CMSC 858F: Algorithmic Game Theory
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Complexity of finding Nash equilibrium: PPAD completeness results

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1 Overview

Nash's theorem guarantees the presence of mixed nash equilibrium in every finite game. We are concerned here with the problem of *finding* such an equilibrium *efficiently*. In other words, does there exist an algorithm to find the mixed nash equilibrium in polynomial time? The answer to this question is not known but it is known finding mixed Nash equilibrium is PPAD complete, which implies some sort of hardness.

2 Introduction

We start by recalling the Nash's theorem:

Theorem 1 *Every finite game has a mixed Nash equilibrium.*

Nash proved the existence of a mixed equilibrium but the computational complexity of finding a mixed equilibrium, which is of obvious algorithmic importance, is unknown. To be more precise, is the problem of finding a mixed Nash equilibrium in **P** ? The answer to this question is unknown. We also do not know whether the problem is NP complete but it has been recently proven that the problem is PPAD complete [1].

3 Two similar problems

Another interesting problem which is well known for its non-constructive nature and which is PPAD complete is the Brouwer's fixed point problem.

Definition 1 Brouwer's Fixed Point Problem.
Given a continuous function $f : \mathbb{B}_n \rightarrow \mathbb{B}_n$, where $\mathbb{B}_n$ is a n-dimensional unit ball, there exists a fixed point in $\mathbb{B}_n$, i.e., a point x such that $f(x) = x$.

The theorem clearly is an existential one and similar to the situation of Nash's equilibrium, finding the fixed point is *hard* in some way - it is PPAD complete.

We first prove the following lemma.

Lemma 1 Sperner's Lemma
A triangle Δ and its triangulation are given. Each vertex of the Δ is given a distinct color, say $\{0, 1, 2\}$. We color rest of the vertices under the following restriction - If a vertex is located on an edge of Δ, then it should be colored using the colors of one of the two end points of the edge. Under this restriction, given any arbitrary coloring of other vertices, there always exists a tri-chromatic (colored using 3 distinct colors) atomic (without any smaller triangles inside it) triangle.

Proof: For each inner triangle formed by the triangulation, we add a vertex interior to that triangle. We add a special vertex a outside the triangle Δ. Now we construct a graph on these vertices as follows - We start from a and draw an arc joining a to the internal vertex (say v_1) of some inner triangle (say Δ_1) with an edge (along an edge of Δ) colored $0, 1$. The arc is drawn in such a way that it cuts the edge colored $0, 1$. Now if triangle Δ_1 has another edge colored $0, 1$, we draw an arc from v_1 to v_2 cutting this edge, where v_2 is the internal vertex of some triangle Δ_2. We continue this process as long as possible. Observing that we never enter a triangle twice and given that there are finite number of inner triangles, our process will come to a halt in finite time. Now noting that every graph has even number of vertices of odd degree and that a is one such vertex, we deduce that there is a internal vertex (v_c) correspoding to some triangle Δ_c with odd degree. But the maximum degree of any vertex is 2. Hence the degree of v_c must be 1. The only possibility of this happening is when Δ_c is trichromatic. Hence there exists a trichromatic triangle. ■

Exercise. An interesting exercise would be to prove the Brouwer's theorem for the one dimensional case.

4 Proof of Brouwer's theorem in 2-D.

We prove the Brouwer's theorem when the domain of the continuous function is a triangular region in euclidean plane which is homotopic to the disk(the 2-dimensional case of the Brouwer's theorem). In other words, consider a continuous function $f : \Delta \rightarrow \Delta$, where Δ represents a triangular region. We prove that there exists a point $x \in \Delta$ such that $f(x) = x$.

By the convexity of a triangular region, every point $x \in \Delta$ can be written in the form

$$x = a_0x_0 + a_1x_1 + a_2x_2 \tag{1}$$

where $a_0 + a_1 + a_2 = 1, a_i \geq 0$ and x_i are the vertices of Δ. Now, we define three sets S_0, S_1, S_2 in the following way - Given $a = (a_0, a_1, a_2)$ and $f(a) = (a'_0, a'_1, a'_2)$, if for some $i \in \{0, 1, 2\}$ $a'_i \leq a_i$ then $a \in S_i$. We observe that, if there is a point $a \in S_i$ $\forall i \in \{1, 2, 3\}$ then, clearly, $f(a) = a$, i.e., a is a fixed point. Our aim is to show that the three sets have a common point.
Given an arbitrary triangulation of T, we assign labels S_0, S_1, S_2 to the vertices to triangles of T. A vertex is labelled S_i only if it belong to S_i. We observe that every point $a = (a_0, a_1, a_2)$ with $f(a) = (a'_0, a'_1, a'_2)$ can be assigned some label. Indeed owing to the fact that $a_0 + a_1 + a_2 = 1 = a'_0 + a'_1 + a'_2$, it is not possible that $a'_i > a_i$ for every i. This implies $\exists j$ such that $a'_j \leq a_j$ and we can therefore label a with S_j. Therefore every point can be labelled with some $S_j, j \in \{0, 1, 2\}$. It is clear that we can label x_0 with S_0, x_1 with S_1 and x_2 with S_2. Thus the labels or "colors" of the vertices of Δ are distinct. Also, the points on the edge of Δ opposite to the vertex x_i must have the i^{th} co-ordinate 0. Since the i^{th} co-ordinate of such a point cannot decrease under f, we can choose some label other than i for those points (In other words, $a_i = 0 \implies \exists j \neq i \ni a'_j \leq a_j \implies$ the point belongs to S_j and hence can be labelled S_j). Hence, the resulting labelling is proper, i.e., it satisfies the requirements of the sperner lemma and we can use the lemma to find a smaller triangle which is colored distinctly at all its nodes. Repeating this process on the smaller triangle and continuing to do so, it can be proven that we will converge to a fixed point.

