An Investigation of Representations of Combinatorial Auctions

David Loker and Kate Larson
University of Waterloo
200 University Ave. W
Waterloo, ON, Canada N2L 3G1
{dloker,klarson}@cs.uwaterloo.ca

ABSTRACT

Combinatorial auctions (CAs) are an important mechanism for allocating multiple goods while allowing self-interested agents to specify preferences over bundles of items. Winner determination for a CA is known to be NP-complete. However, restricting the problem can allow us to solve winner determination in polynomial time. These restrictions sometimes apply to the CA's representation. There are two commonly studied, and structurally different graph representations of a CA: bid graphs and item graphs. We study the relationship between these two representations.

We show that for a given combinatorial auction, if a graph with maximum cycle length three is a valid item graph for the auction, then its bid graph representation is a chordal graph. Next, we present a new technique for constructing item graphs using a novel definition of equivalence among combinatorial auctions. The solution to the WDP for a given CA can easily be translated to a solution on an equivalent CA. We use our technique to simplify item graphs, and show that if a CA's bid graph is chordal, then there exists an equivalent CA with a valid item graph of treewidth one, for which a solution to the WDP is known to be efficient. This result demonstrates how CA equivalence can simplify the structure of item graphs and lead to more efficient solutions to the WDP, which are also a solutions to the WDP for the original auctions.

Categories and Subject Descriptors

I.2 [**Artificial Intelligence**]: Miscellaneous—*combinatorial auctions*

General Terms

Theory and Design

Keywords

Auction and mechanism design, combinatorial auctions, bid graphs, item graphs

1. INTRODUCTION

Cite as: An Investigation of Representations of Combinatorial Auctions, David Loker and Kate Larson, *Proc. of 9th Int. Conf. on Autonomous Agents and Multiagent Systems (AAMAS 2010)*, van der Hoek, Kaminka, Luck and Sen (eds.), May, 10–14, 2010, Toronto, Canada, pp. XXX-XXX.
Copyright © 2010, International Foundation for Autonomous Agents and Multiagent Systems (www.ifaamas.org). All rights reserved.

Auctions are used throughout today's economy to allocate goods, resources, and services to agents, and there are numerous investigations into auction design [10, 5, 8, 11, 9]. The process of determining the allocation of items is known as the *winner determination* problem (WDP) [1, 4].

We consider a type of auction where multiple items are sold. An agent's value for a bundle may not always be the sum of the agent's value for each individual item in that bundle. We ideally would want a mechanism that allows agents to group items together into a single bid; *combinatorial auctions* (CAs) allow us to model this behaviour [4]. Solving WDP for the general CA is NP-complete [10]. However, certain restrictions can allow us to solve WDP in polynomial time. These restrictions sometimes apply to the represenations of the combinatorial auctions [12, 2, 7].

Two common representations of CAs are bid graphs and item graphs. A *bid graph* is a graph in which each vertex represents a single atomic bid and is labeled with the highest bid for its bundle of items. Two vertices are adjacent in a bid graph if and only if they have an item in common. A *valid item graph* of the CA is a graph where the bids induce connected subgraphs. There have been many investigations into solving WDP using both item graphs and bid graphs [10, 11, 12, 4, 2, 9, 3, 7]. In particular, when the item graph of a CA has bounded treewidth, WDP is polynomial [2].

Despite the fact that item graphs and bid graphs represent the same underlying auctions, there is a stark contrast between the two. By better understanding the relationships between item graphs and bid graphs, we may be able to create more efficient algorithms for solving WDP and, as we will see, find a new way of looking at CAs in general. There is little previous work on the relationships between these two graph representations. The only such results we are aware of are by Sandholm and Suri, who observe that if a path is a valid item graph for a given CA, then the bid graph representation of the auction is an interval graph [12]. As a corollary, they point out that if a cycle is a valid item graph, then the bid graph is a circular arc graph [12].

We present an indepth study of relationships between these two representations. For our first result, we present a new relationship between item graphs and bid graphs for fixed combinatorial auctions. In our second result, we obtain a stronger relationship under our novel notion of equivalence between combinatorial auctions.

In our first result, we find that if there exists a valid item graph representing a CA such that the maximum length of any cycle in the graph is three, then the bid graph for the auction is chordal. We show that the result does not extend

to cycles of length four or item graphs of treewidth two. Further, the converse does not hold. This impass lead to our next avenue of investigation: modifying combinatorial auctions.

The motivation behind studying the modification of CAs begins with the fact that item graphs are hard to construct. As shown by Conitzer *et al.*, constructing a valid item graph with the fewest edges is NP-complete [2]. Given a CA, Conitzer *et al.* construct a valid item graph of treewidth one in polynomial time, if the graph exists [2]. However, as shown by Gottlob and Greco, it is NP-hard to decide whether or not a combinatorial auction has a valid item graph of treewidth three [7]. On the other hand, bid graphs can always be constructed in polynomial time.

Throughout the literature, it is assumed that a CA is fixed before the construction of a valid item graph. We take a novel perspective by modifying the CA in order to achieve a valid item graph of smaller treewidth, where the solution to the WDP for the modified CA can easily be mapped to a solution for the original CA. We define equivalence among CAs, and derive a new method for constructing valid item graphs of bounded treewidth.

