# CMSC 858F: Algorithmic Game Theory
# Fall 2010
# Price of Anarchy for Non-Atomic Selfish Routing Games

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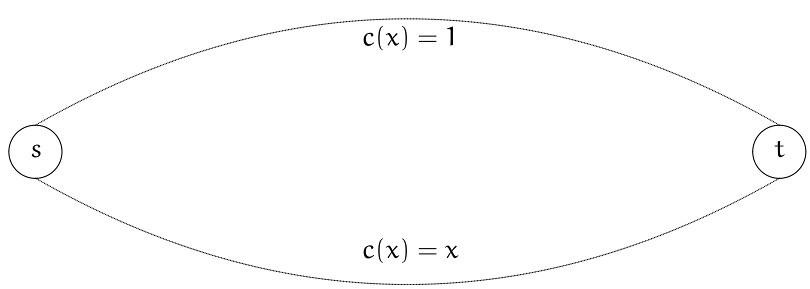
## 1 Overview

Recall the definitions of the "price of anarchy" and "price of stability" from the Lecture 1, Section 8 notes.

This lecture will focus on understanding the price of anarchy in non-atomic selfish routing games (also known as "non-atomic congestion games"). We will first see some examples of such games, and then define these games formally. Last, we will prove a least upper bound result on the price of anarchy in non-atomic selfish routing games.

## 2 Examples

**Example 1** (Pigou's Example, 1920) *On each edge,* $c(x)$ *denotes the cost of traveling on that edge. There is a total flow of* $1$ *starting at* $s$ *that flows to* $t$*, consisting of infinitely many infinitesimally small (i.e. measure* $0$*) decision makers choosing a path. Each decision maker tries to minimize his/her individual travel cost.*



*The (unique) Nash equilibrium in this game is that everyone takes the lower path and faces a travel cost of* $1$. *To see why this is the unique Nash equilibrium, suppose some positive fraction* $\alpha \in (0, 1]$ *choose to take the upper path. If so any one traveler on the upper path has a profitable deviation to the lower path, so we are not at a Nash equilibrium.*

*The social optimum minimizes expected average travel time, in that it requires* $\alpha^* \in [0, 1]$ *to take the upper path, and* $1 - \alpha^* \in [0, 1]$ *to take the lower path, such that*

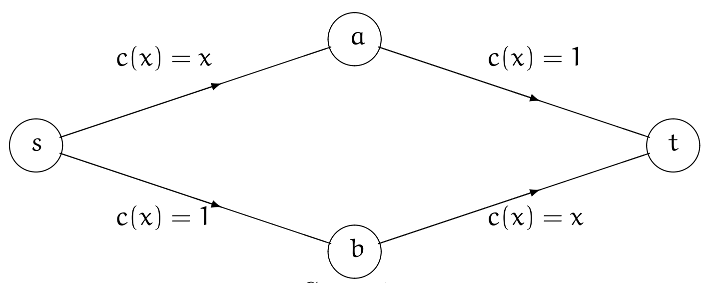
$$\alpha^* = \operatorname{argmin}_\alpha \alpha + (1 - \alpha)^2$$

*Solving for* $\alpha^*$*:*

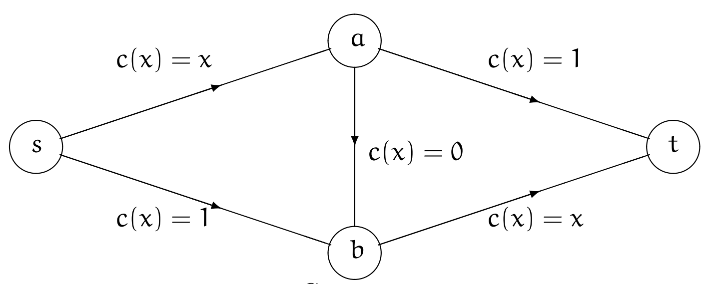
$$\alpha^* = 1/2$$

*In this case, the expected average travel time is* $\alpha^* + (1 - \alpha^*)^2 = 1/2 + (1 - 1/2)^2 = 3/4$. *The price of anarchy is the Nash equilibrium payoff divided by the socially optimal average payoff, i.e.* $1/(3/4) = 4/3$. *Since the Nash equilibrium is unique, the price of anarchy is also the price of stability.*

