CMSC 858F: Algorithmic Game Theory
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Introduction to Algorithmic Game Theory

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1 Overview

2 Cooperative Game Theory

In game theory as we have studied it so far this semester, games are assumed to be interactions between completely independent and selfish parties. For example, Nash equilibria were assumed to be stable, since no single player could alter their strategy and benefit. However, there could, for example, be a situation where there are two Nash equilibria. In the current equilibrium state, no player benefits from changing their strategy, but two players together could be enough to switch the equilibrium to a different, more beneficial equilibrium. This mean that the first equilibrium is not in fact stable in the case of cooperating adversaries. When cooperating groups are fixed, we deal with this by simply considering cooperating players to be a single player that controls all members of the cooperating group. However, this does not deal with questions of how players choose to cooperate or not with each other. We study this group formation through *cooperative games*.

A cooperative game has two main components. The first is a set of agents, $N = \{1, 2, \ldots, n\}$. The second is a value function. We use 2^N to denote the power set $\mathcal{P}(N)$. The value function maps subsets of N to non-negative real numbers, $V : 2^N \to \mathbb{R}^+ \cup \{0\}$. Intuitively, one should think of the value function as representing the payout that a subset of players can achieve as a result of cooperating.

The players in this game are making no choices other than who to cooperate with. The outcome of the game is a subset of cooperating players, along with a payment vector $\vec{x} = (x_1, \ldots, x_n)$ that specifies what portion of the payout is given to each player in the game.

Having defined a cooperative game itself, we now define several related concepts and attributes:

Definition 1 *A cooperative game is super-additive if for* $S, T \subseteq N$ *we have that* $V(S \cup T) \geq V(S) + V(T)$.

Definition 2 *A cooperative game is monotonic if for all* S, T *with* $S \subseteq T$ *we have* $V(T) \geq V(S)$.

Definition 3 *A subgame of a cooperative game is simply the same game but where the potential participating players are limited to a subset* S*. Formally, there is a new value function* $V_S : 2^S \to \mathbb{R}^+ \cup \{0\}$ *where* $V_S(T) = V(T)$ *for any* $T \subseteq S$.

Definition 4 *The dual of a cooperative game is a game with the same* N *players but with value function* $V^*(S) = V(N) - V(N \setminus S)$.

Definition 5 *A simple cooperative game is one where* $V(S)$ *is 0 or 1 for all* S.

Definition 6 *A cooperative game is symmetric if only the size of the coalition matters. Formally, in a symmetric game we have* $V(S \cup \{i\}) = V(S \cup \{j\})$ *for all subsets* $S \subseteq N \setminus \{i, j\}$.

Definition 7 *An outcome is efficient if the entire payout is distributed to players, meaning* $\sum_{i \in N} x_i = V(N)$.

Definition 8 *An outcome has individual rationality if no individual can do better on their own than with the coalition, i.e. if* $x_i \geq V(\{i\})$ *for all* i.

Definition 9 *An outcome has group rationality if no group can do better as its own separate coalition, i.e. if* $\sum_{i \in S} x_i \geq V(S)$ *for all* S.

Definition 10 *The core of a game is the set of outcomes that have both group rationality and efficiency.*

Definition 11 *The power of player* i *over player* j*, denoted* s_{ij}*, is the amount that* i *can get without cooperating with* j*. Power is specific to a particular outcome, and is formally*

$$s_{ij} = \max_S \{V(S) - \sum_{k \in S} x_k : S \subseteq N, i \in S, j \notin S\} \tag{1}$$

Definition 12 *The prekernel of a game is the set of outcomes where all pairs of players have equal power over each other, meaning that* $s_{ij} = s_{ji}$ *for all* i *and* j.

The prekernel and core are natural concepts of interest to economists. It has previously been shown, and we take as given, the fact that the prekernel is always nonempty, and also the following theorem:

Theorem 1 *If the core of a game is nonempty, then so is the intersection of the core and the prekernel. Furthermore, this intersection can be calculated in polynomial time as long as any s_{ij} value can be calculated in polynomial time.*

In the remainder of these notes, we will show that this result from economics implies important results in network bargaining games that were first shown by computer scientists in a more complex way.

