Subcubic Equivalence between Graph Centrality Problems

All-Pair Shortest Path (APSP): compute $dist(x,y)$ for all pairs x,y (a very essential problem in graph algorithms)

Here assume, given a directed/undirected n-node, m-edge, graph $G=(V,E)$ with (absolute) edge weights $\{0,1,\dots,M\}$ (i.e., complete graphs)

Two Centrality Measures:

1) Radius R of a graph is $\min_v \max_u dist(v,u)$ 2) Median M of a graph is $\min_v \sum_u dist(v,u)$

Center of a graph is $\arg\min_v \max_u dist(u,v)$

Also Diameter D of a graph is $\max_{u,v} dist(u,v)$

Above measures have many applications in social, transportation, computer, biological networks.

Floyd-Warshall Algorithm for APSP: $O(n^3)$

n iterations of Dijkstra's: $O(mn+n^2\log n) = O(n^3)$ in the worst case

better than $O(n^3)$? Fredman obtained $\frac{n^3\sqrt[3]{\log\log n}}{\sqrt[3]{\log n}}$ in year 1976. After several subsequent work Williams obtained $\frac{n^3}{2^{\Omega(\sqrt{\log n})}}$ in 2014.

Big Open problem: APSP is truly subcubic time, i.e., $n^{3-\varepsilon}, \varepsilon>0$?

APSP conjecture: APSP requires $n^{3-\varepsilon}$, for constant $\varepsilon>0$.

Note that after solving APSP, we only need $O(n^2)$ to go over all $dist(u,v)$ to compute Radius, Median, or diameter.

Main questions:

1) Can we compute these centrality measures without resorting to APSP?
   Answer: Not really [Abboud, Grandoni, Williams'14]
2) Can we compute them in truly subcubic time?
   Answer: If ~~and~~ only if APSP can be solved in truly subcubic time. [AGW'14] (see above)

Another definition: Betweenness Centrality:

$$BC(v) = \sum_{s,t\in V-\{v\},\, s\neq t} BC_{s,t}(v) \text{ where } BC_{s,t}(v) = \text{fraction of shortest } s\text{-}t \text{ paths using } v$$

If all shortest paths are unique:

$BC(v)$ = number of s-t pairs with a shortest path passing through v.

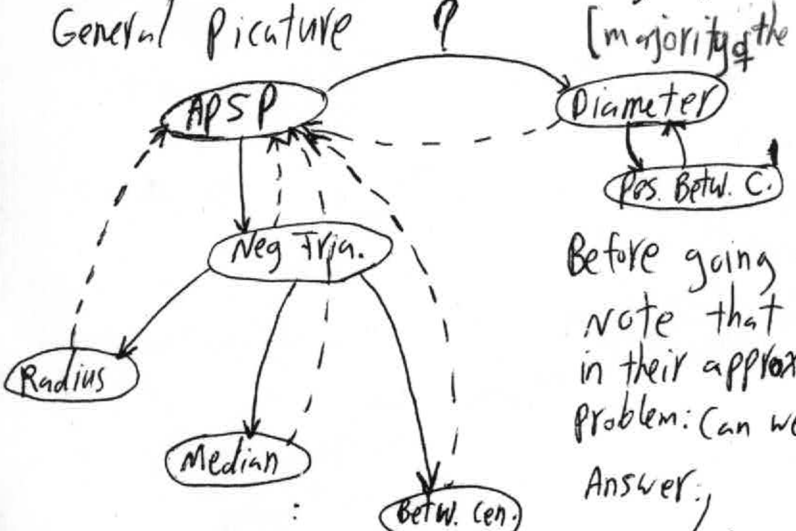
Subcubic reduction from A to B:

Assuming an algorithm of truly subcubic for B, we can obtain an algorithm of truly subcubic for A.

Main Thm: Radius, Median and Betweenness Centrality are subcubically equivalent (thus improving any of these problems is a major breakthrough.

Note that centrality measures only require computing a single number!

General Picature ? [majority of the results are from Abboud, Grandoni and Vassilevska SODA'15]



Before going through some proofs:

Note that all centrality measure are interesting in their approximate version.

Problem: Can we approximation them in truly subcubic time?

Answer:

1. Positive Betweenness Centrality Problem is to determine whether some shortest path uses a given v as an intermediate node

✓ We can compute $(1+\varepsilon)$-aprox APSP in time $\tilde{O}(n^{\omega}/\varepsilon)$, where ω < 2.3728639 is fast matrix multiplication exponent [Zwick-FOCS'98]

This gives truly subcubic PTASs for Radius and Median, but does not help for betweenness

Thm*: A $\frac{3}{2}$-approx of Diameter and Betweenness Centrality in $O(m^{2-\varepsilon})$ implies SETH is false

(where Strong Exponential Time Hypothesis (SETH) say SAT Problem on n-variables cannot be solved in time $O((2-\varepsilon)^n)$ for any constant $\varepsilon > 0$.) or equivalently $O(2^{\delta n})$ for $\delta < 1$ [Roditty, Vassilevska, STOC'13]

Open Problems

1: Is Diameter equivalent to APSP under subcubic reductions?

