Fault Tolerant Facility Location Problems

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Introduction

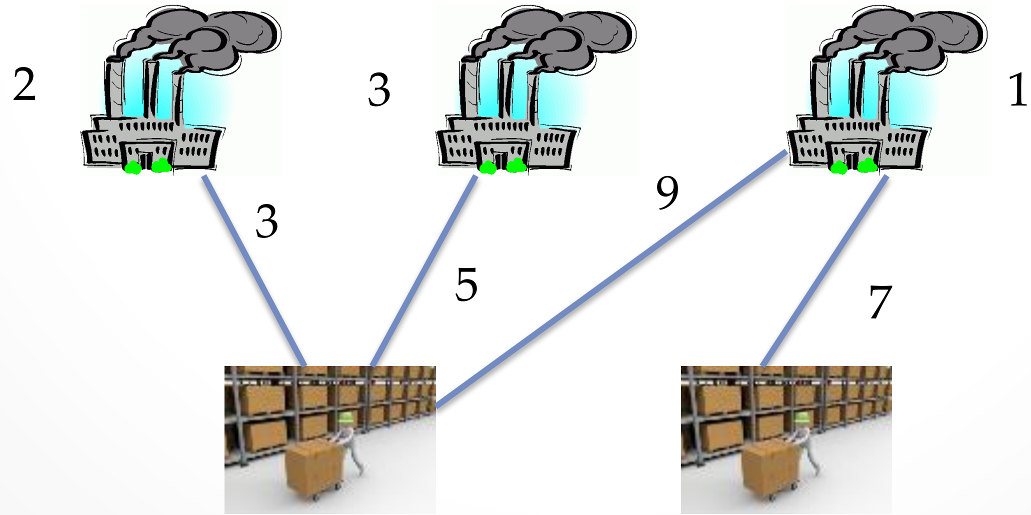
- Facility Location problems significant problem in various domains
  - Operations Research
  - Industrial Engineering

- Various variants of this class of problems studied in the past

- Many questions yet to be answered

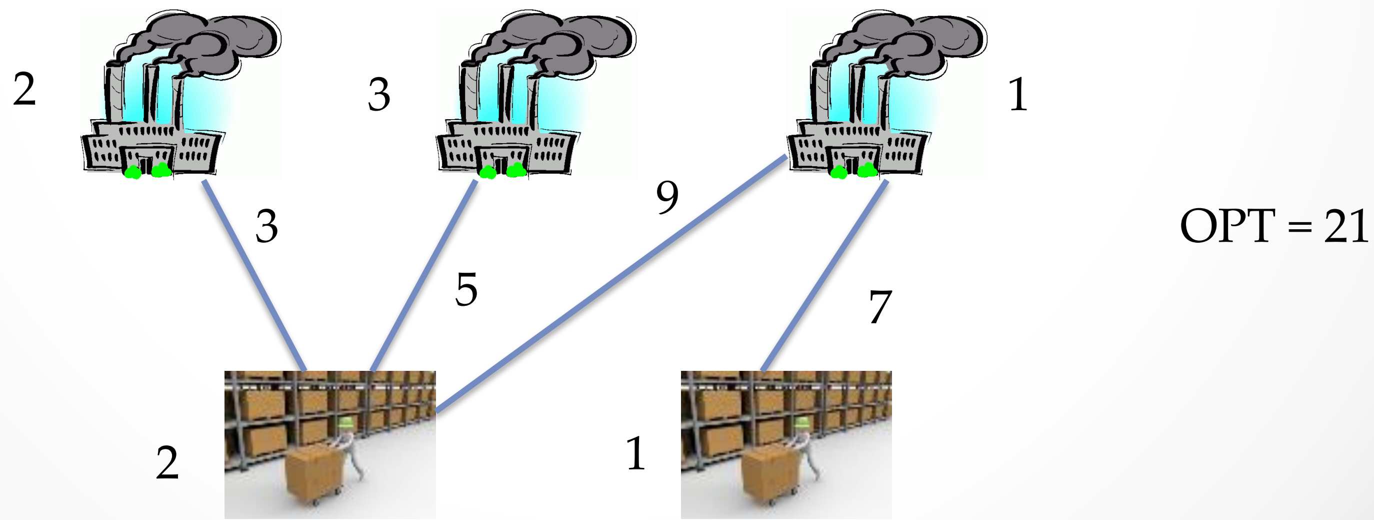
Definitions

- Uncapacitated Facility Location
  - Most basic version of the problem
  - A set of facilities F, each having a cost of opening f_i
  - A set of clients C, each having a cost of connection to the j^{th} facility c_{ij}
  - Objective is to open a subset of facilities F, such that cost of opening facilities + the sum of minimum cost of connection for every client to this subset of facilities is minimized