3 Network Bargaining

We now turn from general definitions to a specific class of games. We look at network bargaining games. Here we consider a graph where the vertices are the n agents. Each agent i can participate in at most c_i contracts. Potential contracts are represented as edges in the graph, with weights w_{ij} representing the profit that would be generated by that contract. We focus on the matching varient, where $c_i = 1$ for all i.

An outcome of such a game is a pair $(M, \{z_{ij}\})$. M represents a matching, the set of contracts utilized, while the set of z_{ij} values sets how the profit from each contract is split. (The subscripts on the edges w_{ij} are not sensitive to order, but the subscripts on z_{ij} are.) In particular, we have that $z_{ij} + z_{ji} = w_{ij}$ for all $(i, j) \in M$.

This outcome can trivially be converted to an outcome as defined for general cooperative games. In the general network bargaining case $x_i = \sum_{j \in N} z_{ij}$. In the specific matching case, $x_i = z_{ij}$ if there is a j such that $(i, j) \in M$ and is zero otherwise.

Definition 13 *The outside option α_i is the best that i could achieving by joining a contract other than that specified by the matching M. It is assumed that a potential contract partner k will join the contract if and only if they are paid an amount equal to the amount x_k that they would be receiving otherwise. Formally,*

$$\alpha_i = \max_{k:(i,k) \in E \setminus M} \{w_{ik} - x_k\}. \tag{2}$$

Definition 14 *An outcome is stable if $x_i > \alpha_i$ for all i.*

Definition 15 *An outcome is balanced if for all edges (i, j) in the matching M, $x_j - \alpha_j = x_i - \alpha_i$.*

Our main goal is to prove the following theorem:

Theorem 2 *If we already know one stable solution, then we can find a balanced and stable solution in polynomial time.*

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4 Proof of Theorem 2

We prove Theorem 2 by showing that it is a special case of the more general Theorem 1. We first define $V(S)$ to be the value of the maximum-weight matching possible on $G[S]$, the induced subgraph on $S \subseteq N$. We will show that in this formulation, a solution is stable if and only if it is in the core, and it is both stable and balanced if and only if it is in the intersection of the core and the prekernel. Given those results, Theorem 2 follows immediately from Theorem 1. (In this case it is trivial that s_{ij} can be computed in polynomial time.

Lemma 1 *Outcome* $(M, \vec{x})$ *is stable if and only if the payoff vector* $\vec{x}$ *is in the core.*

Proof: This lemma is proved through linear programming using the strong duality theorem and complementary slackness conditions.

First, we prove that if $(M, \vec{x})$ is stable then $\vec{x}$ is in the core. Note that efficiency is trivially true, so we only need to show group rationality. Also note that group rationality in this case is equivalent to the claim that M is a maximum-weight matching. We consider the following primal linear program P, using V and E to represent the vertices and edges of the graph G, respectively:

$$P = \min \left\{ \sum_i x_i \right\}$$

$$\begin{aligned} x_i + x_j &\geq w_{ij} \quad \forall (i,j) \in E \\ x_i &\geq 0 \quad \forall i \in V \end{aligned}$$

And also its dual, D:

$$D = \max \left\{ \sum_e w_e y_e \right\}$$

$$\begin{aligned} \sum_{j: e=(i,j) \in E} y_e &\leq 1 \quad \forall i \in V \\ y_e &\geq 0 \quad \forall e \in E \end{aligned}$$

$\vec{x}$ and M define a feasible solution for both linear programs (taking $y_e = 1$ if $e \in M$ and $y_e = 0$ otherwise). We can conclude from the strong duality theorem that M is a maximal-weight matching. This implies efficiency, since the sum of x_i values is the weight of matching M and is therefore equal to the value of the cooperating subset of players. Furthermore, for $e \in M$ we have that $x_i + x_j = w_e$ by definition. For $e \notin M$ the definition of an outside option implies that $\alpha_i \geq w_e - x_j$ and the definition of stable says $x_i \geq \alpha_i$. Rearranging gives us $x_i + x_j \geq w_e$. This can be summed over the edges of any matching on the graph. This implies group rationality. This means that we have shown that stability implies both conditions of membership in the core.