Intuitively, two bids graphs are equivalent if they are isomorphic, and two CAs are equivalent if their bid graphs are equivalent. Using our notion of CA equivalence, we then prove that if a bid graph of a CA is chordal, then there exists an equivalent CA that has a tree as a valid item graph. The item graph for the equivalent CA can be found in polynomial time, given the bid graph of the original auction. With our notion of CA equivalence, if the bid graph is chordal, then even if the smallest treewidth for the item graph is arbitrarily larger than one, there still exists an equivalent auction with a tree as a valid item graph. Further, a solution to the WDP found using the item graph of treewidth one gives us a solution to the original, equivalent auction.

Understanding the relationship between bid graphs and item graphs is a very important step to better understanding combinatorial auctions and what makes the problem difficult to solve. Such an understanding may lead to new approaches to constructing item graphs and better algorithms for solving restricted instances of the WDP for CAs.

Here is an outline of the paper. First, we formally define combinatorial auctions, the winner determination problem, bid graphs and item graphs. Next, we present a relationship between item graphs and bid graphs where the combinatorial auction is fixed, and describe some limitations that result from fixing the auction. We then present our novel definition of combinatorial auction equivalence. We then use CA equivalence to show that if the bid graph of a CA is chordal, then there exists an equivalent auction that has a tree as a valid item graph. We finish with conclusions and some open problems.

2. PRELIMINARIES

2.1 Combinatorial Auctions

A CA consists of a set of n agents, and m items to be auctioned. The set of agents will be denoted by $N = \{1, 2, \ldots, n\}$ and the set of items by M. For any subset $S \subseteq M$, any agent i can place a bid $b_i(S) \in \mathbb{Z}$ on S. We assume that $b_i(S) \geq 0$ for all $S \subseteq M$ and that the agents are self-interested.

Sandholm notes that only the highest bid for each bundle needs to be considered in order to solve winner determination [11]. Thus, define $b^*(S)$ as follows:

$$b^*(S) = \max_{i \in N} b_i(S)$$

An atomic bid, denoted (S, p), includes a set of items $S \subseteq M$ and its bid value, $p \geq 0$. We assume that for each agent i, bids are submitted as a set of atomic bids, $\{(S_{i1}, p_{i1}), \ldots, (S_{ir_i}, p_{ir_i})\}$.

Calculating the allocation of goods to agents can be done by solving an integer program, which is NP-hard. We define a *valid outcome* $\mathcal{X} = \{S_1, S_2, \ldots, S_l\}$ as a set of bundles of items, where $S_j \subseteq M$, $|S_j| \geq 1$ for all j and for every $S_j, S_k \in \mathcal{X}, j \neq k$, we have $S_j \cap S_k = \emptyset$. The solution to the winner determination problem is the following:

$$\max_{\mathcal{X}} \sum_{S \in \mathcal{X}} b^*(S)$$

where $\mathcal{X}$ is a valid outcome. It finds a set of disjoint subsets of M that maximizes the sum of the bids. Note that some items may not be included in any of the subsets $S \in \mathcal{X}$ and a valid outcome does not explicitly state who wins each of the subsets. However, this can be determined by giving every bundle $S \in \mathcal{X}$ to the agent who placed the highest bid for S.

Any items that are not included in a valid outcome we will consider to be in set S' with $b^*(S') = 0$. With this in mind, the summation can be maximized by a valid outcome $\mathcal{X}^*$ such that all items are included in the outcome.

An *exhaustive valid outcome* is a valid outcome where every item is included in exactly one subset of the outcome. A solution to WDP using the integer programming solution is an exhaustive valid outcome that maximizes the summation of bids on the subsets of that outcome.

2.2 Bid Graphs and Item Graphs

In this section, we first briefly define bid graphs, followed by item graphs. Put simply, each vertex in a bid graph is a bid, and an edge exists between two vertices if the two bids they represent share an item. Recall that N is our set of agents, and M our set of items. We denote the bid for agent i as V_i, which consists of atomic bids $(S_{i1}, p_{i1}), \ldots (S_{ir_i}, p_{ir_i})$ where $S_{ij} \subseteq M$ and $p_{ij} \geq 0$. For each agent i, we define r_i to be the number of atomic bids in V_i and let M_i represent the total number of items used over all atomic bids, counting each item exactly once. Lastly, we let R represent the number of distinct atomic bids across all V_i; we distinguish between two atomic bids by their item bundles. Equivalently, R represents the number of distinct subsets of items $S \subseteq M$ that have at least one atomic bid placed by an agent.

See Figure 1 for an example bid graph. The vertices are labelled $\{i, S_{ij}, p_{ij}\}$, where $i \in N$. WDP is equivalent to finding a maximum weighted independent set on the constructed graph.

