually disjoint atomic bids whose bid price sum is at least k for B, k-WD if and only if the original k'-WEIGHTED SET PACKING problem has a family of mutually disjoint sets whose sum of their associated weights is also at least k'. □

It then follows that parameterizing by the number of atomic bids per agent is not worthwhile, as the problem remains hard. However, if we parameterize the number of atomic bids per agent *and* the number of agents we can achieve fixed-parameter tractability because even our naive algorithm was exponential only in n and B. In real-world applications, enforcing this parameterization is likely more feasible than parameterizing the total number of atomic bids over all agents.

3.1.1 "Single-Minded" Agents

An often studied special case of CAs restricts agents to bidding on at most a single bundle of items. In the literature the agents are referred to as "single-minded" when faced with this restriction. Due to the significance of this special case, we should note that Proposition 3.1.1 gives us a proof of the hardness when agents are "single-minded". In fact, the case is equivalent to forcing the value of B to 1.

Proposition 3.1.1 states that B, k-WD is $W[1]$-hard, but more importantly, since our proof of this proposition is for the case where $B = 1$, it is exactly the case where agents are "single-minded". Thus, even if each agent is "single-minded", solving the winner determination problem for the resulting combinatorial auction is just as hard as solving it in the general case. We can conclude that for combinatorial auctions, the auction designer does not gain any simplicity from restricting agents in such a way, and given that it is a severe restriction from the point of view of the agent, it serves no purpose in this case. For a further and more in-depth discussion of single-minded agents, we refer the reader to Lehmann *et al* [8].

3.2 Restricting the Graph Class

As an additional avenue of investigation, we consider what happens if we restrict agents to certain types of bids. That is, we restrict each agent to bidding such that the bid graph of the agent's bids maintains a desired structure. The question we then ask is how this structure affects the hardness of the overall combinatorial auction; is the WDP for the auction fixed-parameter tractable, or does it remain $W[1]$-hard? It is important to note that while we are restricting the agents' bids for proof of fixed-parameter tractability, we are not necessarily proposing that the agents be restricted in practice; rather, we are investigating which structures do, and which do not reduce the complexity of the problem. We will comment further on this at the end of this section.

First we present a general problem definition, in which we restrict our problem using β and parameterize by k, where β is some desired graph class. We require that the graph class of each agent i's bid graph be a graph from class β. The parameter k will be used in the typical way we have seen previously.

β, k-WINNER DETERMINATION (β, k-WD)

Input:	A set of agents N, items M, and bids $V = V_1, V_2, \ldots V_n$, where the bid graph generated by V_i belongs to graph class β.
Graph Class:	β.
Parameter:	Positive integer k.
Question:	Does there exist a set of mutually disjoint atomic bids $\{(S_{i_1}, p_{i_1}), (S_{i_2}, p_{i_2}), \ldots, (S_{i_\ell}, p_{i_\ell})\}$ such that $\sum_{j=1}^{\ell}(p_{i_j}) \geq k$?

Analyzing the general problem is of course very difficult. We cannot immediately generalize all graph classes and give one result. First we introduce two well-known graph classes, interval and chordal graphs, and then consider the β, k-WD problem where β is restricted to one of these classes.

DEFINITION 3.2.1 (INTERVAL GRAPH). *Let $I_1, I_2, \ldots, I_n$ be intervals on the real line. Then, we define the* interval graph $G = (V, E)$ *representing the intervals $I_1, \ldots, I_n$ as follows:*

$$V = \{I_1, I_2, \ldots, I_n\}, \text{ and}$$
$$\{I_i, I_j\} \in E \iff I_i \cap I_j \neq \emptyset.$$

DEFINITION 3.2.2 (CHORDAL GRAPH). *A graph G is a* chordal graph *if and only if for any cycle C in G containing at least four vertices there exists an edge between two vertices in C that is not part of the cycle.*

Rothkopf *et al.* discussed a combinatorial auction where the items can be ordered and it is required that bids be placed on consecutive items [11]. While Rothkopf *et al.* [11] did not make use of interval graphs directly, they were actually specifying that the agents' bid graphs be interval graphs of one common interval. By definition, the bid graph of the resulting CA is also an interval graph. Thus, we have the following proposition.

PROPOSITION 3.2.3 ([11]). *If β is the class of interval graphs defined by interval $I = (1, 2, \ldots, m)$, the β, k-WD problem is fixed-parameter tractable.*

Rothkopf *et al.* showed that for interval graphs, the β, k-WD problem can be solved in linear time. It is not always possible to restrict β and result in a CA that is easier to solve. A good illustration of this is for β as the class of chordal graphs.

