CMSC 858F: Algorithmic Lower Bounds
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Introduction to Algorithmic Game Theory

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1 Overview

In this lecture, we delve into some basics about Game Theory and Algorithmic Game Theory. In particular, we first introduce some basic game theory definitions and then explain Nash's theorem, Brouwer's fixed point problem and Spenser's lemma which are all in a complexity class PPAD. Finally, we provide a proof for Brouwer's problem using Spenser's lemma and give an intuition for PPAD-completeness of fining Nash's equilibrium.

2 Algorithmic Game Theory

Game Theory is an attempt to study systems by modeling them as *games*. A game can be defined as a situation where a set of *agents* interact or affect each other's outcomes.

Algorithmic Game Theory is slightly newer than general Game Theory, and is primarily concerned with smart, selfish agents who are interested in maximizing their own utility. Algorithmic Game Theory is an attempt at making Game Theory more "algorithmic," by coordinating the agents with *Mechanism Design* to socialize so that they may generate something good for the society as a whole [1].

The goal of Mechanism Design is to design and impose rewarding rules to encourage selfish agents to change their strategy and behave socially. With Mechanism Design, we try to get an approximate optimal *solution*.

A solution is an outcome of a game. Typically, we are interested with stable solutions or *equilibria* (or equilibrium points).

Definition 1 *An equilibrium point or just equilibrium is a state in which no person involved in the game wants any change. More precisely, an equilibrium* *is simply a state of the world where economic forces are balanced and in the absence of external influences, the (equilibrium) values of economic variables will not change.[1]*

The existence of equilibrium is a subject of study in economics. The performance of output (or approximation factor) is studied in both computer science and economics. And convergence (running time) or non-convergence is a subject of study in computer science.

There are two important classes of equilibria: Market equilibrium and Nash equilibrium. The former is related to the games in which we have a set of sellers and a set of buyers and want to put price on goods such that everyone become happy at the end of the process. In this lecture we talk more about Nash equilibrium.

3 Nash Equilibrium

A *Nash equilibrium* is a solution concept (a condition which identifies the equilibrium) of a game involving two or more players in which no player has anything to gain by changing only his or her own strategy unilaterally. In fact, such a solution is self-enforcing in the sense that once the players are playing such a solution, it is in every player's best interest to stick to his strategy [1, 3].

Theorem 1 *Any game with a finite set of players and finite set of strategies has a Nash equilibrium of mixed(randomized) strategies.*

Nash proved the existence of a mixed equilibrium but the computational complexity of finding a mixed equilibrium, which is of obvious algorithmic importance, is unknown. To be more precise, is the problem of finding a mixed Nash equilibrium in P? The answer to this question is unknown. We also do not know whether the problem is NP-complete but it has been recently proven that the problem is PPAD-complete [2].

4 Prisoners' dilemma

Two prisoners are on trial for a crime and each face the choice of confessing to the crime or remaining silent. If they both remain silent, the authorities will not be able to charge them for this particular crime, and they will both face two years in prison for minor offenses. If one of them confesses, his term will be reduced to one year, but he will have to bear witness against the other, who will be sentenced to five years. If they both confess, they will both get a small break and be sentenced to four years in prison (rather than five).

We can summarize the four outcomes and the utility with the following *cost matrix*:

P1 \ P2	Confess	Silent
Confess	4 4	5 1
Silent	1 5	2 2

The only *stable solution* in this game is when both confess. In each of the other three outcomes, a prisoner can switch from being silent to confessing in order to improve his own payoff. The social optimum in this case is when both remain silent; however, this outcome is not stable. In this game, there is a unique optimal selfish strategy for each player, independent of what other players do. One way to specify a game in algorithmic game theory is to explicitly list all possible strategies and utilities of all players. Expressing the game in this form is called the *standard form* or matrix form, and it is convenient to represent two-player games with a few strategies in this form, as demonstrated for the Prisoner's dilemma game [3].