2: Can we get a similar result to Thm* for Radius and Median as well.

Now let's see some proofs:

First another definition: Negative triangle: given an undirected graph G with integer edge weight $\{-M, \dots, M\}$, the Negative Triangle, problem is to determine whether G contains a triangle of Negative weight (on w(e) and not dist)

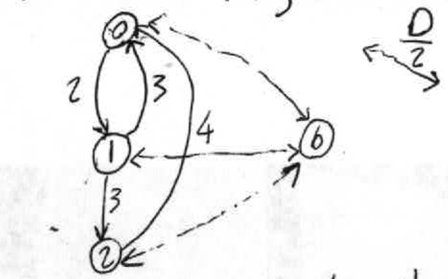
let's start with easier reductions:

Lm: Given a $T(n,m)$-time algorithm for Positive Betweenness Centrality, there is $\tilde{O}(T(n,m))$-time algorithm for Diameter ($\tilde{O}$ means O upto polylog factors)

Pf: By multiplaying all distances by 2, we assume Diameter and distances are even. Now add a new vertex b and put all distances to b, $\frac{D}{2}$.

Now diameter is the largest value of D such that $BC(b)>0$.

example



Lm: Given a $T(n,m,M)$-time algorithm for Diameter, there is an $\tilde{O}(T(n,m,M))$-time algorithm for Positive Betweenness Centrality.

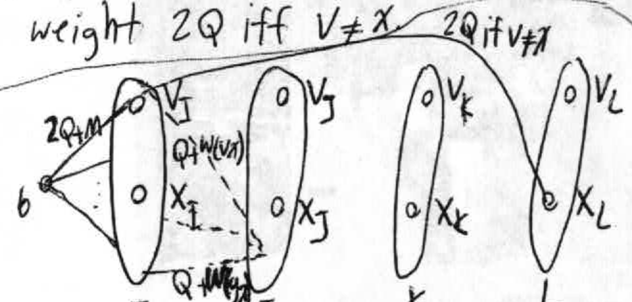
Pf: for each original vertex v, add two extra vertices v_A and v_C with $w(v_A,v) = D'-dist(v,b)$ and $w(v,v_C) = D'-dist(b,v)$, where the edges here are directed and D' is chosen larger than $2diam(G)$, e.g. $D'=2diam(G)$. If the distance is ∞ we put D' in this reduction. Now it is easy to see $D = dist(s_A,t_C) = 2D' - dist(s,b) - dist(b,t) + dist_G(s,t) \leq 2D'$ and equality holds if path sbt is a shortest path from s to t going through b, i.e., $BC(b)>0$. Note that diameter of the new graph will be at most five times ($4diam+1diam$) before.

Put $w(v,v_A)=0$, $w(v_C,v)=0$

Note that all dist (v,b), dist(v,b) can be computed in $O(n^2)$ by Dijkstra

→ Now let's prove the main thm for Radius

First note that Vassilevska and Williams'13 proved that APSP and Neg Triangle are equivalent under subcubic reductions. Also it is folklore that there is a subcubic reduction from Radius to APSP. Thus here we give a subcubic Reduction from Neg Triangle to Radius. So we are given a Neg Triangle instance G with $w(e)$ in $\{-M,\dots,M\}$. We creat a graph H on layers I, J, K, L. Each $v \in V$ has copies v_I, v_J, v_K, v_L. For edge $\{v,x\}$ of G, we add $\{v_I x_J, v_J x_K, v_K x_L\}$ of weight $Q+w(vx)$ where $Q=3M$. We add a node b of distance $2Q+M$ to all nodes of I. Add (v_I, x_L) of weight $2Q$ iff $v \neq x$. For any other edge, put weight $\infty = 6Q$, for example



claim: The radius of the new graph H is $<9M$ iff G contains a negative triangle. H has $O(n)$ nodes and edge weight $O(M)$

Proof:

1) Now, if $R<9M$, Center is in I as all other nodes are at dist $3Q=9M$ from b

2) v_I is at distance at most most $2Q+2M=8M$ from all nodes except v_L

3) If v is in a negative triangle distance $(v_I,v_L) < 3Q = 9M$ → for triangle $v_I x_J y_K v_L$

4) If v is not in a neg triangle, $dist(v_I,v_L)$ is $\min(3Q, 4Q-2M) = 9M$ → $2Q$ for back & forth between K & L → for two negative edges