PROPOSITION 3.2.4. *If β is the class of chordal graphs, the β, k-WD problem is $W[1]$-hard.*

It is important we note that in the proof of Proposition 3.2.4, we reduce from an arbitrary instance of a CA and its bid graph. This is valid because chordal graphs place no restrictions on how different agents' bids intersect.

PROOF PROOF OF PROPOSITION 3.2.4. In order to show that this problem remains $W[1]$-hard we give a fixed-parameter reduction from k'-WD to β, k-WD. Given an *arbitrary instance* of a CA and its bid graph $G = (V, E)$, we construct a new CA where each agent's individual bid graph belongs to the class of chordal graphs. We let the number of agents in our new CA, n, be the same as the number of vertices of G, and consider each vertex in G to belong to a distinct agent. Each agent's bid in the new CA consists of a single atomic bid involving only the labels of the edges adjacent to the agent's corresponding vertex in G. A graph of one vertex belongs to the class of chordal graphs, and as such every agent's individual bid graph also belongs to the class of chordal graphs. Thus, we have constructed a new CA with $n = |V|$ agents, each of which places exactly one atomic bid. This reduction requires time polynomial in the size of the original CA to complete, and finally we note that $k = k'$. □

Given that we cannot restrict β as the class of chordal graphs, a natural question to ask is for which graph classes the β, k-WD problem remains $W[1]$-hard. We will provide a partial characterization, but first, we need to define what it is to be a minimal β graph. We choose to quantify the size of a graph by the sum of the number of vertices and edges in the graph.

DEFINITION 3.2.5 (MINIMAL β GRAPH). *A* minimal β graph *is a graph contained in graph class β whose size is at most the size of any other graph from graph class β.*

Intuitively, it is a graph from graph class β whose size is minimal. It is possible for a minimal β graph to have infinite size, and so our first restriction is that β have a minimal β graph of finite, constant size. Further, we require that β imposes no restriction on the interaction between the atomic bids of different agents. We will show that under these restrictions, the resulting β, k-WD problem remains $W[1]$-hard. The idea is that the minimal β graph has finite, constant size, and thus has a maximum weighted independent set that can be determined in constant time.

THEOREM 3.2.6. *If β has a minimal β graph of finite, constant size and imposes no restriction on the interaction between the atomic bids of* different *agents, then the resulting β, k-WD problem is $W[1]$-hard. A minimal β graph representation is given as G_β, and does not need to be discovered.*

PROOF. To show that β, k-WD is $W[1]$-hard, we reduce from k'-WD. Given an instance of k'-WD, we consider its bid graph G. We aim to construct a bid graph G' that represents a β, k-WD problem. For this, we consider each vertex in G to be a distinct agent for our new problem. We now have a single atomic bid per agent, but an agent's individual bid graph is not necessarily a valid β graph. To remedy this, we replace each vertex in G' with a copy of G_β. Further, we assign to each vertex in the copy of G_β a weight equal to that of the weight of the original single vertex it replaced from G. Each agent in our new problem is now associated with exactly one distinct minimal β graph in G', and exactly one distinct vertex in G. If two vertices v_1, v_2 were adjacent in G, then every vertex in agent v_1's minimal β graph must be made adjacent to every other vertex in agent v_2's minimal β graph in G'. We describe this as agent v_1's minimal β graph being adjacent to agent v_2's minimal β graph.

G' now represents a new CA where the number of agents is equal to the number of vertices of the original graph, and each agent has an individual bid graph that is a copy of the minimal β graph G_β, where each atomic bid has the same weight. We denote the size of a maximum independent set for G_β as I_β. Note that since the vertices' weights are all equal in each agent's bid graph, I_β must consist of the same number of vertices as would a maximum weighted independent set for each bid graph. Further, I_β can be found in constant time because G_β is of constant size, which is necessary for the reduction to remain a fixed-parameter reduction.