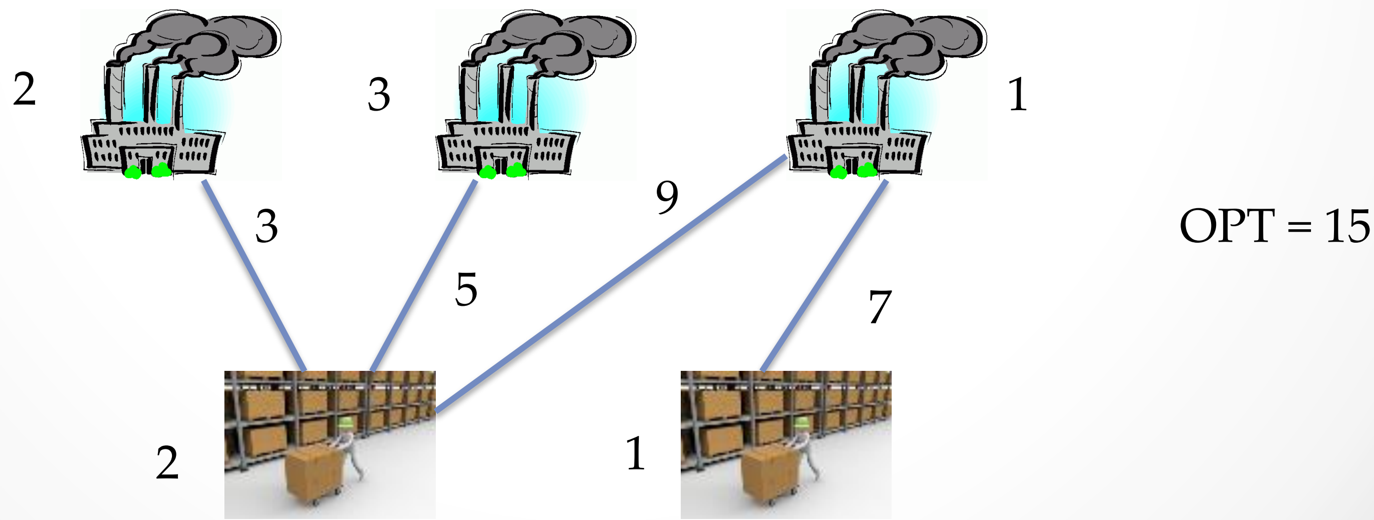


OPT = 13

- Fault Tolerant Facility Location (FTFL)
  - A variant of the UFL Problem
  - Setting similar to UFL: Facilities F, with opening costs, Clients C with connection cost
  - Difference : With every client there is an associated demand $r_{j,}$ every client should be connected to $r_{j,}$ different facilities in the subset opened.



- Fault Tolerant Facility Placement (FTFP)
  - A variant of the FTFL Problem
  - Setting similar to FTFL: Facilities F, with opening costs, Clients C with connection cost
  - Difference : Same facility can be opened multiple times. We may pay the cost of opening for each time it is opened.



Known Bounds

- Lower Bounds:
  - UFL – 1.463 [Guha, Khuller][SODA '98]
  - FTFL – Uniform demands, r=2, 1.278 [Byrka, Rybicki][ArXiv' 13]
- Upper Bounds
  - UFL – 1.488 [Li][ICALP '13]
  - FTFL – 1.725 [Byrka, Srinivasan, Swamy][IPCO '10]
  - FTFP – for r>=2, 1.439 [Byrka, Rybicki][ArXiv '13]

New Lower Bound FTFP

- Extend the idea of Byrka, Rybicki for FTFL to FTFP
- For the uniform case, r = 2, get a lower bound of 1.18
- For uniform case, generalized r, lower bound solution to equation