We now show the reverse, that given $\vec{x}$ in the core, we can show a matching M such that $(M, \vec{x})$ is stable. We need a matching M where for all $e \in M$ we have $x_i + x_j = w_e$. Note that $\vec{x}$ is a feasible solution for linear program P. Consider any maximum-weight matching M and set $y_e = 1$ if $e \in M$ and $y_e = 0$ otherwise. $V(V)$ (abusing notation, using V as the value function and the vertex set) and the weight of M are equal to $\sum x_k$. As a result, the strong duality theorem says that $\vec{x}$ and $\vec{y}$ are optimal solutions to both linear programs. The complementary slackness conditions then tell us that $y_e(x_i + x_j - w_e) = 0$. This is equivalent to saying that if $y_e > 0$ (i.e., if $e \in M$) then $x_i + x_j = w_e$. This is therefore a valid outcome to the game. It remains to show that it is stable. Take j to be the player such that a contract with j is the best outside option for player i. By definition $\alpha_i = w_e - x_j$. However, we have that $x_i + x_j \geq w_e$, which can be rearranged as $x_i \geq w_e - x_j = \alpha_i$. Therefore this is a stable solution. ■

Corollary 1 *A graph has a non-empty core (or equivalently, a stable outcome) if and only if the associated linear program defined above has an integrality gap of 1 for finding a maximum-weight matching.*

This corollary gives a useful characterization of graphs with stable outcomes.

Lemma 2 *An outcome $(M, \vec{x})$ is stable and balanced if and only if $\vec{x}$ is in the intersection of the core and the prekernel.*

Proof: We already have proven that the $(M, \vec{x})$ is stable if and only if $\vec{x}$ is in the core and that if $\vec{x}$ is in the core we can construct a maximum-weight matching consistent with $\vec{x}$. We now prove that if $(M, \vec{x})$ is balanced, $\vec{x}$ is in the prekernel. We now give a simplified definition of s_{ij}:

$$s_{ij} = \max\{w_{ik} - x_i - x_k : (i, k) \in E, k \neq j\} \tag{3}$$

The condition is weaker because it is not dependent on the choice of S, but this new definition of power gives a value no higher than the previous definition. For a set S with only two vertices, the definitions are equivalent. In any non-empty maximum-weight matching M on a set S, each edge e contributes w_e to $V(S)$ and at least w_e to $\sum x_k$, so it cannot increase the value. We will show that this new value is always zero. The group rationality condition (which is assumed true here) says that the true power values (by the original definition) are at most zero, and since our new value is a lower bound, this will imply that the power values by the old definition are also always zero.

To see that this new s_{ij} is always zero, we consider two cases for each edge. If the edge is in M, then the same k that maximizes $w_{ik} - x_i - x_k$ should maximize α_i as well. Similarly, the same k' should maximize both $w_{jk'} - x_j - x_{k'}$ and α_j. This means that the balance condition, $x_i - \alpha_i = x_j - \alpha_j$ implies that $s_{ij} = s_{ji}$. Alternatively, if the edge is not in M, then take k such that $(i, k) \in M$. Then $s_{ij} \geq w_{ik} - x_i - x_k = 0$. However, the conditions of being in the core again tell us that $s_{ij} \leq 0$, so $s_{ij} = 0$. If there is no such k, $x_i = 0$ and we can instead consider any $k \neq j$ with $(i, k) \in E$. We know $x_k = w_{ik}$, so $s_{ij} \geq w_{ik} - x_i - x_k = 0$.

Again, this implies that $s_{ij} = 0$. We now have shown that in all cases $s_{ij} = 0$ and therefore have completed this portion of the proof.

All that remains is to prove that if $\vec{x}$ is in the prekernel then $(M, \vec{x})$ is balanced. This is simple. By definition we know that $s_{ij} = s_{ji}$. For an edge $(i, j) \in M$ the simplified definition gives us $s_{ij} = \alpha_i - x_i$ and $s_{ji} = \alpha_j - x_j$. This means that $s_{ij} = s_{ji}$ is exactly equivalent to the balance condition. ■

As stated previously, these lemmas are sufficient to complete the larger proof of Theorem 2, given that Theorem 1 is assumed.

5 Homework Assignment

The current recession is in large part due to the collapse of the housing market. There were many reasons for this collapse. One is that if housing prices were rising, it was in a person's interest to buy a house even if they couldn't afford its actual price, because they could hold it for a little while and resell it, making a profit greater than what they would owe for the mortgage. Use game theory to suggest some changes that could reduce the potential for a similar collapse to reoccur in the future.