Item graphs are a representation of CAs altogether different from bid graphs. We will be interested in restricting our attention to CAs that can be represented by an item graph with a specific structure, in order to gain insight into the structure of the auction's bid graph. Informally, in a valid item graph of the CA, the bids must be connected induced subgraphs of the item graph. Formally, we have the following definition:

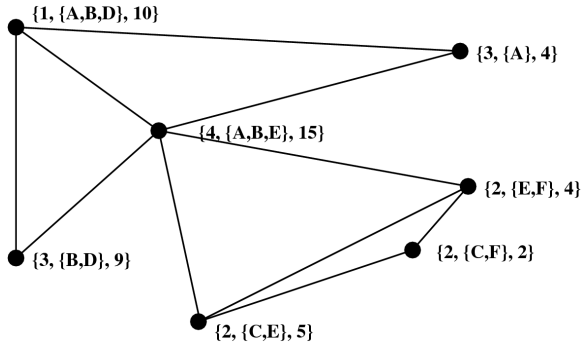

Figure 1: An example of a bid graph representing a CA. The CA consists of 4 agents, 6 items $\{A, B, C, D, E, F\}$, and 7 distinct atomic bids.

Definition 2.2.1. *Given a CA, a* valid item graph $G = (M, E)$ *representing the given CA must satisfy the following conditions: each item is represented by exactly one vertex in the graph, and for each atomic bid (S, p), the induced subgraph of G on the vertices contained in $S \subseteq M$ must be a connected graph.*

For example, let's assume we are working with a CA on four items consisting of the following atomic bids: $(\{1\}, 1)$, $(\{1, 2, 3\}, 5)$, $(\{2\}, 4)$, and $(\{2, 3, 4\}, 6)$. For simplicity, we assume that we have only one bidder. The graphs in Figure 2 are all examples of valid item graphs for this CA. We note how in each example there are exactly four vertices (corresponding to the total number of items in the CA). Further, for each example, if we restrict our attention to the vertices corresponding to any particular atomic bid, then the subgraph induced on those vertices is connected.

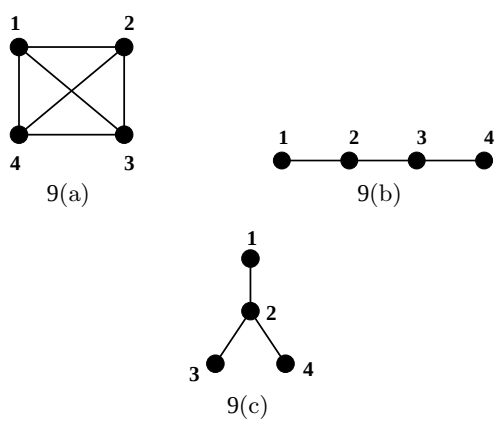

Figure 2: Examples of valid item graphs for the same combinatorial auction.

There are other examples of valid item graphs for the above CA. Specifically, the addition of edge $\{1, 3\}$ to the item graph in Figure 2(b) or 2(c) results in two additional valid item graphs for the above CA. Further, there are many other CAs for which Figure 2(a) is also a valid item graph. In particular, any CA on four items. Consequently, given an item graph we cannot know for certain which CA it represents. In contrast, one may translate a bid graph to and from a CA and in polynomial time. Any CA obtained from a bid graph yields the same solution to WDP, ignoring the case where multiple agents submitted the same highest bid on a bundle of items. Since an item graph may represent many CAs and allows for many different combinations of bids and bid values, the solution to WDP could be different for each CA that the item graph may represent. Obviously then, we cannot formulate a specific bid graph given an item graph. However, we can prove various properties about the bid graph of any CA that may result in a given item graph. This will be one of our main goals.

3. RELATING ITEM GRAPHS AND BID GRAPHS FOR FIXED CAS

While item graphs and bid graphs differ in how they represent CAs, the underlying instance of the auction they represent is the same. We wish to determine how bid graphs and item graphs relate to one another. In this section we investigate classes of item graphs and whether they enforce enough structure on their CAs to constrain the class of bid graphs associated with these auctions. First, we give a result by Conitzer *et al.*

Theorem 3.0.2 (Conitzer et al. [2]). *Suppose we are given a CA problem instance, together with a tree decomposition T with treewidth tw of an item graph G_I. Then the optimal solution to WDP can be determined in $O(|T|^2(R+1)^{tw+1})$ using dynamic programming.*

The algorithm presented by Conitzer *et al.* is vastly different from any algorithm designed to work on bid graphs, and it is beyond the scope of this paper whether or not there is some relationship between the two, in the most general sense. If we consider only CAs that have item graphs of treewidth one, then the item graph is a forest, for which we can solve WDP on each component separately. As stated, we know of a polynomial-time solution to WDP for an item graph that is a tree [12], which easily extends to an item graph that is a forest. We are interested in investigating the bid graph expression of all CAs that have a valid item graph representation with a treewidth of one. Rather than starting by investigating item graphs of treewidth one, we will consider a simpler example where our item graphs are paths. First, however, we discuss whether or not the item graph is connected.

Lemma 3.0.3. *If it is valid for an item graph G_I of a given combinatorial auction to be disconnected, then either the bid graph of the auction must be disconnected, or the combinatorial auction contains only bids on items in one component of G_I.*

Proof. Consider any bid B_1, which must contain only items from one connected component of G_I. Any other bid B_2 that is adjacent to B_1 in the bid graph must contain at least one item from B_1. Therefore, B_2 must contain only items from the same connected component of the item graph as B_1. If it did not, then the item graph would not be valid as the induced subgraph of G_I on the items of B_2 would be disconnected. If the bid graph is connected, then there is a path from B_1 to any other bid, and hence all bids must contain only items from the same connected component of G_I as B_1. If all bids contain only items from the same component of G_I, then this is valid. If not, we have a contradiction. □

We will assume that our item graph representation is connected, as otherwise our theorems will hold for each of the connected components of the item graph.