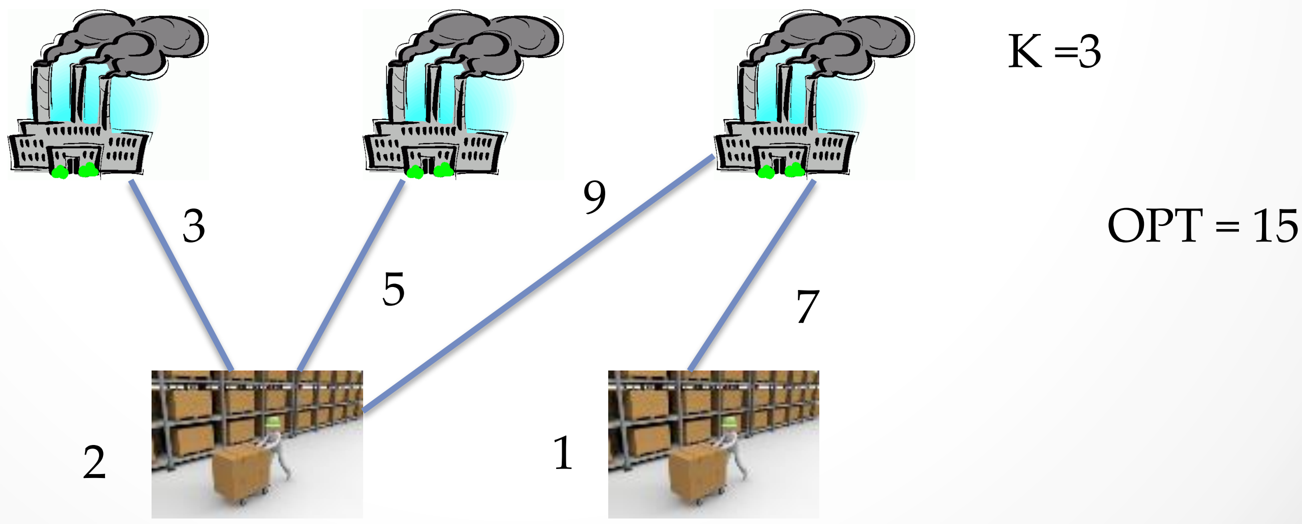
$$X = 1+2e^{-x.r}$$

Overview of proof

- Given approximation algorithm for FTFP, use it to get approximation for set cover
- Set cover cannot be approximated better than O(ln n)
  - Use this to show lower bound on approximation for FTFP
- For every set, have a facility, for every element have a client
  - For elements contained in a set a edge of cost 1 added between client and facility, else edge of cost 3

- Guess optimal solution of set cover
- Cost of j^{th} facility is a parameter (q) fixed later
- Find Lower bound on cost of any solution - L
- Find optimal cost of solution - U
- Approximation ratio should be at least the ratio of L by U
- Maximize this with respect to the parameter q

- Fault Tolerant K-Median Facility Location(FTkMFL)
  - A variant of the FTFL Problem
  - In this case, we have facilities F, but no cost associated with them
  - We have clients C, having demands r_j and connection costs c_{ij}.
  - Since facilities have no cost, hence only connection cost appears in objective



