(1)

* Why do we study game theory? Algorithmic Game Theory: incentive-aware algorithms
  - we have selfish agents or self-interested agents, which optimize their own objective functions.
  - Goal of Mechanism Design: encourage selfish agents to act socially by designing rewarding rules such that when agents optimize their own objective, a social objective is met.

Note: Game Theory is NOT a tool. It is a concept.

* How do we study these systems?
  - First model the system (usually a network) as a game
  - We analyze equilibrium points and then compare the social value of equi. points to global optimum
* An equilibrium point or just equil. is a state in which no person involved in the game wants any change. More precisely an equib. is simply a state of the world where economic forces are balanced and in the absence of external influences the (equib) values of economic variables will not change.

- Two Important class of Equilibrium: Nash equib and Market equib. We introduce both of them in this session (or maybe the next) and give some important results for them. (first Nash and then Market)

change the order

- Important Factors:
  - Existence of equib as a subject of study ← economics
  - Performance of the output (Approximation Factor) ← both CS & economics
  - Convergence (Running time) ← Computer Science
    - OR do we converge at all?

Some important game theory concepts (some of them will be discussed in this course)

- Lack of coordination in networks and equilibrium concepts
- Price of Anarchy (or stability) in Games such as load balancing game, Selfish Routing games, Congestion games, Market sharing games, network creation games, Network formation games, etc.
- Coordination Mechanisms to obtain better price of Anarchy
- Convergence and best response dynamics and their outcomes e.g. in sink equilibrium
- Market Equib. and applications e.g. in wireless networks
- Interdomain routing and stable paths problems, Gao-Rexford Conditions
- Auctions, VCG, truthfulness, Sponsored Search Auctions, Online Auction
- Cost sharing, privacy and complexity (hardness)

Nash equib. is a solution concept (a condition which identifies the equib) of a game involving two or more players, in which no player has anything to gain by changing only his or her own strategy unilaterally. In other words, if each player has chosen a strategy and no player can benefit by changing his or her strategy while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs (profits, rewards) constitute a Nash Equib. It is named after John Nash. In this case, each player either knows the strategies of othere players or can derive these.

lets first start with a two-player game (to be more formal)

some examples first:

Example ①: Prisoners' dilemma: two prisoners are on a trial for a crime and each one faces a choice of confessing to the crime or remaining silent. If they both remain silent, the authorities will not be able to prove charges against them and they will both serve a short prison term, say 2 years, for minor offenses. If only one confesses, he will get a reduced 1 year prison and he will be used as a witness against the other who will get 5 years. Finally if they both confesses, they both will get a small break of cooperating with the authorities and will have to serve prison sentences of 4 years each (rather than 5). we can succinctly summarize the costs incurred in these four outcomes via the following two-by-two matrix, which is called a cost matrix because it contains the cost incurred by the players for each choice of their strategies.

P_1 \ P_2	confess	silent
confess	4, 4	1, 5
silent	5, 1	2, 2

[Expressing games in this form is the standard form or the matrix form is good for small # player and # strategies however not good for big games thus we use the implicit form]

note → (confess, confess) is the only stable solution in each of the other three cases, at least one of the players can switch from silent to confess and improve his own payoff.

on the other hand the social optimum choice is the (silent, silent) case, which is not stable

Prisoners' Dilemma arise naturally in a lot of different situations with many players (see ISP games in chap 1 of the book of Nisan et al.)

Example ② we might have multiple outcomes which are stable

Battle of sexes: consider two players, a boy and a girl, are deciding on how to spend their evening. They both consider two possibilities: going to a baseball game or going to a softball game: the boy prefers baseball and the girl prefers softball, but they both would like to spend the evening together rather than

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separately. The payoffs (benefits) are as follows.

Girl \ Boy	B	S
B	5, 6	1, 1
S	2, 2	6, 5

Here both attending the same game, whether it is softball or baseball, are both stable solutions.

This Battle of the Sexes is an example of coordination games arise naturally in many contexts such as the context of routing to avoid congestion (see the book)

Example ③: not all games has stable outcomes in the sense that none of players would want to individually deviate from such an outcome.