**Example 2** (Braess's Paradox, ~1940) *Consider the following two games, again with a flow of* $1$ *starting at* $s$ *traveling to* $t$




*Game 1*




*Game 2*

*In Game 1, the Nash equilibrium is that* $1/2$ *the population travels from* $s$ *to* $a$ *to* $t$, *incurring a travel cost of* $1/2$ *from* $s$ *to* $a$ *and* $1$ *from* $a$ *to* $t$, *for a total cost of* $3/2$. *The other* $1/2$ *of the population travels from* $s$ *to* $b$ *to* $t$, *incurring* *a travel cost of* $1$ *from* $s$ *to* $b$ *and* $1/2$ *from* $b$ *to* $t$*, for a total cost of* $3/2$*. The Nash equilibrium is also the social optimum, so the price of anarchy is* $1$*.*

*In Game 2, the Nash equilibrium is that all of the population travels from* $s$ *to* $a$ *to* $b$ *to* $t$*, and everyone incurs a travel cost of* $2$*. It is easy to see that this is worse than the social optimum, which is unchanged from game* $1$*. Therefore, in Game 2, the price of anarchy is* $2/(3/2) = 4/3$*. Paradoxically, adding a no-cost path has made us worse off.*

This observation has applications in urban planning - adding a highway may not make a city better off; indeed, it may make it worse off. Observe that in both examples, we have achieved a price of anarchy equal to 4/3. Can we add an edge to Game 2 that will further increase the price of anarchy? The answer is no; now we will formalize our definition of these routing games, and we will show that 4/3 is the least upper bound on the price of anarchy in these types of games.

## 3 Formal Definitions

**Definition 1** *A* graph $G$ *is a set of vertices* $V$ *and edges* $E$ *connecting elements of* $V$*. We will write* $G = (V, E)$*.*

Edges may have a direction; we will typically deal with directed graphs since we can view any undirected edge as two distinct edges going in opposite directions. In a network design problem, we will associate with each edge $e$ a cost function $c_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that takes as its input the flow $f_e$ along edge $e$ and outputs the cost $c_e(f_e)$ associated to traveling on that edge.

The games we have considered so far have featured flow from $s$ to $t$, known as "single source, single sink" games. Without loss of generality, we normalize flow $d = 1$ from $s$ to $t$ in these games, by which we mean that the flow out of $s$ is $1$, the flow into $t$ is $1$, and the net flow to all other vertices is $0$. If there are $k$ paths $p_1, p_2, \cdots p_k$ paths from $s$ to $t$, and flow $f_k$ along path $p_k$, we can think of any division over the paths as a probability distribution. More generally, we consider multicommodity demand $D = \{(s_1, t_1), (s_2, t_2), \cdots (s_k, t_k)\}$ with flow $d_i$ from $s_i$ to $t_i$.

**Definition 2** *A* non-atomic selfish routing game *is a graph* $G$*, demand* $D$*, and set of cost functions* $C$ *for each edge. Denote a non-atomic selfish routing game by the triple* $(G, D, C)$*.*

Define $\mathbb{P}_i$ to be the set of all paths from $s_i$ to $t_i$ and define $\mathbb{P} = \bigcup_{i=1}^{k} \mathbb{P}_i$. Then for any path $p \in \mathbb{P}_i$, the cost $c(f_p)$ of flow $f_p$ along $p$ is $\sum_{e \in p} c_e(f_e)$, the sum of costs of the flow on each edge $e$ in the path $p$.

**Definition 3** *A* Nash equilibrium *in a non-atomic selfish routing game is a feasible flow* $f$ *(between all pairs* $s_i$ *and* $t_i$*) such that for every pair* $p, \tilde{p} \in \mathbb{P}_i$ *of* $s_i \to t_i$ *paths with* $f_p > 0$*, we have* $c_p(f) \leq c_{\tilde{p}}(f)$*.*

Essentially, a Nash equilibrium requires that (1) flow is minimized on each path and (2) it is equal on all paths. We can find the Nash equilibria by minimizing total cost:

$$\min C(f) = \min \sum_{p \in \mathbb{P}} c_p(f) f_p = \min \sum_{e \in E} c_e(f_e) f_e$$

# 4 Existence and Uniqueness of Equilibrium Flows

Now we want to show that with reasonable cost functions, a non-atomic selfish routing game admits a unique Nash equilibrium. First, we an intuitive preliminary result.