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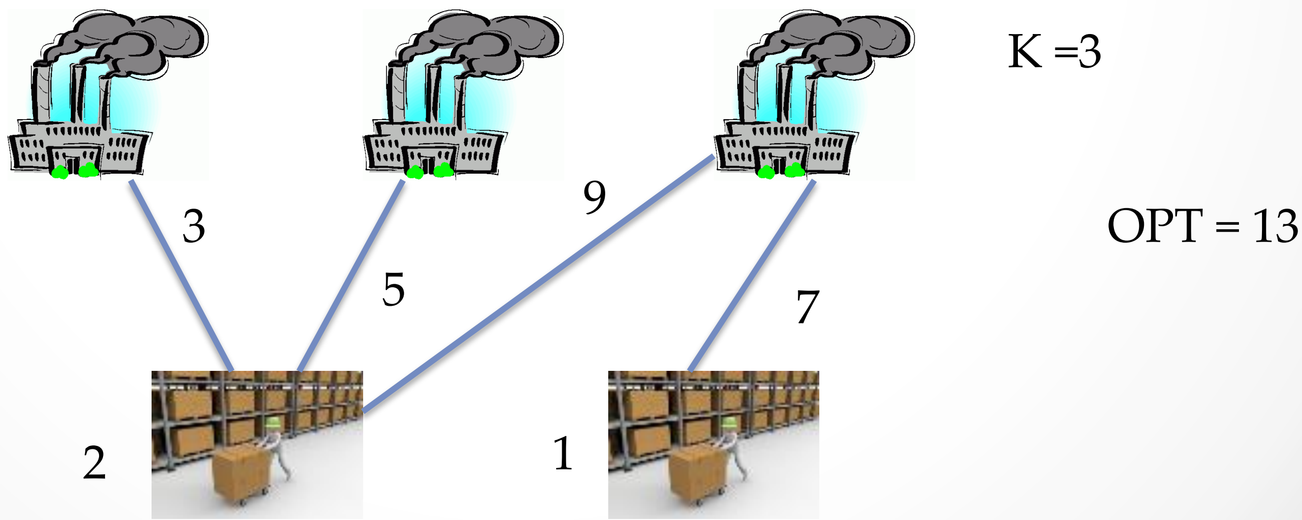
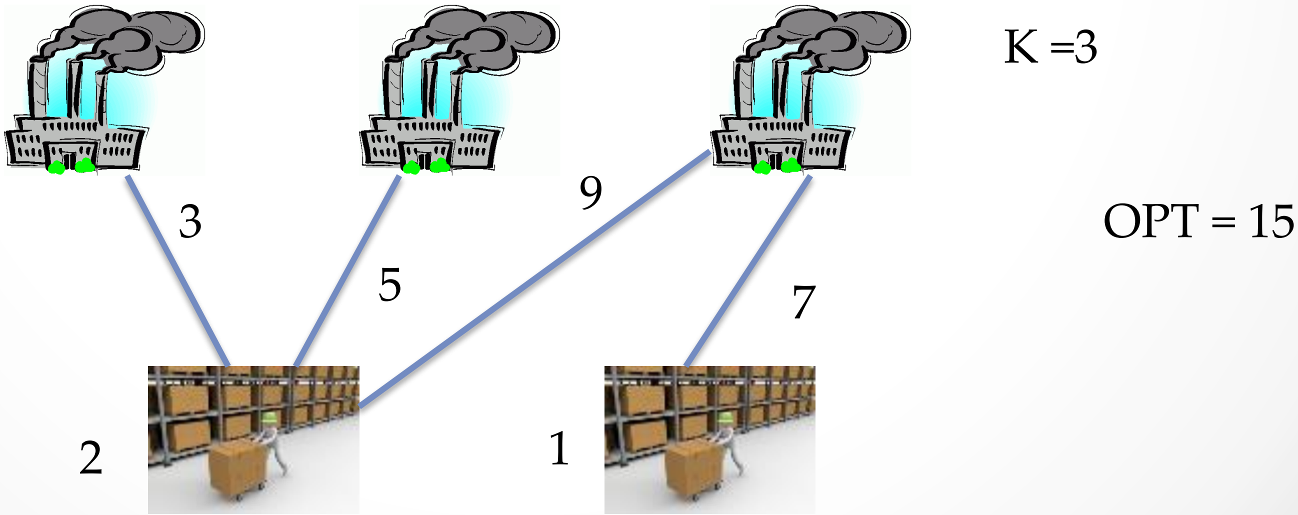


k-median

- Remove facility costs.
- Add constraint:
  Use <= k facilities.
- Hardness reduction is simpler.