NOTE 3.0.4. *If a path is a valid item graph for a given combinatorial auction, then the bid graph representation of the auction is an interval graph.*

PROOF. Let us consider an arbitrary combinatorial auction that has a valid item graph representation as a path. Each bid of the combinatorial auction must induce a connected subgraph of the path. The path has length $|M|$. Consider one of the vertices of degree one, u. Starting at u and walking along the path, without repeating any edge or vertex, relabel each vertex starting with the integer 1, and increasing the label value by 1 with each subsequent vertex. To be more specific, the vertex adjacent to u will have label 2; the next vertex in the path will have label 3, and so on. With this labeling, it is easy to see that each bid in the combinatorial auction is a sub-interval of the interval $[1, |M|]$. The bid graph is then an intersection graph of the sub-intervals of the interval $[1, |M|]$, and therefore is an interval graph. □

Note 3.0.4 was also noted by Sandholm and Suri, although no proof was given [12]. As a corollary to Note 3.0.4, we see that if a cycle is a valid item graph, then the bid graph is a circular arc graph. A cycle can be viewed as a circle, with each bid an arc of the circle. The bid graph is then the intersection graph of those arcs, and hence is a circular arc graph. This observation was also made by Sandholm and Suri [12]. The maximum weighted independent set of a circular arc graph can be found in time $O(l|\mathcal{S}^*|)$, where l is the minimum number of arcs passing through some point on the circle [13]. This is comparable to the algorithm by Sandholm and Suri mentioned above, which has a running time of $O(\min\{s, |M|\}(R + |M|))$ and applies to valid item graphs that are cycles.

Drawing conclusions about an item graph with treewidth one is more complicated. First of all, a graph has treewidth one if and only if it is a forest. Since we are assuming that our item graph is connected, then our item graph must be a tree.

THEOREM 3.0.5. *For a given combinatorial auction CA, if a graph $G_I = (V_I, E_I)$ of treewidth one is a valid item graph for the auction, then its bid graph representation is a chordal graph.*

This theorem follows directly from a theorem by Gavril [6]. The intersection graph of a family of subtrees of a tree is called a *subtree graph*, where two subtrees intersect if they share a vertex [6]. Gavril's result states that a graph is chordal if and only if it is a subtree graph [6]. Since G_I is a tree and by definition the bids of the combinatorial auction CA it represents induced connected subgraphs of G_I, the bids induce subtrees of G_I. By definition, the bid graph is the intersection graph of the bids of CA, and thus is also the intersection graph of a family of subtrees of G_I. Therefore, the bid graph must be chordal [6].

We cannot use the other direction of Gavril's result to prove the converse of Theorem 3.0.5. Gavril places no restrictions on the structure of the subtree graph for which the result applies, while we require that the vertices are the set of items in order to be a valid item graph for a given bid graph. In Section 4, we will introduce a notion of bid graph equivalence in order to make full use of Gavril's result.

It is not immediately obvious whether or not all CAs whose bid graphs are chordal must have valid item graphs that are trees. To investigate this question, we show that a CA's bid graph must be chordal if it has a valid item graph with maximum cycle length three. We say a graph has *maximum cycle length k* if all cycles in the graph have length at most k.

The proofs of Lemma 3.0.6 and Theorem 3.0.8 focus on one arbitrary cycle of length three in our item graph $G_I = (V_I, E_I)$. Without loss of generality, we label the vertices involved as 1, 2, and 3. Consider the subgraph of G_I constructed by removing edges $\{1, 2\}$, $\{2, 3\}$ and $\{1, 3\}$. This subgraph must have exactly three connected components because the maximum cycle length in G_I is three. Therefore, there exist connected subgraphs $G_i = (V_i, E_i)$, $1 \leq i \leq 3$, of G_I such that $V_I = \bigcup_{i=1}^{3} V_i$, $E_I = (\bigcup_{i=1}^{3} E_i) \cup \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$, $i \in V_i$ for $1 \leq i \leq 3$, $E_i = \{(u, v) \in E_I \mid u, v \in V_i\}$ for $1 \leq i \leq 3$, and for all $1 \leq i < j \leq 3$, $V_i \cap V_j = \emptyset$. See Figure 3 for a depiction of this item graph. We refer back to Figure 3 in the proofs of Lemma 3.0.6 and Theorem 3.0.8.

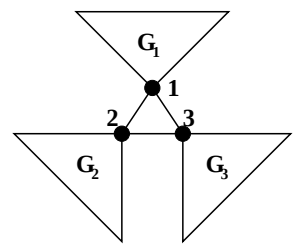

Figure 3: Example of an item graph with at least one cycle of length three. Subgraph G_i is assumed to have maximum cycle length three.