**Proposition 1** *Let* $(G, D, C^*)$ *be a non-atomic selfish routing game such that for every edge* $e$ *the function* $c_e^*$ *is convex and continuously differentiable. Let* $c_e^{*\prime}$ *denote the marginal cost function of edge* $e$*. Then* $f^*$ *is an optimal flow of* $(G, D, C^*)$ *if and only if for every commodity* $i \in \{1, \cdots, k\}$ *and every pair* $p, \tilde{p} \in \mathbb{P}_i$ *of* $s_i \to t_i$ *paths with* $f_p^* > 0$, $c_p^{*\prime}(f^*) \leq c_{\tilde{p}}^{*\prime}(f^*)$.

**Proof:** One can verify this result by writing the optimization problem longhand and taking first order conditions. ■

**Theorem 1** *Let* $(G, D, C)$ *be a non-atomic selfish routing game where* $c_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *are non-negative, continuous, non-decreasing functions. Then*

1. $(G, D, C)$ *admits at least one equilibrium flow*
2. *If* $f, \tilde{f}$ *are equilibrium flows for* $(G, D, C)$ *then* $c_e(f_e) = c_e(\tilde{f}_e)$ *for every edge* $e$.

**Proof:** Note that from Proposition 1, we can write $c_e^*(x) = \int_0^x c_e^{*\prime}(y)dy$. Define the potential function

$$\phi(f) = \sum_{e \in E} c_e^*(f_e) = \sum_{e \in E} \int_0^{f_e} c_e^{*\prime}(y)dy$$

Observe that $\phi(\cdot)$ inherits continuous differentiability from the $c_e^*$. By definition of Nash equilibrium in a non-atomic selfish routing game, optimal $f^*$ is a global minimizer of $\phi(\cdot)$. Note also the that constraint set is compact, since if $f$ is the total flow through the whole game, the domain of $\phi(\cdot)$ is contained in $\bigoplus_{e \in E}[0, f]$. By Weierstrass's Theorem, a continuously differentiable function on a convex, compact set achieves a unique minimum on that set. Furthermore, since the $c_e$ are nondecreasing, that minimum is unique. ■

Note that if the Nash equilibrium is unique (item 2), then the price of anarchy is equal to the price of stability.

# 5 Least Upper Bound on Price of Anarchy

First, we need a lemma.

**Lemma 1** *The inner product* $< c(f^{Eq}), f^{Eq} - f > \leq 0$ *for all flows* $f$.

**Proof:** Suppose not. Then expanding the inner product

$$c(f^{Eq})f^{Eq} > c(f^{Eq})f$$

Expanding both sides as sums over $p \in \mathbb{P}$ :

$$\sum_{p \in \mathbb{P}} c_p(f^{Eq})f_p^{Eq} > \sum_{p \in \mathbb{P}} c_p(f^{Eq})f_p$$

Since cost along any edge is nonnegative and flow on any source-sink path is nonnegative, with positive flow on at least one path between some pair $s_i$ and $t_i$, it follows that there exists at least one path $p$ such that

$$c_p(f^{Eq})f_p^{Eq} > c_p(f^{Eq})f_p$$

In other words, there is some path where the equilibrium flow is not the cheapest option, contradicting our assumption that $f^{Eq}$ is an equilibrium flow. So the lemma holds.

■

**Theorem 2** *Suppose* $(G, D, C)$ *is a non-atomic selfish routing game and for every edge* $e$*, the cost function* $c_e$ *is affine (i.e.* $c_e(x) = c_e x + b_e$ *for nonnegative* $c_e, b_e$*). Then the price of anarchy in* $(G, D, C)$ *is less than or equal to* $4/3$.