5 Price of Anarchy (PoA) and Price of Stability (PoS)

The "price of anarchy" is a popular measure of inefficiency of an equilibrium. It is defined as the ratio between the worst objective function value of an equilibrum of the game and that of an optimal outcome (social optimum). We are interested in a "price of anarchy" which is close to 1, i.e., all equilibriums are good approximations of an optimal solution. For example in Prisoner's dilemma we have $\mathsf{PoA} = \frac{4+4}{2+2} = 2$. A game with multiple equilibria has a large "price of anarchy" even if only one of its equilibria is highly inefficient. The "price of stability" of a game is the ratio between the best objective function value of one of its equilibrium and that of an optimal outcome. In games with a unique equilibrium, $\mathsf{PoA} = \mathsf{PoS}$ [1].

6 Two similar problems

Another interesting problem which is well known for its non-constructive nature and which is PPAD complete is the Brouwers fixed point problem.

Definition 2 *Given a continuous function* $f : Bn \to Bn$*, where* Bn *is an n-dimensional unit ball, there exists a fixed point in* Bn*, i.e., a point* x *such that* $f(x) = x$.

The theorem clearly is an existential one and similar to the situation of Nash's equilibrium, finding the fixed point is hard in some way - it is PPAD-complete.

We first prove the following lemma.

Lemma 1 *(Sperners Lemma) Given a triangle* Δ *and a triangulation of it. Each vertex of* Δ *is given a distinct color, say* $\{0, 1, 2\}$*. We color rest of the vertices under the following restriction - If a vertex is located on an edge of* Δ*, then it should be colored using the colors of one of the two end points of the edge. Under this restriction, given any arbitrary coloring of other vertices, there always exists a tri-chromatic (colored using* 3 *distinct colors) atomic (without any smaller triangles inside it) triangle.*

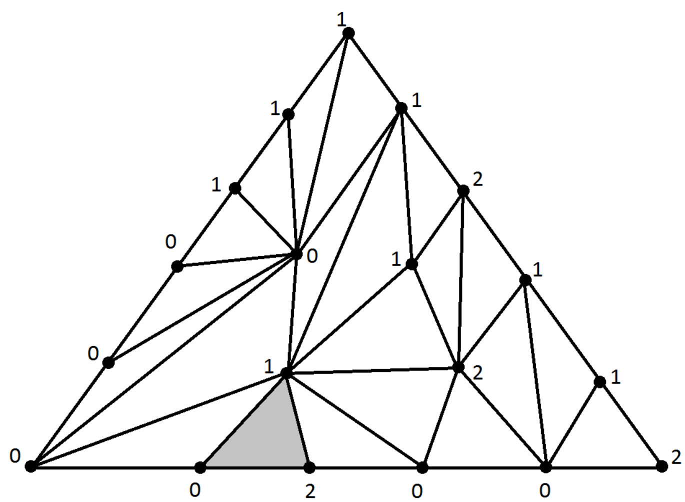

Figure 1: An arbitrary coloring of a triangle and existence of a trichromatic triangle.

Proof: For each inner triangle formed by the triangulation, we add a vertex interior to that triangle. We add a special vertex a outside the triangle Δ. Now we construct a graph on these vertices as follows - We start from a and draw an arc joining a to the internal vertex (say v_1) of some inner triangle (say Δ_1) with an edge (along an edge of Δ) colored $0, 1$. The arc is drawn in such a way that it cuts the edge colored $0, 1$. Now if triangle Δ_1 has another edge colored $0, 1$ we draw an arc from v_1 to v_2 cutting this edge, where v_2 is the internal vertex of some triangle Δ_2. We continue this process as long as possible. Observing that we never enter a triangle twice and given that there are finite number of inner triangles, our process will come to a halt in finite time. Now noting that every graph has even number of vertices of odd degree and that a is one such vertex, we deduce that there is a internal vertex (v_c) correspoding to some triangle Δ_c with odd degree. But the maximum degree of any vertex is 2. Hence the degree of v_c must be 1. The only possibility of this happening is when Δ_c is trichromatic. Hence there exists a trichromatic triangle. ■

7 Proof of Brouwer's theorem in 2-D

We prove the Brouwer's theorem when the domain of the continuous function is a triangular region in euclidean plane which is homotopic to the disk(the 2-dimensional case of the Brouwer's theorem). In other words, consider a continuous function $f : \Delta \to \Delta$, where Δ represents a triangular region. We prove that there exists a point $x \in \Delta$ such that $f(x) = x$.