First, we show that if a tree with one extra edge resulting in a cycle of length three is a valid item graph for a CA, then the auction's bid graph must be chordal. To show this, we use a lemma about CAs when their bid graphs are chordal. Lemma 3.0.6 was derived from the idea of removing an edge in any cycle of length three from a valid item graph, and then adding that edge back. Of course, with the removal of an edge in the item graph, we can no longer have certain bids and so these bids are removed to result in a modified combinatorial auction. We show that given any cycle, if for each edge in that cycle the modified combinatorial auction has a chordal bid graph, then so does the original auction.

For brevity, we say a bid (S, p) *contains item set* $S' \subseteq M$ if $S' \subseteq S$. Further, we say a bid (S, p) *does not contain item set* $S' \subseteq M$ if $S' \not\subseteq S$. We introduce notation to denote when a bid contains a set of items S^+, while not containing another set of items S^-. For all $1 \leq i \leq |M|$, we let i^- denote an alternate label for item i. Hence, label i^- represents the same item as i. The label i is used to denote items that are contained in a bid, whereas the label i^- is used to denote items that are not contained in a bid. We let $M^- = \{1^-, 2^-, 3^-, \ldots, |M|^-\}$, and let $S^* = S^+ \cup S^-$, where $S^+ \subseteq M$, $S^- \subseteq M^-$, and $\{i \mid i \in S^+, i^- \in S^-\} = \emptyset$. We say a bid (S, p) *contains item set* $S^* = S^+ \cup S^-$ if $S^+ \subseteq S$ and $\{i \mid i \in S, i^- \in S^-\} = \emptyset$. For example, we define bid

$B = (S, p)$ by $S = \{1, 2, 4\}$ and p arbitrary. Bid B contains item sets $\{1, 2\}$, $\{1, 4\}$, $\{1, 2, 4\}$, and $\{1, 3^-, 4\}$. Bid B does not contain item set $\{4^-\}$, $\{1, 2, 3\}$, or $\{1, 2^-, 3^-, 4\}$.

LEMMA 3.0.6. *We let CA denote a combinatorial auction that has a valid item graph $G_I = (V_I, E_I)$ such that the maximum length of any cycle in G_I is three and G_I has at least one cycle. We denote the bid graph of CA by G_B. We let C_I be any cycle in G_I, and we label its vertices as 1, 2, and 3. We let $\{i, j\}$ be any edge in cycle C_I, and CA' be the modified combinatorial auction obtained by removing all bids containing $\{i, j\}$, but not containing $\{1, 2, 3\}$, from CA. Given any cycle C_I in G_I, if for any edge $\{i, j\}$ and CA' as defined above the bid graph of CA' is chordal, then the bid graph of CA is chordal.*

PROOF. We refer the reader to Figure 3 for a depiction of G_I, as previously discussed.

We assume that for any $\{i, j\}$ in C_I, the combinatorial auction resulting from the removal of all bids containing $\{i, j\}$, but not containing $\{1, 2, 3\}$, has a chordal bid graph. For each $\{i, j\}$, the resulting combinatorial auction may be different, but they all have a chordal bid graph. Without loss of generality, let CA' be the combinatorial auction resulting from the removal of all bids containing $\{1, 2, 3^-\}$ from CA. The bid graph of CA' is chordal. We have only to consider what happens when we add back all the removed bids containing $\{1, 2, 3^-\}$, and prove that the result is a chordal bid graph.

Assume that the bid graph is no longer chordal after adding back either one or two bids containing $\{1, 2, 3^-\}$. We do not consider adding back more bids because any chordless cycle can involve no more than two bids containing $\{1, 2, 3^-\}$. Specifically, by our definition of a bid graph in Section 2.2, if two bids share an item then they are adjacent in G_B. Therefore, if a cycle involves three bids containing $\{1, 2, 3^-\}$ then by definition they must all be pairwise adjacent to one another. Hence, the cycle cannot be chordless. If we are assuming that the bid graph is no longer chordal, then it must have at least one cycle C_B of length at least four with no chords. Cycle C_B must involve at least one bid containing $\{1, 2, 3^-\}$ because the original bid graph was chordal, but not more than two such bids because otherwise it would not be chordless. We note that the bids that we add back do not contain item 3.

We claim that cycle C_B must also involve a bid containing item set $\{1^-, 2, 3\}$. If not, we construct a bid graph G'_B by removing all bids containing $\{1^-, 2, 3\}$ from G_B. None of the bids removed are from C_B because C_B does not involve a bid containing item set $\{1^-, 2, 3\}$. The resulting bid graph is not chordal because the original bid graph was not chordal and we removed none of the bids in the chordless cycle C_B. However, by assumption, the bid graph must be chordal, which is a contradiction. By the same logic, the cycle C_B must involve a bid containing $\{1, 2^-, 3\}$. Specifically, we can derive another contradiction; the bid graph resulting from removing all bids containing $\{1, 2^-, 3\}$ from G_B is not chordal by construction, but was assumed to be chordal.

Therefore, the cycle C_B must have the following three distinct bids: one containing $\{1, 2, 3^-\}$, one containing $\{1^-, 2, 3\}$, and another containing $\{1, 2^-, 3\}$. These three bids form a clique of size three within C_B, which is not possible because C_B is chordless. Therefore, the bid graph resulting in the addition of one or two bids containing both items 1 and 2 (but not containing item 3) must be chordal.