Assume G has an independent set I_G of weight at least k', and denote the vertices of I_G as $v_1, v_2, \ldots, v_\ell$. The vertices of I_G map to ℓ distinct agents in our new CA, each with a bid graph that is a copy of G_β. Originally, each vertex $v_i \in I_G$ contributed weight p_i, which represents the price bid on the set of items associated with v_i in G. For of our new CA, we construct an independent set $I_{G'}$ from G'. For each agent, we add the vertices associated with I_β from the agent's copy of G_β in G' to our independent set

$I_{G'}$. These maximum independent sets contribute a total weight of $\sum_{i=1}^{\ell} I_\beta \cdot p_i = I_\beta \cdot k'$. By construction, $I_{G'}$ is an independent set and therefore we have an independent set in our new bid graph of weight at least $k = I_\beta \cdot k'$.

Now we will show that if G' contains an independent set $I_{G'}$ of weight at least $k = I_\beta \cdot k'$, then the original graph contains an independent set of weight at least k'. Consider only a single vertex in G' from each copy of G_β that has a vertex in $I_{G'}$, and denote this set of vertices as I_G. As I_β is maximal by definition, we have that the vertices of I_G must have a weight of at least k' from our assumption that $k = I_\beta \cdot k'$. Further, each vertex of I_G maps directly back to a distinct vertex in the original graph, G, and we denote this set as I_G^*. Since the edge adjacencies of G were preserved in our construction, the agents' minimal β graphs are adjacent to each other in G' if and only if the agents' corresponding vertices were adjacent to each other in G. Therefore, I_G^* is an independent set of G of weight at least k'.

Finally, the construction of G' is simply a multiplication of the size of the minimal β graph, which is a constant, and the size of the original graph G. Hence, it is a fixed-parameter reduction from k'-WD to β, k-WD and therefore β, k-WD is $W[1]$-hard. □

As a final note on the construction of G' in our proof of Theorem 3.2.6, both stipulations stated in the Theorem are necessary. Our assumption in the beginning of the proof was that we were given an arbitrary instance of k'-WD, and hence for G' to be a valid bid graph of an instance of β, k-WD we must allow arbitrary interactions between different agents' minimal β graphs. Further, it was clearly stated in the proof where a minimal β graph of finite, constant, size becomes necessary for the reduction to remain a fixed-parameter reduction.

With Theorem 3.2.6, we have a partial characterization of our β, k-WD problem where we have outlined special cases for which the problem remains fixed-parameter intractable. We have also shown that when β is the class of interval graphs defined by the interval $I = (1, 2, \ldots, m)$, then β, k-WD is fixed-parameter tractable; in fact it can be solved in polynomial time. In the case of interval graphs, a maximum independent set could be found for the bid graph in linear time.

In the investigation of combinatorial auctions as they apply to specific economic areas, one may find that agents tend to bid such that their individual bid graphs belong to a particular graph class. If this is the case, it is natural for a seller to consider restricting all agents to bid in this manner in exchange for a faster solution to the WDP. From our work in the section, we now have an example where finding said particular graph class is useful information, and a few examples where it is not. Should β, k-WD remain fixed-parameter intractable when restricted to β, it is useless for the seller to restrict agents to bid graphs belonging to graph class β.

As a final example of a fixed-parameter reduction, we investigate the WDP when the CA's bid graph is a graph such that the number of vertices in each disconnected component is bounded from above.

3.2.1 Disconnected Components of Bounded Vertices

We previously restricted each agent's bid graph, and then evaluated the parameterized complexity of β, k-WD. We now restrict the CA's bid graph to a graph of disconnected components of bounded vertices. More precisely, the number of vertices in each component will be bounded from above. In the end of this section, we show how this can be achieved. Consider a CA's bid graph, G, and denote $C_1, C_2, \ldots, C_q$ as the disconnected components of G. By our construction in Section 2.1.1, every vertex of G represents a distinct atomic bid on a bundle of items, and hence there are at most R components. As the components are disconnected, to solve the WDP we must find a maximum weighted independent set for each of the components. We omit the proof of Lemma 3.2.7 due to lack of space, though it follows easily.

Lemma 3.2.7. *A maximum weighted independent set of a graph G has the same weight as the union of a maximum weighted independent set from each disconnected component of G. Let I denote any particular maximum weighted independent set of G, and let $W(I)$ denote its weight. Further, let I_U denote the union of maximum weighted independent sets, one for each disconnected component of G, and let $W(I_U)$ denote the combined weight of all vertices in I_U. Then $W(I) = W(I_U)$.*

C-Component ℓ, k-WINNER DETERMINATION (C, ℓ, k-WD)

Input:	A set of agents N, items M, and bids $V = V_1, V_2, \ldots V_n$, where the bid graph generated by V is a disconnected graph of C components, each of which have at most ℓ vertices.
Graph Class:	C disconnected components, each of which have at most ℓ vertices.
Parameter:	Positive integers k, ℓ.
Question:	Does there exist a set of mutually disjoint atomic bids $\{(S_{i_1}, p_{i_1}), (S_{i_2}, p_{i_2}), \ldots, (S_{i_\ell}, p_{i_\ell})\}$ such that $\sum_{j=1}^{\ell}(p_{i_j}) \geq k$?

