# Subexponential Time Algorithms via Concentration Bounds

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# Outline

# Motivation

- Computation of equilibria in 2-player games is a hard problem.
- However, if strategies are **sparse** and **uniform** then the search space is sub-exponential.
- Althofer Lemma guarantees existence of such a grid in strategy space of players.
- We will give a novel application of this sparsification technique in the second half.

# Sparse Solutions to Linear Systems: Althofer's Lemma

- Let $p = (p_1, \ldots, p_n)$ be any probability vector, i.e., $\sum_{i=1}^{n} p_i = 1, p_i \geq 0$.
- Let $A$ be any $m \times n$ matrix with all entries between 0 and 1. Let $A_i$ be the $i^{\text{th}}$ row of $A$.
- Then the linear transform $Ap$ can be approximated by $Aq$, where probability vector $q$ is:

approximate representation of $p$

$$|A_i \cdot (p - q)| \leq \epsilon, i \in \{1, \ldots, n\}.$$

sparse at most $k = \frac{log 2m}{2\epsilon^2}$ entries are non-zero.

uniform all entries are of form $q_i = \frac{k_i}{k}$, $k_i$ is integral

# Proof Outline

- Construct a sparse representation of $p$ by sampling.
- Use Hoeffding bound and union bound to prove that with non-zero probability, the constructed representation is an approximate representation of $p$.
- This proved existence of a sparse representation.

# Proof Outline : Construction of Sparse Representation

- Let $p = (p_1, \ldots, p_n)$ be a given probability vector.
- Sample $k$ times with replacement from set $\{1, \ldots, n\}$ with probability of sampling $i$ given by $p_i$.
- Let $S$ be the multiset representing the result of the above sampling.
- Let $n_i$ be the number of times $i$ occurs in $S$.
- Then $q_i = \frac{n_i}{k}$.

# Proof Outline : Properties of Sparse Representation

Probability vector $q = (q_1, \ldots, q_n)$:

is sparse at most $k$ non-zero entries.

is uniform all entries are multiples of $\frac{1}{k}$.

preserves dot products $E(a \cdot q) = a \cdot p$.

# Proof Outline

- To prove existence of sparse $q$, we need to choose appropriate $k$.
- Dot products are preserved in expectation, therefore, the linear transform $Ap$ is preserved in expectation.
- We use Hoeffding bound and union bound to choose appropriate value of $k$.