We continue to add back bids containing $\{1, 2, 3^-\}$ two at a time until only one remains. Finally, we add the last remaining bid containing $\{1, 2, 3^-\}$. From above, at no point during the addition of one or two bids containing $\{1, 2, 3^-\}$ can the resulting bid graph become non-chordal. Therefore, the bid graph resulting from adding back all bids containing $\{1, 2, 3^-\}$ is chordal. Hence, the original bid graph G_B is chordal. □

With the result of Lemma 3.0.6, we have Lemma 3.0.7, which will be used as a base case of Theorem 3.0.8.

LEMMA 3.0.7. *For a given combinatorial auction CA, if a tree with one extra edge resulting in a cycle of length three is a valid item graph for the auction, then its bid graph representation is a chordal graph.*

PROOF. We denote the valid item graph as G_I and the bid graph as G_B. Because G_I is a tree with one extra edge resulting in a cycle of length three, we can depict the graph as in Figure 3. For simplicity and without loss of generality, we have labeled the items of the cycle as 1, 2 and 3. Subgraphs G_1, G_2 and G_3 of G_I are trees. Consider the item graph resulting from the removal of edge $\{1, 2\}$, $\{2, 3\}$, or $\{1, 3\}$. This item graph is valid for a modified combinatorial auction resulting from the removal of all bids containing $\{1, 2, 3^-\}$, $\{1^-, 2, 3\}$, or $\{1, 2^-, 3\}$, respectively, from CA. From Theorem 3.0.5, the bid graph of such an auction is chordal.

We now apply the result of Lemma 3.0.6. Given any cycle (in this case there is only one) and for each edge $\{i, j\}$ in that cycle, the modified combinatorial auction resulting from the removal of all bids containing $\{i, j\}$, but not containing $\{1, 2, 3\}$, from CA has a chordal bid graph. Therefore, by Lemma 3.0.6 the original combinatorial auction CA has a chordal bid graph. □

THEOREM 3.0.8. *For a given CA, if there exists a valid item graph $G_I = (V_I, E_I)$ representing the auction such that the maximum length of any cycle in G_I is three, then the bid graph for the auction is chordal.*

PROOF. For this proof, when we refer to a valid item graph, the length of all cycles in the graph is three. Theorem 3.0.5 handles the case when there are no cycles, so we assume that there is at least one cycle of length three.

The proof is by induction on the number of cycles of length three. Lemma 3.0.7 proves the base case, when the number of cycles of length three is one. We assume the result holds for all valid item graphs with at most k cycles of length three. We will show the result holds for all valid item graphs with $k + 1$ cycles of length three.

Consider a valid item graph that has $k+1$ cycles of length three. We focus on one arbitrary cycle C_I of length three in G_I, and without loss of generality, we label the vertices involved as 1, 2, and 3. Recall Figure 3 for a depiction of G_I, as previously discussed.

If we remove any edge $\{i, j\}$ in C_I from the item graph, and remove all bids from the combinatorial auction that contain $\{i, j\}$, but not $\{1, 2, 3\}$, then by construction the resulting item graph is valid for the new auction. By our inductive hypothesis, the new combinatorial auction's bid graph is chordal because the number of cycles in the item graph decreased by one. Therefore, by Lemma 3.0.6 the original combinatorial auction's bid graph is chordal. □

As a result of Lemma 3.0.7 and Theorem 3.0.8, it appears that we can conclude that if a bid graph of a CA is chordal, then a tree is not necessarily a valid item graph for the auction. For example, the cycle of length three in Figure 4(a) is a chordal bid graph, and the item graph in Figure 4(b) is the only valid item graph for such a bid graph.

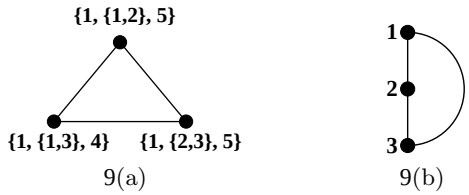

Figure 4: Example of (a) a bid graph that is a cycle of length three, and (b) its only valid item graph.

To conclude our discussion on the relationships between item graphs and bid graphs, we consider the CA with the following atomic bids, which has a valid item graph of treewidth two: $(\{1,2\},5)$, $(\{2,3\},5)$, $(\{3,4\},5)$, and $(\{1,4\},4)$. The bid graph is a cycle of length four, and a cycle of length four is also a valid item graph, as depicted in Figure 5. This example is interesting because the item graph has treewidth two, which means that given a CA for which an item graph of treewidth two is valid, the auction's bid graph is not necessarily a chordal graph.

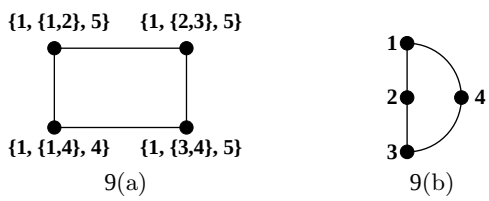

Figure 5: Example of (a) a non-chordal bid graph, and (b) one of its valid item graphs.

We leave open the general relationship between item graphs of treewidth two and bid graphs. Note that whatever classification of bid graphs may arise from such a relationship, the classification must encompass the class of chordal graphs, because any item graph of treewidth one is also considered to have treewidth two, by the definitions of treewidth and tree decompositions.