MAX-COVER

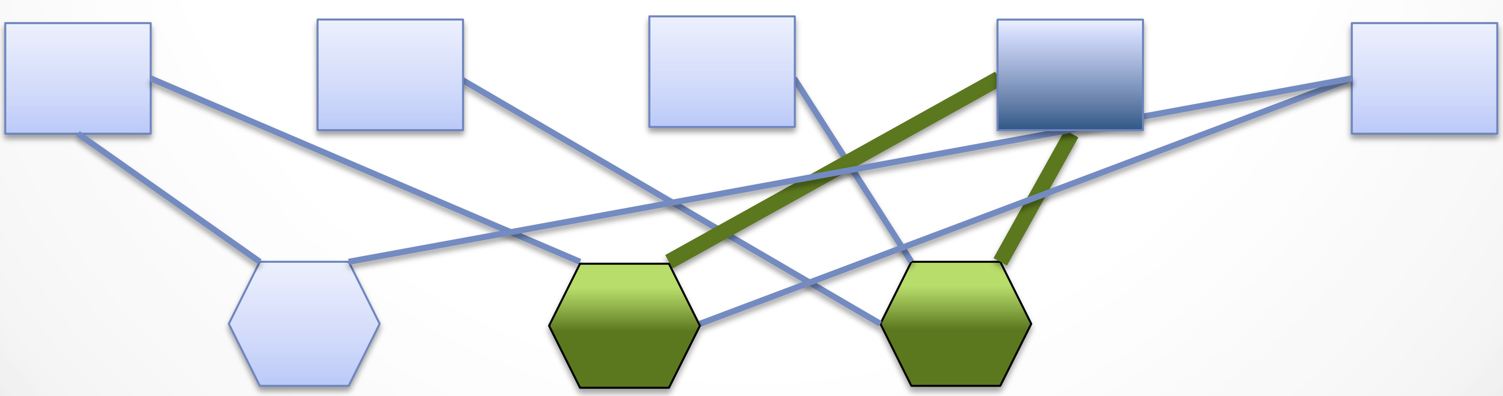
- Input:
- Sets S
- Elements X

- Goal: cover many elements using k sets.

- Hardness: 1-1/e
  - Proof by reduction to Set Cover.
  - Run MAX-COVER repeatedly, k=OPT_SC
  - If 1-1/e- ε approximation, we stop in <ln(n) steps
  - Use less than ln(n)OPT sets.