By the convexity of a triangular region, every point $x \in \Delta$ can be written in the form:

$$x = a_0x_0 + a_1x_1 + a_2x_2$$

where $a_0 + a_1 + a_2 = 1$, $a_i \geq 0$ and x_i are the vertices of Δ. Now, we define three sets S_0, S_1, S_2 in the following way - Given $a = (a_0, a_1, a_2)$ and $f(a) = (a'_0, a'_1, a'_2)$, if for some $i \in \{0, 1, 2\}$ $a'_i \leq a_i$ then $a \in S_i$. We observe that, if there is a point $a \in S_i$ $\forall i \in \{1, 2, 3\}$ then, clearly, $f(a) = a$, i.e., a is a fixed point. Our aim is to show that the three sets have a common point.

Given an arbitrary triangulation of T, we assign labels S_0, S_1, S_2 to the vertices to triangles of T. A vertex is labeled S_i only if it belong to S_i. We observe that every point $a = (a_0, a_1, a_2)$ with $f(a) = (a'_0, a'_1, a'_2)$ can be assigned some label. Indeed owing to the fact that $a_0 + a_1 + a_2 = 1 = a'_0 + a'_1 + a'_2$, it is not possible that $a'_i > a_i$ for every i. This implies $\exists j$ such that $a'_j \leq a_j$ and we can therefore label a with S_j. Therefore every point can be labeled with some $S_j, j \in \{0, 1, 2\}$. It is clear that we can label x_0 with S_0, x_1 with S_1 and x_2 with S_2. Thus the labels or colors of the vertices of Δ are distinct. Also, the points on the edge of Δ opposite to the vertex x_i must have the i^{th} coordinate 0. Since the i^{th} coordinate of such a point cannot decrease under f, we can choose some label other than i for those points (In other words, $a_i = 0 \Rightarrow \exists j \neq i : a'_j \leq a_j \Rightarrow$ the point belongs to S_j and hence can be labeled S_j).

Hence, the resulting labeling is proper, i.e., it satisfies the requirements of the Sperner's lemma and we can use the lemma to find a smaller triangle which is colored distinctly at all its nodes. Repeating this process on the smaller triangle and continuing to do so, it can be proven that we will converge to a fixed point.

We observe that the graph constructed in the sperners lemma has a "path like structure", i.e., every vertex has degree 1 or 2. We can assign directions to the edges of the graph in the following way - Starting from source, we assign directions in such a way that every vertex has an in-degree at most 1 and out-degree at most 1. The existence proof of the Nash equilibrium has the following abstract structure. A directed graph is defined over the vertices of the polytope where all strategies are easily recognizable and represented. Each one of these vertices has in-degree at most 1 and out-degree at most 1. Hence the graph is a collection of paths and cycles. By necessity, there is one vertex with no in-coming edge and one out-going edge - such a vertex is called the *standard source*. By the basic properties of directed graphs we conclude that there must be a vertex with out-degree 0. This sink vertex is our Nash equilibrium.

The above argument suggests a simple algorithm to find a solution - start from the source and follow the path until you find a sink. Unfortunately, this is not an efficient algorithm because the number of vertices in the graph could be exponentially large. We note that even in this case the following three problems are efficiently solvable:

- Is v a vertex of the graph?
- Is u a neighbor of v in the graph?
- Which vertex is the predecessor and which vertex is the successor of a given vertex?

Apart from NASH there are a host of problems which are PPAD-complete like the Sperner's lemma on an exponentially large set of vertices, finding Brouwer's fixed point etc. It is unknown whether PPAD belongs to P or not. Similar to the class NP, which has NP-complete class as a set of problems which are interreducible in polynomial time (i.e., if one of these problems is solved in polynomial time, so are the rest), the class PPAD has the class of PPAD-complete class. Problems like NASH, Sperner, Brouwer, Arrow-Debreu equilibrium etc., are PPAD-complete. PPAD-completeness is weaker than NP-completeness because even if PPAD = P it is not clear that NP = P.

References

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[3] Nisan, Noam et al. *Algorithmic Game Theory.* New York: Cambridge University Press, 2007.