Matching Pennies Game: Two players, each having a penny, are asked to choose from among two strategies - heads (H) and tails (T). The row player (Player 1) wins if the two pennies match while the column player (player 2) wins if they do not match

1 \ 2	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

1: win
-1: loss

It is easy to see that this game has no stable solution and the best for the players to randomize (with prob $\frac{1}{2}$) between strategies (Randomized (mixed) strategies)

All examples ①, ②, and ③ are one-shot simultaneous move games, in that all players simultaneously chose an action from their set of possible strategies. **We might have repeated Games.**

☆ Formally, we have n player $\{1, 2, \ldots, n\}$. Each player i has his own set of possible strategies, say S_i. To play the game, each player i selects a strategy $s_i \in S_i$. We will use $s = (s_1, \ldots, s_n)$ to denote the vector of strategies selected by the players and $S = \times_i S_i$ denote the set of all strategies, each has a payoff for each player, i.e. $\underline{u_i(s)}$ where $s \in S$ is a vector.

☆ We say a game has a <u>dominant strategy solution</u> if each player has a unique best strategy, independent of the strategies played by the other players. <u>More formally</u>, for a strategy vector $s \in S$ we use s_i to denote the strategy played by player i and s_{-i} to denote the $(n-1)$-dimensional vector of the strategies played by all other player

— We say a strategy vector $s \in S$ is a <u>dominant strategy solution</u> if for each player i, and each alternate strategy vector $s' \in S$, we have that $u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i})$. For example, Prisoners Dilemma has a solution that always confess, but note that a dominant strategy solution may not give an optimal payoff to any of the players.

Since games rarely possess dominant strategy solutions, we need to seek a less stringent and more widely applicable solution concept. (see e.g. the Battle of Sexes game)

Nash equib. captures the notion of a stable solution:

A strategy vector $s \in S$, is said to be a Nash equib. if for all players i and each alternate strategy $s_i' \in S_i$, we have that

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

Note that a solution is self-enforcing in the sense that once the players are playing such a solution, it is in every player's best interest to stick to his or her strategy.

Also clearly, a dominant strategy solution is a Nash Equib.

we may have several Nash equib. e.g. in the battle of sexes game.

★ mixed strategy Nash Equib.:

So far every thing was deterministic and thus called pure strategy equib

but matching Pennies game did not possess any pure Nash equib.

If each player picks each of his two strategies with prob. $\frac{1}{2}$ then we obtain a stable solution in a sense, since the expected payoff of each player now is 0 and neither player can improve on this by choosing a different randomization.

To define randomized strategies formally, let us enhance the choices of players so each one can pick a probability distribution over his set of possible strategies, such a choice is called a mixed strategy

we assume players independently select strategies using the probability distribution which leads to a probability distribution of strategy vectors S.

Thm (Nash 1951): Any game with a finite set of players and finite set of strategies has a Nash equib of mixed strategies.

If we do not have finite sets, there is not a mixed Nash necessarily (see the pricing game in the book chap).

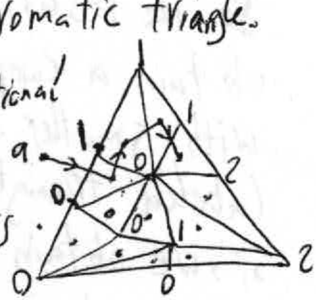
The price of anarchy (PoA) is the most popular measure of the inefficiency of equib. is defined as the ratio between the worst objective function value of an equi of the game and that of an optimal outcome (social optimum).

We are interested in a price of anarchy which is close to 1, i.e., all equib. are good approximations of an optimal solution.

complexity of Finding Nash Equil., PPAD-Completeness

Nash's Theorem (1951): Every finite game has a mixed Nash equilibrium.
Can we find it efficienly (in polynomial-time) or It is NP-complete?
(still we dont know, we only know it is PPAD-complete).

Proof of Nash's theorem is based on Brouwer's fixedpoint theorem. Stating that every continuous function f from the n-dimensional unit ball to itself has a fixed point, a point x such that $f(x)=x$.
Nash's proof is a clever reduction of the existence of a mixed equilibrium to the existence of such a fixed point.
Brouwer's theorem is well-known for its non-constractive nature and finding it was known to be hard problem (1989), but recent proofs shows Nash is precisely (PPAD-complete) as hard as fixed Brouwer's fixed point.
Sperner's Lemma: A triangle and its triangulation is given. Each of the vertices of the big triangle has a unique color and each vertex on an edge of the big triangle can only have the color of one of the two colors of the end points of its edge. In any such scenario there exists a trichromatic triangle.