Note that C is not a parameter, but rather a variable that is bounded above by R.

Proposition 3.2.8. *C, ℓ, k-WD is fixed-parameter tractable.*

Proof. Using our result from R, k-WD in Section 3.1, we break down the decision problem into components. The number of vertices in each component is bounded from above, and thus a maximum weighted independent set can be found for each component separately, requiring running time $O(\ell^2 \cdot 2^\ell)$. As was discussed in Section 3.1, this running time can be improved slightly with various techniques that can also be applied here. Lemma 3.2.7 states that we can combine the vertices from each component's maximum weighted independent set to form a maximum weighted independent set for the entire bid graph. It follows that the required running time of a solution to the WDP on a CA where the bid graph G is such that the vertices in each disconnected component of G are bounded from above by ℓ, is $O(n \cdot R^2 \cdot m^2 + C \cdot \ell^2 \cdot 2^\ell)$. □

To apply this parameterization in practice, the seller must enforce a global restriction resulting in a bid graph of C disconnected components, each having a bounded number of vertices. There are at least two ways of accomplishing this. First, recognizing the structure of such a bid graph can be accomplished in polynomial time. In particular, the seller may construct the bid graph and then determine if it is a graph of C disconnected components, each of which have at most ℓ vertices. If the graph follows the necessary restrictions, then we can solve the problem.

As an alternate method, if it is feasible to do so then the seller may separate items into groups of bounded size, and no agent may place items from different groups in the same atomic bid. By separating items into groups of size at most ℓ', we divide the m items into $\lceil \frac{m}{\ell'} \rceil$ groups of size at most ℓ'. Agents must be instructed that none of their atomic bids may include items from more than one group. With this restriction we have bounded the number of possible bids per group to at most $2^{\ell'}$. By setting $\ell = 2^{\ell'}$, we have constructed a valid instance of C, ℓ, k-WD, where k still needs to be set.

4. CONCLUSIONS

Combinatorial auctions (CAs) are important and very useful for allocating items to agents who wish to communicate preferences over bundles of items. Finding a solution to the winner determination problem (WDP) is NP-complete in the general case, and it is also equally difficult to find approximate solutions. Under special circumstances that enforce structure on the CA, the WDP can become easier to solve. In this paper, we made use of a new approach to analyzing the complexity of structured versions of the WDP. By using parameterized complexity theory, we formulated parameterizations of the WDP and show when these parameterized problems result in fixed-parameter intractable algorithms. Further, in four of our parameterizations of the WDP, we demonstrated fixed-parameter tractability.

For our main result, we restricted graph class β such that the resulting β, k-WD problem remained $W[1]$-hard. More precisely, if β has a minimal β graph of finite, constant size and imposes no restriction on the interaction between the atomic bids of *different* agents, then the resulting β, k-WD problem is $W[1]$-hard. This restriction of β is lax enough to allow many different graph classes. It is useful because in the investigation of CAs as they apply to specific economic areas, one may find structure in the bid graphs of individual agents. We show a number of these structures that do not reduce the complexity of the WDP. Knowing which structures are *not* helpful is often just as important as finding ones that lead to fixed-parameter tractability.

Our results are of particular relevance to auction designers since they provide information as to what types of bidding-restrictions are effective for simplifying the winner determination problem, and which would simply limit the expressiveness of the agents while not providing any additional computational gains. For example, considering "single-minded" agents in the context of combinatorial auctions yields no computational benefits even though it restricts each agent to only a single bid. With each negative result, we find ourselves closer to a more complete characterization of the WDP for CAs and gives a deeper insight into what makes the problem difficult.

The main contribution of this paper is its demonstration of the use of parameterized complexity in the investigation of the WDP for CAs. With parameterized complexity theory, it is possible to parameterize the WDP and discover new, more efficient algorithms, and prove when parameterizations are as hard as any solution to the general-case the WDP. We hope that this methodology will inspire additional questions and ideas that lead to improved ways of solving and structuring the WDP for CAs.

5. ACKNOWLEDGEMENTS

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