4. MODIFYING COMBINATORIAL AUCTIONS USING A NOTION OF EQUIVALENCE

In this section, we turn our attention beyond the direct relationship between item graphs and bid graphs, and discuss the possibility of modifying a CA such that the bid graph of the new auction is equivalent to the bid graph of the original auction. Equivalent CAs are useful because a solution to WDP for one auction can be easily translated into a solution to WDP for any equivalent CA. We will show a method for modifying a CA while maintaining bid graph equivalence, and further demonstrate how such modifications can be used to find valid item graphs of smaller treewidth.

To begin, we define bid graph equivalence. Intuitively, two bids graphs are equivalent if they are isomorphic, and two CAs are equivalent if their bid graphs are also equivalent. We now give the formal definition.

Definition 4.0.9 (Bid Graph Equivalence). *Given two bid graphs, $G = (V, E)$ and $G' = (V', E')$, we say that G is* equivalent *to G' (and vice-versa) if there exists an isomorphism f of G and G' and for all $v = (S, p) \in V$, $f(v) = (S', p') \in V'$ we have $p' = p$. If two CAs have equivalent bid graphs then we say the auctions are* equivalent.

We note that because the bid graphs are isomorphic and the weights on all of the vertices are the same, a solution to either auction gives us a solution to the other auction. More importantly, if we construct an item graph using the equivalent bid graph, and solve the WDP using this item graph, then that solution can be used to solve the WDP on the original bid graph by using the isomorphism to map the bids back to the original bid graph.

Theorem 4.0.10. *We let CA denote a combinatorial auction that has a valid item graph $G_I = (V_I, E_I)$. If the bid graph G_B of CA is chordal, then there exists a combinatorial auction CA' equivalent to CA that has a tree G'_I as a valid item graph.*

Proof. A result by Gavril states that a graph is chordal if and only if it is a subtree graph [6]. Gavril gives an efficient algorithm for constructing an item graph $G'_I = (V'_I, E'_I)$ that is a tree for any given bid graph $G_B = (V_B, E_B)$ that is chordal [6]. We let CA be the combinatorial auction represented by G_B. Gavril shows that $|V'_I| \leq |V_B| = R$ and that G'_I can be found in at most $O(R^4)$ time [6]. The construction considers each maximal clique C in G_B, of which there are at most R because G_B is chordal, and constructs a vertex in G'_I that represents C [6]. For each $B_i \in V_B$, consider all maximal cliques $C_1, C_2, \ldots, C_k$ in G_B, where C_j contains B_i. Gavril constructs G'_I such that the cliques containing B_i induce a tree in G'_I. This item graph is valid for a combinatorial auction that is equivalent to CA.

Since G'_I is a tree, the bids induce subtrees of G'_I. By definition, the bid graph is the intersection graph of the bids of CA', and thus is also the intersection graph of a family of subtrees of G'_I. Therefore, Gavril's result applies and the bid graph of CA' must be chordal. Since CA' is equivalent to CA, which we now show, their bid graphs are isomorphic and thus the bid graph of CA must also be chordal.

Consider the vertices of G'_I as items. Copy the bid graph G_B (to ensure our bid graphs are equivalent), and change the contents of every bid B_i of our copy to be exactly the set of items of G'_I that represent all the cliques to which B_i belongs. Specifically, B_i is now a bid on items $C_1, C_2, \ldots, C_k$, where C_j is a vertex of G'_I representing a clique of G_B, which contains vertex B_i. The resulting bid graph is equivalent to G_B and by construction is a subtree graph of G'_I [6]. Therefore, G_I is a valid item graph for a combinatorial auction that is equivalent to CA. Therefore, if we find any solution to WDP using the newly constructed item graph G_I, this solution can be easily translated to a solution for the original combinatorial auction. □

It is important we note that with our notion of CA equivalence, Gavril's result helps us show that even if the smallest treewidth for G_I is arbitrarily larger than one, there still

exists a CA equivalent with a tree as a valid item graph, given that the bid graph is chordal. This perspective on CA equivalence and the existence of item graphs of small treewidth has never been shown before.

COROLLARY 4.0.11. *For a given combinatorial auction CA, the bid graph of the auction is chordal if and only if there exists a combinatorial auction that is equivalent to CA for which a graph of treewidth one is a valid item graph.*

With the result of Theorem 4.0.10, it becomes difficult to gauge the usefulness of item graphs. If it is possible for a CA to have a valid item graph of treewidth one while another equivalent CA does not, then it is possible that we are translating our auction to a less efficient form. Further, there may or may not exist a CA with a chordal bid graph for which all valid item graphs have treewidth at least tw, for some large tw. That is, it is unclear how big of an improvement is possible using the equivalent CA technique. We have seen an example of a CA with a chordal bid graph for which all valid item graphs have treewidth at least two, but the improvement could be much larger. Given this potentially large lack of consistency between item graph representations of equivalent CAs, the construction process, as previously described in the literature, appears to be flawed.

5. CONCLUSIONS AND FUTURE WORK

Finding a solution to the winner determination problem (WDP) is NP-complete in the general case, and it is also equally difficult to find approximate solutions. Under special circumstances that enforce structure on the CA, WDP can become easier to solve. One such circumstance involves the treewidth of the item graph representation of the auction.