**Proof:** Let $f^{Eq}$ denote Nash equilibrium flow and let $f$ denote socially optimal flow. By Lemma 1, $f^{Eq}$ is equilibrium flow we must have $< c(f^{Eq}), f^{Eq} - f > \leq 0$ for all $f$. Expanding this inner product:

$$c(f^{Eq})f^{Eq} - c(f^{Eq})f \leq 0$$
$$c(f^{Eq})f^{Eq} \leq c(f^{Eq})f$$

Expanding both sides as sums over $e \in E$ :

$$\sum_{e \in E} c_e(f_e^{Eq})f_e^{Eq} \leq \sum_{e \in E} c_e(f_e^{Eq})f_e$$

Adding and subtracting $\sum_{e \in E} c_e(f_e)f_e$ on the right hand side:

$$\sum_{e \in E} c_e(f_e^{Eq})f_e^{Eq} \leq \sum_{e \in E} c_e(f_e)f_e + \sum_{e \in E} (c_e(f_e^{Eq}) - c_e(f_e))f_e$$

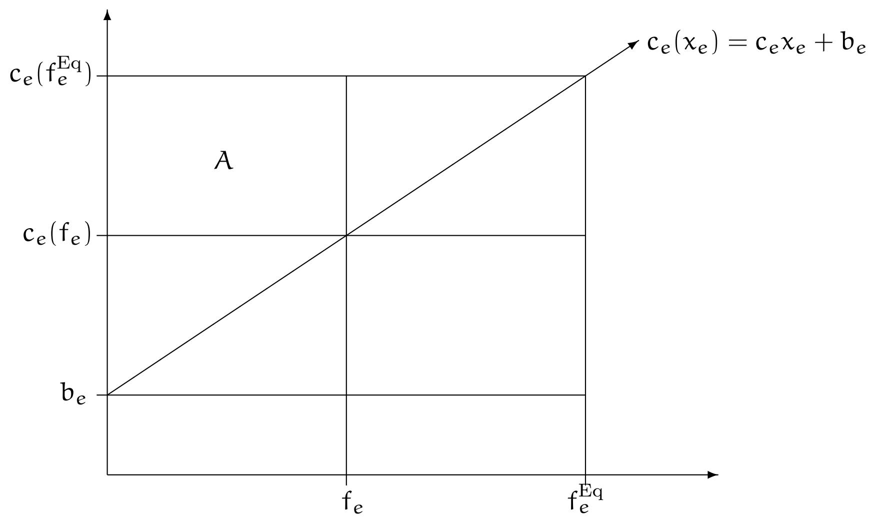
Now we can see that it is sufficient to show that

$$\sum_{e \in E} (c_e(f_e^{Eq}) - c_e(f_e))f_e \leq \frac{1}{4}c(f_e^{Eq})$$

since if so

$$c(f^{Eq}) \leq c(f) + \frac{1}{4}c(f^{Eq})$$
$$\frac{3}{4}c(f^{Eq}) \leq c(f)$$
$$\frac{c(f^{Eq})}{c(f)} \leq \frac{4}{3}$$

We now prove that for every edge, $c_e(f_e^{Eq}) - c_e(f_e) \leq \frac{1}{4}c_e(f_e^{Eq})$, which, summing over edges, immediately implies the sufficient condition.




In the picture, the area above the cost function clearly accounts for less than half the area of the large rectangle, i.e. the region bounded by the points $(0,0), (f_e^{Eq}, 0), (f_e^{Eq}, c_e(f_e^{Eq}))$, and $(0, c_e(f_e^{Eq}))$. The area of rectangle $A$ is at most half of the area above the cost function. Computing the area of $A$ and the large rectangle we obtain $(c_e(f_e^{Eq}) - c_e(f_e))(f_e) \leq \frac{1}{4}(c_e(f_e^{Eq}))(f_e^{Eq})$, as desired. ∎

# 6 Homework

Same problem as last week. The point of these exercises is to figure out how to formulate the problem as a research question.

## References

[1] Nisan, N. Roughgarden, T.; Tardos, E.; and V. Vazirani. *Algorithmic Game Theory.* Cambridge University Press, 2007.

[2] Hajiaghayi, M. T. *Algorithmic Game Theory: Lecture 2 Hand- written notes.* 1 September 2010. University of Maryland.

[3] Wunder, M.,*Algorithmic Game Theory: Scribe Notes.* 9 February 2009. Rutgers University.