Pf: First there is an edge of colors 0-1 on the side 01 (two-dimentional Sperner). Add a vertex a out side and one vertex inside each small triangle. start from a and always only go across edges that are colored 0-1. since the triangles that we start and we see have no color 2, then they should be 0,1,1 or 0,0,1. so if enter with 0-1, there is always another edge 0,1 that we exit. But since we see each trajngle once and their number is limited (why?) we should see a triangle of degree one (which is monocromatic). To avoid unlimited case and be more formal, In the graph above connect every two triangles (indeed vertices corresponding to triangles) with a common edge 0-1. vertex a has odd degree and any other vertex has even degree if the is no monocromatic triangle which is impossible, since # of velties of odd degree in any undirected graph is even. ⊠

Application of Sperner's Lemma to the proof of two dimensional Brouwer fixed-point ②
theorem: Brouwer's theorem in 2D can be interpreted (by homotopy) as saying that a continuous mapping from a triangular region T to itself must have a fixed point. Suppose the corners of T are points (vectors) x_0, x_1, x_2. Due to convexity we can express each $x \in T$ uniquely as a weighted average of the corners: $x = a_0 x_0 + a_1 x_1 + \dots + a_i x_i$ where $\sum a_i = 1$ and each $a_i \geq 0$. So we can specify x by its vector of coefficients $a = (a_0, a_1, a_2)$
Now define sets S_0, S_1, S_2 for each mapping f by saying that $a \in S_i$ if $a_i' \leq a_i$ where $f(a) = a'$. Because the coefficients of each point sum to one, every point in T belongs to at least one of the sets, and a point belongs to all three sets if and only if it is a fixed point for f. We want to show that the three sets have a common point.
Given an arbitrary simplicial subdivision (triangulation) of T, for each node a choose a label i such that $a \in S_i$. Note that the points on the edge of T opposite x_i have ith coordinate 0. Since their ith coordinate cannot decrease under f, we can choose some label different from i for each point on that edge. Note that $x_i \in S_i$ since coefficient of x_i is one (max) when we write x_i in terms of x_1, x_2, x_3.
So the resulting labeling is proper and we can apply Sperner's lemma to obtain a completely labeled cell. We can repeat the process using triangulation with smaller and smaller cells, we obtain a sequence of smaller and smaller completely labeled triangles; call them $\{x_i, y_i, z_i\}$ receiving labels 0, 1, 2, respectively. In each S_i, we obtain an infinite sequence of points. The rest is topology to say since f is continuous $\{x_i\}, \{y_i\}, \{z_i\}$ converges to x_{ik}, y_{ik}, z_{ik} and since the distances between them approaches zero (since triangles are inside each others), x_{ik}, y_{ik}, z_{ik} converges to the same point which thus belongs to all S_0, S_1, S_2 and we are done ☒
Thus Sperner's Lemma $\Longrightarrow$ Fixed point Brouwer's **Theorem** $\Rightarrow$ Nash's Theorem.
Note that in the proof of Sperner's lemma, the graph has a very simple "path-like" structure: All vertices (except the source) have either one or two edges incident upon them. The important point is that there is definitely at least one known endpoints of the path (i.e., the source vertex of odd-degree in the outside. We must conclude that there is another end point of degree one, i.e. the **triangle** with all three colors.

We can make the graph directed such that starting from the source vertex we can assign a direction to its incident edges, at most one coming in and at most one going out and do it such that it is consistent from one vertex to another. Indeed the existence proof of Nash's theorem (for two player games though something similar holds for the the general case as well) has the following abstract structure. A directed graph is defined on vertices of the polytope where all strategies are easily recognizable and represented). Each one of these vertices has indegree and outdegree at most one; therefore the graph is a set of paths and cycles (even simpler that Sperner's lemma). By necessity there is one vertex with no incoming edges and one outgoing edge, called a standard source (in the polytope of 2-player Nash, the all-zero vertex). We must conclude that there must be a sink; a nash equilibrium. Any such proof suggest a simple algorithm for finding a solution: start from the standard source, and follow the path until you find a sink (in case of 2-player Nash is called the Lemke-Howson algorithm).
Unfortunately this is not an efficient algorithm because the number of vertices in the graph is exponentially large (and indeed this happens in 2-player Nash). Indeed besides Nash, there is a host of of other computational problems such as Sperner's with exponentially large set of vertices or finding an approximate Brouwer's fixed point in which whose solution space can be set up as the set of all sinks (and all non standard) sources in a directed graph defined on a finite but exponentially large set of vertices with each vertex has indegree and out degree at most one in which 1) telling a vertex 2) its neighbors and 3) telling which one is the predecessor and successor (direction of each edge are computationally easy and we are given an standard source and we want to find a sink or any non-standard source.
All these problems comprise the complexity class called PPAD. Solving a PPAD problem is to telescope the long path and arrive at a sink fast without exhaustive search. We do not know PPAD belongs to P or not