We showed that if a graph of treewidth one is a valid item graph for a combinatorial auction, then its bid graph representation is a chordal graph. Moreover, we generalize this result, showing that if there exists a valid item graph representing a CA such that the maximum length of any cycle in the graph is three, then the bid graph for the auction is chordal. However, we demonstrate that the result does not extend to cycles of length four or item graphs of treewidth two. Further, the converse does not hold when the auction is fixed. This impass leads to our next avenue of investigation: modifying combinatorial auctions.

While solving WDP using an item graph of fixed treewidth can be done in polynomial time [2], deciding whether a CA has a valid item graph of treewidth three is NP-hard [7]. In Section 4, we circumvent this result by providing a new technique for constructing item graphs.

We present a new technique for constructing item graphs using our new notion of CA equivalence. Currently the item graph construction process, for which the results by Conitzer *et al.* and Gottlob and Greco hold [2, 7], involves finding an item graph for a *fixed* CA. We demonstrate that this is not necessarily optimal. We leave open whether the construction process presented in Section 4 can be extended beyond chordal bid graphs. It is important to note that any solution to an item graph for the equivalent CA can easily be mapped to a solution to WDP for the original auction.

Understanding the relationship between bid graphs and item graphs is a very important step to better understanding combinatorial auctions and what makes the problem difficult to solve. Such an understanding may lead to new approaches to constructing item graphs and better algorithms for solving restricted instances of the WDP for CAs.

Since it is NP-hard to find a valid item graph of treewidth $tw \geq 3$ [7], it may be better to find equivalent CAs for which finding a valid item graph of treewidth tw is easy. This brings to question the practicality of studying item graphs of bounded treewidth. The new construction technique that we introduce opens another avenue of investigation for determining the practicality of item graphs of bounded treewidth. We initiate the study of combinatorial auction modification and for future work, it may be interesting to consider alternative constructions.

It would also be interesting to investigate relationships between bid graphs and item graphs of treewidth greater than one. Our new notion of CA equivalence and the resulting construction technique may be useful in this investigation. Further, Gottlob and Greco recently introduced a new method for qualifying hypergraphs of CAs, which they refer to as hypertrees of bounded hypertree width [7]. Item graphs of bounded treewidth are a special case of hypertrees of bounded hypertree width, and as such it would be interesting to see the parallels, if any, between their relationships with bid graphs. Despite the fact that item graphs of bounded treewidth are a special case of this new model, we can construct a hypertree of bounded hypertree width for a given CA, should one exist, in time that is polynomial in the size of the auction [7]. A natural question would be to ask how does our construction technique affect the new model by Gottlob and Greco? Does the new construction method account for why an item graph of bounded treewidth was previously NP-hard to construct, for treewidth larger than two, while a hypertree of bounded hypertree width is polynomial to construct for any fixed hypertree width?

6. REFERENCES

[1] A. Andersson, M. Tenhunen, and F. Ygge. Integer programming for combinatorial auction winner determination. *ICMAS*, pages 39–46, 2000.

[2] V. Conitzer, J. Derryberry, and T. Sandholm. Combinatorial auctions with structured item graphs. In *AAAI*, pages 212–218, 2004.

[3] P. Cramton, Y. Shoham, and R. Steinberg, editors. *Combinatorial Auctions*. MIT Press, 2006.

[4] S. de Vries and R. Vohra. Combinatorial auctions: A survey. *Informs Journal on Computing*, 15(3):284–309, 2003.

[5] Y. Fujishima, K. Leyton-Brown, and Y. Shoham. Taming the computational complexity of combinatorial auctions: Optimal and approximate approaches. *IJCAI*, pages 548–553, 1999.

[6] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *J. of Comb. Theory, Series B*, 16:47–56, 1974.

[7] G. Gottlob and G. Greco. On the complexity of combinatorial auctions: Structured item graphs and hypertree decompositions. In *ACM Conference on Electronic Commerce*, pages 152–161, 2007.

[8] N. Nisan and A. Ronen. Computationally feasible vcg mechanisms. In *ACM Conference on Electronic Commerce*, pages 242–252, 2000.

[9] Noam Nisan. Bidding languages for combinatorial auctions. In Shoham Cramton and Steinberg, editors,

Combinatorial Auctions. MIT Press, 2006.

[10] M. H. Rothkopf, A. Pekeč, and R. M. Harstad. Computationally manageable combinatorial auctions. *Management Sci.*, 44(8):1131–1147, 1998.

[11] T. Sandholm. Algorithm for optimal winner determination in combinatorial auctions. *Artificial Intelligence*, 135:1–54, 2002.

[12] T. Sandholm and S. Suri. Bob: Improved winner determination in combinatorial auctions and generalizations. *Artificial Intelligence*, 145:33–58, 2003. Earlier version: Improved Algorithms for Optimal Winner Determination in Combinatorial Auctions and Generalizations. *AAAI-00*, pp. 90–97.

[13] J. P. Spinrad. *Efficient Graph Representations*, volume 19 of *Fields Institute Monographs*. American Mathematical Society, 2003.