Network Design

Sina Dehghani, Soheil Ehsani, Saeed Seddighin

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1 Introduction

The problem of satisfying connectivity demands on a graph while respecting given constraints has been a pillar of the area of network design since the early seventies. The problem of DEGREE-BOUNDED SPANNING TREE, introduced in Garey and Johnson's *Black Book* of NP-Completeness, was first investigated in the pioneering work of Fürer and Raghavachari (Allerton'90). In the DEGREE-BOUNDED SPANNING TREE problem, the goal is to construct a spanning tree for a graph $G = (V, E)$ with n vertices whose maximal degree is the smallest among all spanning trees. Let b^* denote the maximal degree of an optimal spanning tree. Fürer and Raghavachari give a parallel approximation algorithm which produces a spanning tree of degree at most $O(\log(n)b^*)$.

Agrawal, Klein, and Ravi() consider the following generalizations of the problem. In the DEGREE-BOUNDED STEINER TREE problem we are only required to connect a given subset $T \subseteq V$. In the even more general DEGREE-BOUNDED STEINER FOREST problem the demands consist of vertex pairs, and the goal is to output a subgraph in which for every demand there is a path connecting the pair. They design an algorithm that obtains a multiplicative approximation factor of $O(\log(n))$. Their main technique is to reduce the problem to minimizing congestion under integral concurrent flow restrictions and to then use the randomized rounding approach due to Raghavan and Thompson(, STOC'85).

Shortly after the work of Agrawal*et al.*, in an independent work in SODA'92 and later J.of Algorithms'94, Fürer and Raghavachari significantly improved the result for DEGREE-BOUNDED STEINER FOREST by presenting an algorithm which produces a Steiner forest with maximum degree at most $b^* + 1$. They show that the same guarantee carries over to the *directed* variant of the problem as well. Their result is based on reducing the problem to that of computing a sequence of maximal matchings on certain auxiliary graphs. This result settles the approximability of the problem, as computing an optimal solution is NP-hard even in the spanning tree case.

In this paper, we initiate the study of degree-bounded network design problems *in an online setting*, where connectivity demands appear over time and must be immediately satisfied. We first design a deterministic algorithm for ONLINE DEGREE-BOUNDED STEINER FOREST with a logarithmic competitive ratio. Then we show that this competitive ratio is asymptotically best possible by proving a matching lower bound for randomized algorithms that already holds for the Steiner tree variant of the problem.

In the offline scenario, the results of Fürer, Raghavachari and Agrawal, Klein, Ravi were the starting point of a very popular line of work on various degree-bounded network design problems. We refer the reader to the next sections for a brief summary. One particular variant that has been extensively studied is the *edge-weighted* DEGREE-BOUNDED SPANNING TREE.

Initiated by Ravi*et al.* (J.of Algorithms'98), in this version, we are given a weight function over the edges and a bound b on the maximum degree of a vertex. The goal is to find a minimum-weight spanning tree with maximum degree at most b. The groundbreaking results obtained by Goemans(FOCS'06) and Singh and Lau(STOC'07) settle the problem by giving an algorithm that computes a minimum-weight spanning tree with degree at most $b+1$. Slightly worse result are obtained by Singh and Lau(STOC'08) for the Steiner tree variant. Unfortunately, in the online setting it is not possible to obtain a comparable result. We show that for any (randomized) algorithm $\mathcal{A}$ there exists a request sequence such that $\mathcal{A}$ outputs a sub-graph that either has weight $\Omega(n) \cdot \mathrm{OPT}_b$ or maximum degree $\Omega(n) \cdot b$.

1.1 Related Connectivity Problems

The classical family of degree-bounded network design problems have various applications in broadcasting information, package distribution, decentralized communication networks, etc.(see e.g.). Ravi*et al.*(J. Algorithms'98), first considered the general *edge-weighted* variant of the problem. They give a bi-criteria $(O(\log n), O(\log n) \cdot b)$-approximation algorithm, i.e., the degree of every node in the output tree is $O(\log n) \cdot b$ while its total weight is $O(\log n)$ times the optimal weight. A long line of work(see e.g. STOC'00 and SIAM J. C.) was focused on this problem until a groundbreaking breakthrough was obtained by Goemans(FOCS'06); his algorithm computes a minimum-weight spanning tree with degree at most $b+2$. Later on, Singh and Lau(STOC'07) improved the degree approximation factor by designing an algorithm that outputs a tree with optimal cost while the maximum degree is at most $b+1$.

In the *degree-bounded survivable network design* problem, a number d_i is associated with each demand (s_i, t_i). The solution subgraph should contain at least d_i edge-disjoint paths between s_i and t_i. Indeed this problem has been shown to admit bi-criteria approximation algorithms with constant approximation factors(e.g. STOC'08). We refer the reader to a recent survey in . This problem has been recently considered in the node-weighted variant too (see e.g.). The degree-bounded variant of several other problems such as k-MST and k-arborescence has also been considered in the offline setting for which we refer the reader to and references therein.

1.2 Related Online Problems

Online network design problems have attracted substantial attention in the last decades. The online edge-weighted Steiner tree problem, in which the goal is to find a minimum-weight subgraph connecting the demand nodes, was first considered by Imase and Waxman(,SIAM J. D. M.'91). They showed that a natural greedy algorithm has a competitive ratio of $O(\log n)$, which is optimal up to constants. This result was generalized to the online edge-weighted Steiner forest problem by Awerbuch *et al.*(SODA'96) and Berman and Coulston(STOC'97). Later on, Naor, Panigrahi, and Singh(FOCS'11) and Hajiaghayi, Liaghat, and Panigrahi(FOCS'13), designed poly-logarithmic competitive algorithms for the more general *node-weighted* variant of Steiner connectivity problems. This line of work has been further investigated in the prize-collecting version of the problem, in which one can ignore a demand by paying its given penalty. Qian and Williamson(ICALP'11) and Hajiaghayi, Liaghat, and Panigrahi(ICALP'14) develop algorithms with a poly-logarithmic competitive algorithms for these variants.

The online b-matching problem is another related problem in which vertices have degree bounds but the objective is to maximize the size of the solution subgraph. In the worst case model, the celebrated result of Karp *et al.*(STOC'90) gives a $(1-1/e)$-competitive algorithm. Different variants of this problem have been extensively studied in the past decade, e.g., for the random arrival model see for the full information model see and for the prophet-inequality model see. We also refer the reader to the comprehensive survey by Mehta.

Many of the problems mentioned above can be described with an online packing or covering linear program. Initiated by work of Alon *et al.*on the online set cover problem, Buchbinder and Naor developed a strong framework for solving packing/covering LPs fractionally online. For the applications of their general framework in solving numerous online problems, we refer the reader to the survey in . Azar *et al.*generalize this method for the fractional *mixed* packing and covering LPs. In particular, they show an application of their method for integrally solving a generalization of capacitated set cover. Their result is a bi-criteria competitive algorithm that violates the capacities by at most an $O(\log^2 n)$ factor while the cost of the ouput is within $O(\log^2 n)$ factor of optimal. We note that although the fractional variant of our problem is a special case of online mixed packing/covering LPs, we do not know of any online rounding method for Steiner connectivity problems.