IN case of NP, we have a useful notion of difficulty - NP-completeness. similarly we have PPAD-completeness, meaning all problems in PPAD reduces can be reduced this problem and if we can solve one of these PPAD-complete problems efficienty we can solve all. Nash, Brouwer, Sperner, finding Arrow-Debreue equilibria and many more are PPAD-complete. However PPAD completeness is weaker evidence of intractability that NP-completeness. an thus it could be easily that $PPAD = P \neq NP$.

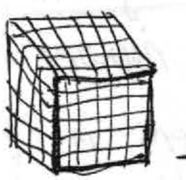
There ~~sure~~ somehow compelling evidence that $PPAD \neq P$: otherwise some non-trivial results happen but then so what? :)

$\boxed{\text{The rest we discuss PPAD-completness of Nash}}$

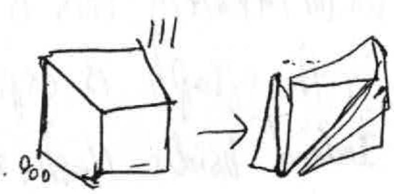
Note that the proof of sperner is PPAD-complete is relatively easy because the problem is essentially the same as the directed graph problem.

We can generalize sperners to 3 and more dimensions.

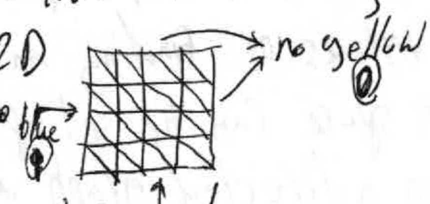
in 3-dimensions



- first we divide each dimension into integer multiples of 2^{-m} for some integer m, each called cublets.
- then simplicization of each cubelet into six tetrahedra all having corner 000, 111.



legal coloring in 2D



in 3D

Sperner's lemma: for a legal coloring (no vertex of the same color as the face) there is an all chromatic simplex.

so it can be proved that 3D-sperner is PPAD complete (reduction to large directed graph).

Next we define the Brouwer fixed point for 3D:

We convert coloring of the cube to the direction of the displacement vector. $f(x) - x$.

color 0 (3-face color) $\longrightarrow \delta_0 = (-1,-1,-1)\, 2^{-m}$

color 1 $\to \delta_1 = (1,0,0)\, 2^{-m}$ color 2 $\to \delta_2 = (0,1,0)\, 2^{-m}$ color 3 $\to \delta_3 = (0,0,1)\, 2^{-m}$.

More precisely, we consider a discrete version of the Brouwer fixpoint theorem. It is presented in terms of a function ϕ from the three-dimensional unit cube to itself. The cube is subdivided into 2^m equal cublets and the function needed only be described at all cubelet centers. At a cubelet center x, $\phi(x)$ can take four values $x+\delta_i$, $i=0,\dots,3$, where δ_i are defined before. We want to find a fixed point which is defined here to be any internal cublet corner point such that among its eight adjacent cublets, all four possible displacement $\delta_{i=0,\dots,3}$ are present.

Essentially the proof of PPAD-completeness for this discrete Brouwer is the proof of PPAD for 3D Sperner (colors go to displacement).

Finally we can reduce Brouwer for Nash with many but constant # players. All these player have just two strategies 0 and 1; therefore we can think of any mixed strategy of a player as a number in [0, 1]. There are three players called leaders who coordinate a point in the cube. The others will respond by analyzing these coordinates to identify the cubelet wherein this point lies and by computing (by a simulation of a boolean circuit) the displacements δ_i at the cubelet and adjacent cublets. The resulting choices by the player will incentivize the 3 leaders to change their mixed strategy - unless the point is a fixed point of ϕ, in which case the three players will not change strategies and we are at a Nash equilibrium.

First these could be simulated by many players, then 4 players, then 3-player and conjectured 2-player is indeed in P. Finally the conjecture was wrong and even 2-player Nash is PPAD-complete.