CMSC 858F: Algorithmic Game Theory
Fall 2010
Introduction to Algorithmic Game Theory

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September 22, 2010

1 Overview

In this lecture, we will discuss auctions and its basic definitions. We introduce the Vickrey Second Price Auction and give incentive-compatibility proofs. We also introduce Vickrey-Clarke-Groves mechanism and prove its incentive compatible. The Clarke-Pivot Payment rule is also stated. Finally, we introduce Combinatorial Auctions.

2 Introduction

An *auction* is a method of selling an asset by competitive bidding. An auction is most useful when the potential price of the asset to be sold is uncertain. Different auction formats exist, varying according to how prices are quoted and bids tendered. The most commonly known of these is the *English Auction*, which is commonly used for artworks and wine.

In an auction, we have a set of buyers and a set of sellers. The setting is similar to the market equilibrium setting described in the previous class. The difference in an auction is that here, there is more negotiation involved and there is a possibility to deviate from the truth (i.e) lie.

Let us start out with the extreme case of there being only a single seller - the auctioneer. The auction rule defines the social choice (i.e) the identity of the winner. *Social choice* is defined as an aggregation of the preferences of different participants towards a single joint decision.

2.1 Setting

We have n players and a set of alternatives/outcomes A. Each player has a *valuation function* $v_i : A \longrightarrow R$ where $v_i(A)$ is the value player i assigns to the alternative A. (v_i is usually in terms of money).

We also assume that if i is chosen, then he is additionally given some quantity of money m. (where m can be positive or negative). Then, the *utility function* becomes $u_i = v_i(a) + m$.

This utility being the abstraction of what the player desires to maximize. These utilities are called *quasi-linear* preferences because they are linearly dependent on m and on other things independent of m.

Note : Here, m can be dependent on outcome and also, m can be dependent on player i. (I.e) in general, it can be $u_i = v_i(a) + m_i(a)$.

English Auction English Auction is a famous auction where the auctioneer raises the current price by small amounts until there is only one bidder remaining at the current price. The highest bidder wins the auction. This is also known as the *ascending auction.*

3 Vickrey's Second Price Auction

3.1 Setting

We have a single item to sell and there are n players bidding for the item. Player i has a real value associated for the item, which would be denoted by w_i. So, if player i wins at Price P, then his utility $u_i = w_i - P$ and if player does not win, his utility would be zero.

Writing it formally,

$$v_i(i \ wins) = w_i \Rightarrow u_i(i \ wins) = w_i - P$$

$$v_i(j \neq i \ wins) = 0 \Rightarrow u_i(j \neq \ i \ wins) = 0 - 0 = 0$$

$$\Rightarrow u_i = v_i - P$$

Considering the fact that w_i of a player i is hidden, we should ensure that we maximize the social welfare (ie) be able to choose i which is $(argmax)_j w_j$. Since all the w_i values are private as stated above, we should ensure that our mechanism decides in such a way that it cannot be manipulated. Such a mechanism is called *truthful* or *strategy proof* or *incentive compatible.*

Why is Vickrey auction needed? Let us consider the following cases and see why they do not work:

- No payment : Just give the item without receiving any payment - If we give the item for free to the highest bidder, the auction will not be truthful as it can be easily manipulated by exaggerating.
- Pay your bid : Similar to first price auction. In this model, under-representing one's bid can help the bidder gain a positive utility rather than zero.

3.2 Formal Definition

Definition 1 *Let the winner be the player* i *with the highest declared value of* w_i *and let* i *pay the second highest declared bid (i.e)* $P^* = max_{j \neq i} w_j$*. This is Vickrey's Second Price Auction.*

Proposition 1 *For every* $w_1, \cdots, w_n$ *and for every* w_i'*(where* w_i' *is the value* i *reports as his bid which could be different from his true value* w_i*), let* u_i *be Player* i*'s utility if he bids* w_i *and* u_i' *be his utility if he bids* w_i'*. Then* $u_i \geq u_i'$*.*

Proof: *Case I* : Let i be the person with the highest bid (i.e) Player i wins the auction. $\Rightarrow u_i = w_i - P^*$

Case 1: $w_i' > w_i$ i wins and pays P^* which remains the same. Therefore, i's utility remains unchanged since $u_i' = w_i - P^* = u_i$.

Case 2: $w_i > w_i' \geq P^*$ Here again, i wins and pays P^* which remains the same. Therefore, i's utility remains unchanged since $u_i' = w_i - P^* = u_i$.

Case 3: $w_i' < P^*$ In this case, player i would lose the auction as there exists a player $j \neq i$ whose $w_j = P^*$ by the definition of P^*. Therefore, the second highest bidder would win the auction. And hence, i's utility would be $u_i' = 0 \leq u_i$.

Thus, player i has no incentive to manipulate his bid if he is the person with the highest bid.