We observe that the graph constructed in the sperner's lemma has a "path like structure", i.e., every vertex has degree 1 or 2. We can assign directions to the edges of the graph in the following way - Starting from source, we assign directions in such a way that every vertex has an indegree at most 1 and out degree at most 1. The existence proof of the Nash equilibrium has the following abstract structure. A directed graph is defined over the vertices of the polytope where all strategies are easily recognizable and represented. Each one of these vertices has in-degree at most 1 and out degree at most 1. Hence the graph is a collection of paths and cycles. By necessity, there is one vertex with no in-coming edge and one out going edge - such a vertex is called the *standard source*. By the basic properties of directed graphs we conclude that there must be a vertex with out-degree 0. This *sink* vertex is our Nash equilibrium.

The above argument suggests a simple algorithm to find a solution - start from the source and follow the path until you find a sink. Unfortunately, this is not an efficient algorithm because the number of vertices in the graph could be exponentially large. We note that even in this case the following three problems are efficiently solvable -

- Is v a vertex of the graph.
- Is u a neighbor of v in the graph.

- Which vertex is the "predecessor" and which vertex is the "successor" of a given vertex.

Apart from NASH there are a host of problems which are PPAD complete like the sperner problem on an exponentially large set of vertices, finding Brouwer's fixed point etc. It is unknown whether PPAD belongs to P or not. Similar to the class NP, which has NP-complete class as a set of problems which are inter-reducible in polynomial time (i.e., if one of these problems is solved in polynomial time, so are the rest), the class PPAD has the class of PPAD complete class. Problems like NASH, Sperner, Brouwer, Arrow-Debreu equilibrium etc., are PPAD complete. PPAD completeness is weaker than the NP completeness because even if PPAD = P it is not clear that NP = P.

5 NASH is PPAD Complete.

Given a unit cube, we divide each dimension into integral multiples of 2^{-n}, for some $n \in \mathbb{N}$. This divides the cube into several *cublets*. We divide each cublet into 6 tetrahedra, this process is called *simplicization*. Given an arbitrary 4 coloring of the cube under the restriction that a face has no "color 1" vertex, an adjacent face has no "color 2" vertex, the face adjacent of both these faces has no "color 3" and the rest faces have no "color 4" vertex - it can be proven that there always exists a tetrahedron with all vertices colored distinctly. This is the Sperner lemma in 3-dimensions. Again, finding such a tetrahedron is PPAD complete.

As the first step in proving the hardness of NASH, we define the following problem and intuitively argue that it is PPAD complete (for rigourous arguement we refer to [1]). We then provide an intuition behind the reduction of this problem to the NASH. The problem is called 3D-Brouwer and is defined as following.

Brouwer. We are given a function ϕ defined from the 3 dimensional cube to itself. Each of the 2^{3n} cubelets defined above can take a value $\phi(x)$ at its center x as $x + \delta_i$, for some $i \in \{0, 1, 2, 3\}$, where -

- $\delta_0 = (-\alpha, -\alpha, -\alpha)$
- $\delta_1 = (\alpha, 0, 0)$
- $\delta_2 = (0, \alpha, 0)$
- $\delta_3 = (0, 0, \alpha)$

Here $\alpha \ll 2^{-n}$. The problem is to find an interior corner vertex of some cublet which has among its eight neighboring cublets, four cublets who centers have values $x + \delta_i$ - one for each of $i \in \{0, 1, 2, 3\}$.

We can prove that the 3D-Brouwer is PPAD complete by reducing it to the problem of 3D-Sperner (by associating color i with δ_i). Finally, the PPAD completeness of NASH with many but constant players can be shown to be PPAD complete from a reduction of 3D-Brouwer to NASH. We give the basic intuition behind this reduction - All the players have only two strategies, 0 or 1. The mixed strategy can thus be represented by a number in range $[0, 1]$ (if p is the mixed strategy, then the player chooses 1 with probability p). Three special points, called the *leaders*, coordinate a point in the cube. The remaining players respond by analyzing the co-ordinates of this point and by computing the displacements δ_i at the centers of the cublet and adjacent cublets. The resulting choices will incentivize the leaders to change their mixed strategy, unless the point is a fixed point of ϕ - in which case they will not change their strategy and we are at a mixed Nash equilibrium.

The PPAD completeness of NASH equilibrium was first proved for the case of 4 or more players. Later it was proven to be PPAD complete for the case of 3 players but it was conjectured that the problem was in P for 2 Players [2]. Proving this conjecture to be false, it was proven that the problem is PPAD complete even in the 2 player case [3]

References

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