Case II : Player i is not the person with the highest bid (i.e) i is not the winner. $\Rightarrow u_i = 0$. Say, Player j with w_j is the winner. This means $w_i \leq w_j$.

Case 1: $w_i' > w_j$ This would mean that player i would now have the highest bid and he would win with $P^* = w_j$. Since $w_i < w_j$, utility of player i would now be $u_i' = w_i - w_j \leq 0$. Therefore, $u_i' \leq u_i = 0$.

Case 2: $w_j > w_i' > w_i$ Player j would still remain the highest bidder. Therefore, there is no change in the utility of player i. Hence, $u_i = u_i' = 0$.

Case 3: $w_i > w_i'$ Similar to Case 2 above, player j would still remain the highest bidder. And for player i, $u_i' = u_i = 0$.

Thus, player i has no incentive to manipulate his bid if he is not the person with the highest bid.

Therefore, in *Vickrey's Second Price Auction*, $u_i \geq u_i'$.

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Note 1 *Here, we assume that there is no externality (i.e) the effect of a player's action on others does not benefit the player. We are concerned with truthfulness of the parties involved because we need true information to optimise our solution. There are scenarios where we consider game theory with malicious players where the mechanism should be designed such that the honest players should get their fair share.*

Exercise Think of different scenarios and check if Vickrey's Second Price Auction would work or not.

Definition 2 *A (direct revelation) auction is a social choice function* $f : V_1 \times \cdots \times V_n \longrightarrow A$ *and a vector of payment functions* $p_1, \cdots, p_n$ *where* $p_i : V_1 \times \cdots \times V_n \longrightarrow R$ *is the amount that player* i *pays.*

Definition 3 *A mechanism* $(f, p_1, p_2, \cdots, p_n)$ *is called incentive compatible (or strategy-proof or truthful) if for every player* i*, every* $v_1 \in V_1, \cdots, v_n \in V_n$ *and every* $v'_i \in V_i$*, if we denote*

$$a = f(v_i, v_{-i}) \quad \text{and} \quad a' = f(v'_i, v_{-i}), \text{then}$$

$$v_i(a) - p_i(v_i, v_{-i}) \geq v_i(a') - p_i(v'_i, v_{-i})$$

Here, v_{-i} *denotes the* $(n-1)$*-dimensional vector where the* i*'th co-ordinate is removed.*

Intuitively, this means that player i prefers telling the truth v_i of his true valuation rather than any "lie" v'_i since this gives him a higher(in the weak sense) utility.

4 Vickrey-Clarke-Groves Mechanism

Definition 4 *A mechanism* $(f, p_1, \cdots, p_n)$ *is called a Vickrey-Clarke-Groves (VCG) mechanism if*

1. $$f(v_1, \cdots, v_n) \in \text{argmax} \sum_i v_i(a)$$

   (i.e) f *maximizes the social welfare.*

2. *for some functions* $h_1, \cdots, h_n$*, where* $h_i : v_{-i} \longrightarrow R$*, ((i.e)* h_i *does not depend on* v_i*) we have that*

$$p_i(v_1, v_2, \cdots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \cdots, v_n)) \tag{1}$$

Note 2 *The social welfare of an alternative* $a \in A$ *is the sum of the valuations of all the players for this alternative (i.e)* $\text{sum}_i v_i(a)$*. This is known as* efficiency*. Also, note that in the VCG definition, the first equation maximizes the efficiency and the second equation says that each player is paid an amount equal to the sum of the values of all the other players (i.e) it is similar to social welfare where you remove* i*'s contribution.*

Theorem 1 *Every VCG mechanism is incentive-compatible.*

Proof: Let the player i have real valuation v_i. We need to prove that his utility does not increase when he uses a different valuation v'_i. Denote $a = f(v_i, v_{-i})$ and $a' = f(v'_i, v_{-i})$. Now,

$$u_i(a) = v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i})$$

where $u_i(a) = v_i(a) - P$.
Similarly,

$$u_i(a') = v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i})$$

Now, since f maximizes the social welfare, $a = f(v_i, v_{-i})$ is maximized over all alternatives.

$$\implies v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a')$$

Subtracting $h_i(v_{-i})$ from both sides,

$$u_i(a) = v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i}) \geq v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) = u_i(a')$$

$$\text{Therefore} \quad u_i(a) \geq u_i(a')$$

Hence, Every VCG mechanism is incentive-compatible.

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Note 3 *Here, the payment of each player is independent of his valuation. This is the typical way to make a mechanism incentive-compatible.*

Example 1 *In the auction of a single item, finding a player with highest value is exactly equivalent to maximizing $\sum_i v_i(a)$ since only a single player gets non-zero value.*

4.1 Clarke-Pivot Payment

Let us consider the choice of the $h_i(v_{-i})$ function. Substituting, $h_i = 0$, the mechanism is simple but this implies that the mechanism pays a good amount of money to the players. Ideally, we would want that the players should pay money to the mechanism that is not more than the profit they make.

Definition 5 *The choice $h_i(v_{-i}) = max_{b \in A} \sum_{j \neq i} v_j(b)$ is called the Clarke Pivot Payment. Under this rule, the payment of player i is $p_i(v_1, \cdots, v_n) = max_b \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$ where $a = f(v_1, \cdots, v_n)$.*

Intuitively, i pays an amount equal to the total damage that he causes the other players - the difference between the total welfare of the others with and without i's participation. The main property of *Clarke Pivot Payment* is that $P_i(V) \geq 0$.

Lemma 1 *A VCG with Clarke Pivot Payments makes no positive transfers (i.e.) that no player is ever paid.*

Proof:

$$P_i(V) = \max_b \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$$

Case 1 : i has the maximum valuation. Now, $\sum_{j \neq i} v_j(a) = 0$. Therefore, $P_i = \text{second maximum price}$.

Case 2 : i does not have the maximum valuation Here, $P_i(v) = \text{highest price} - \text{highest price} = 0$.

Thus, we can see that, $P_i(V) \geq 0$ always. ∎

Note 4 *Clarke Pivot rule does not fit many situations where valuations are negative but it is a fairly general rule for payment.*

Example 2 *If we had two items to sell, considering the following three cases,*

1. Case 1 *:* i *has the maximum value* $\longrightarrow P_i(V) = (\text{second} + \text{third}) - \text{second} = \text{third price}$.

2. Case 2 *:* i *has the second maximum value* $\longrightarrow P_i(V) = (\text{first} + \text{third}) - \text{first} = \text{third price}$.

3. Case 3 *:* i *has neither the maximum nor the second maximum value* $\longrightarrow P_i(V) = 0$.

In general, if we have k *items to sell, we will sell all the items to the first* k *bidders at the* $(k+1)$*th maximum price.*

Exercise If the maximum bidder, gets to pay the second price and the second maximum bidder gets to pay the third price and so on, the mechanism will not be incentive compatible. Prove this.

5 Combinatorial Auction

In combinatorial auctions, a large number of items are auctioned concurrently and bidders are allowed to express preferences for a bundle of items. This is preferable to selling each items separately when there are dependencies between the different items.

Note 5 *If we consider all the cases in a combinatorial auction, we will have combinatorial explosion.*

Definition 6 *A valuation* V *is a real valued function that for each subset* S *of items,* $V(S)$ *is the value that bidder* i *obtains if he receives this bundle of items. A valuation must have free "disposal" (i.e.) be monotone : for* $S \subseteq T$*, we have that* $V(S) \leq V(T)$ *and it should be normalised* $V(\phi) = 0$.

1. We usually have sub-additivity as well (i.e) $V(S \cup T) \leq V(S) + V(T)$.(This is also called the economies of scale). (i.e) if you buy items together, it costs you less money than buying each item individually.

2. We also assume that there are no externalities (i.e) a bidder only cares about the item that he receives and not about how the other items are allocated among the other bidders.

Definition 7 *An allocation of the items among the bidders is* $S_1, \cdots, S_n$ *where* $S_i \cap S_j = \emptyset$ *for every* $i \neq j$. *The social welfare obtained by an allocation is* $\sum_i v_i(S_i)$. *A socially efficient allocation among many bidders with valuations* $(v_1, \cdots, v_n)$ *is an allocation with maximum social welfare among all allocations.*

Note 6 *If we use VCG (with Clarke Pivot payments), then these payments essentially charge each bidder his "externality". So this is incentive compatible in that scenario. But there are some issues which would be discussed in detail in the next lecture.*

Example 3 *Consider a bi-partite graph where the items are on one side of the graph and the bidders on the other. We can represent which item each bidder wants with an edge going from the bidder to the item.*

When this maximizes the social function, the problem is reduced to maximum weighted matching(which is polynomial in n*). The issue with this example is that for combinatorial auction, we cannot solve this always.*

6 Homework Problem

The problem starts with a story. At MIT, there were some thieves who stole a student's knapsack when she was not around. The student on finding it missing placed ads everywhere that say "Knapsack contains half-written thesis. Please return that alone. Will pay 200$. Will not involve police. Meet at 6 PM next week at Lincoln Building." The story goes on to say that the thief returned the thesis but was caught by the police. so, the question is formulated as follows :

How to define a mechanism such that this thief can return the student's thesis and get money from her and be able to avoid the police as well?

One can make rational assumptions to solve the problem. Use some computational complexity ideas. One could consider situations where each page returned gains some money for the thief and look into scenarios where the thief stops appearing (or the student stops paying) after a certain number of pages.

References

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