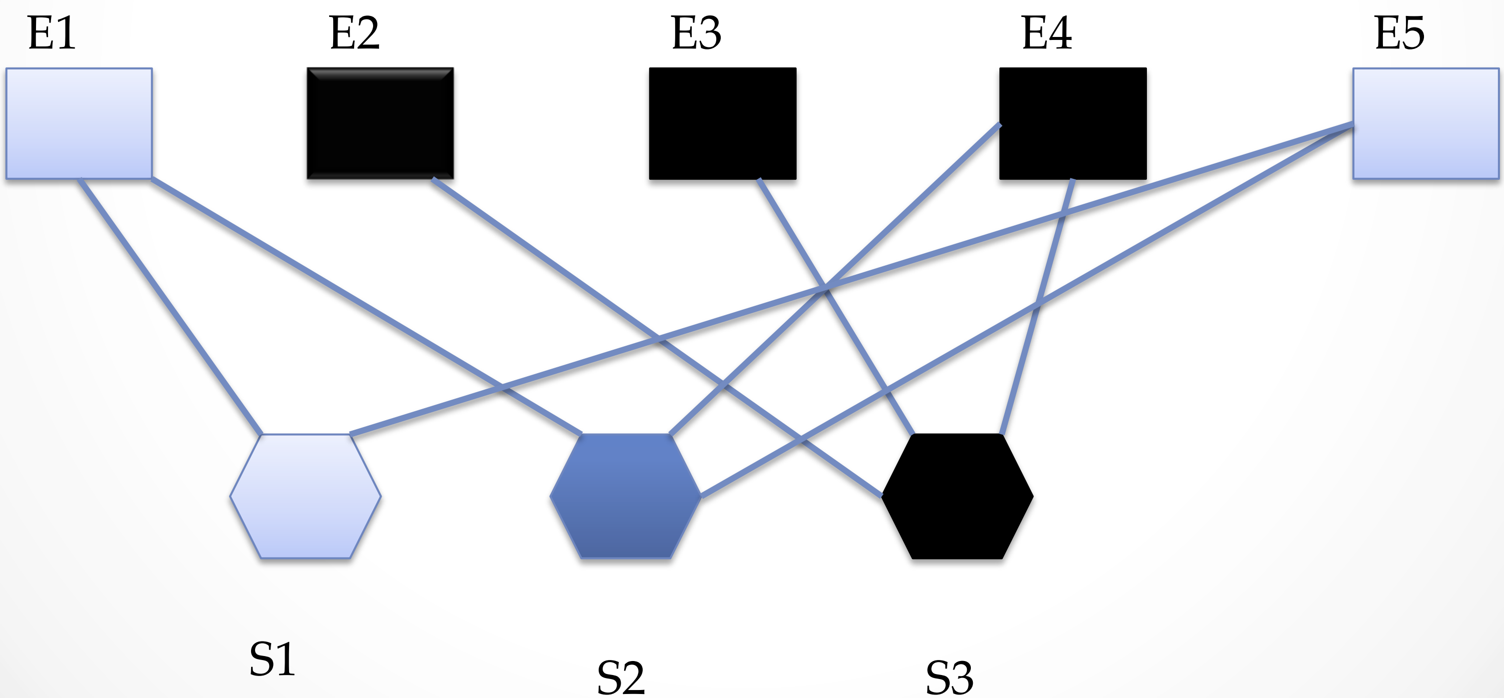
1-1/e Hardness

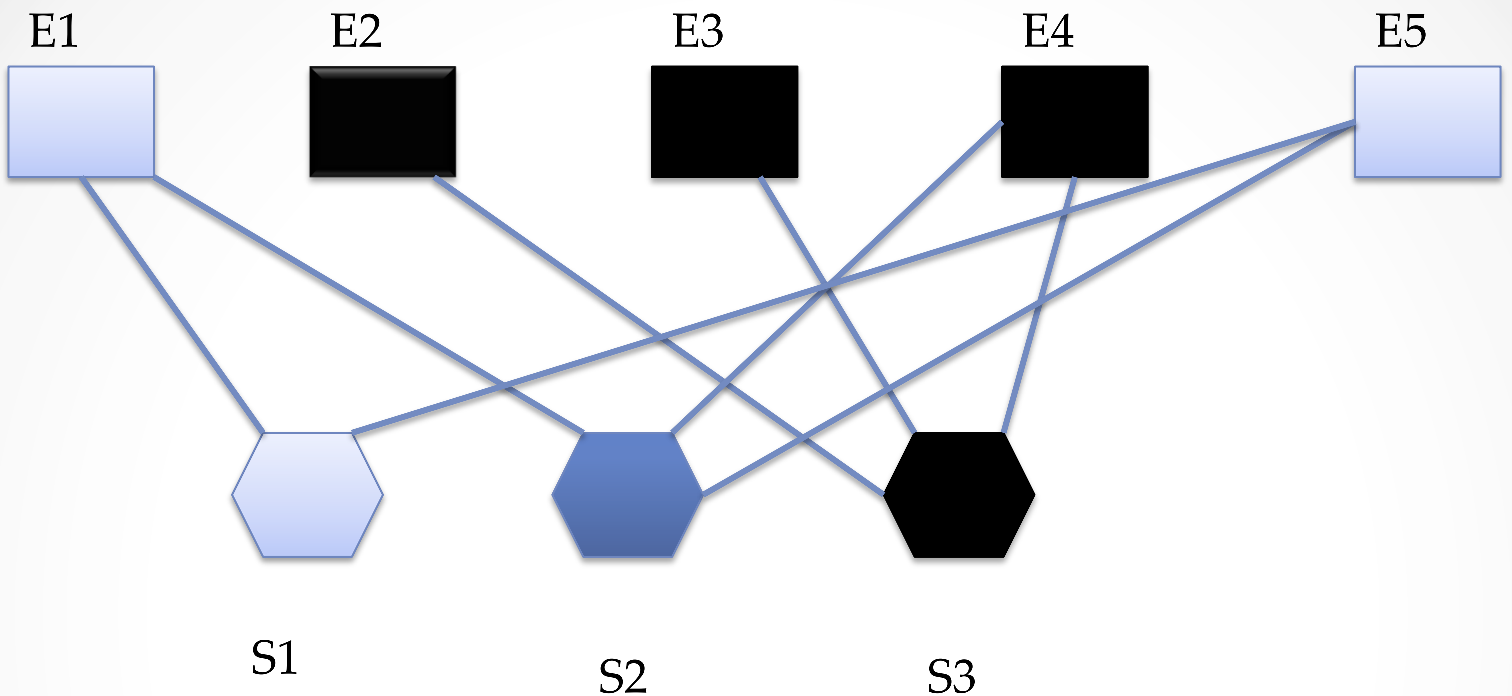
- Run MAX-COVER on SET COVER instance, $k=OPT_{SC}$
  - Remove covered elements and run again
- Suppose >1-1/e approximation
  → $1/e^t$ uncovered elements after t steps.
  → we stop in <ln(n) steps
  → Use less than ln(n)OPT sets.



Simple Lower Bound for k-median

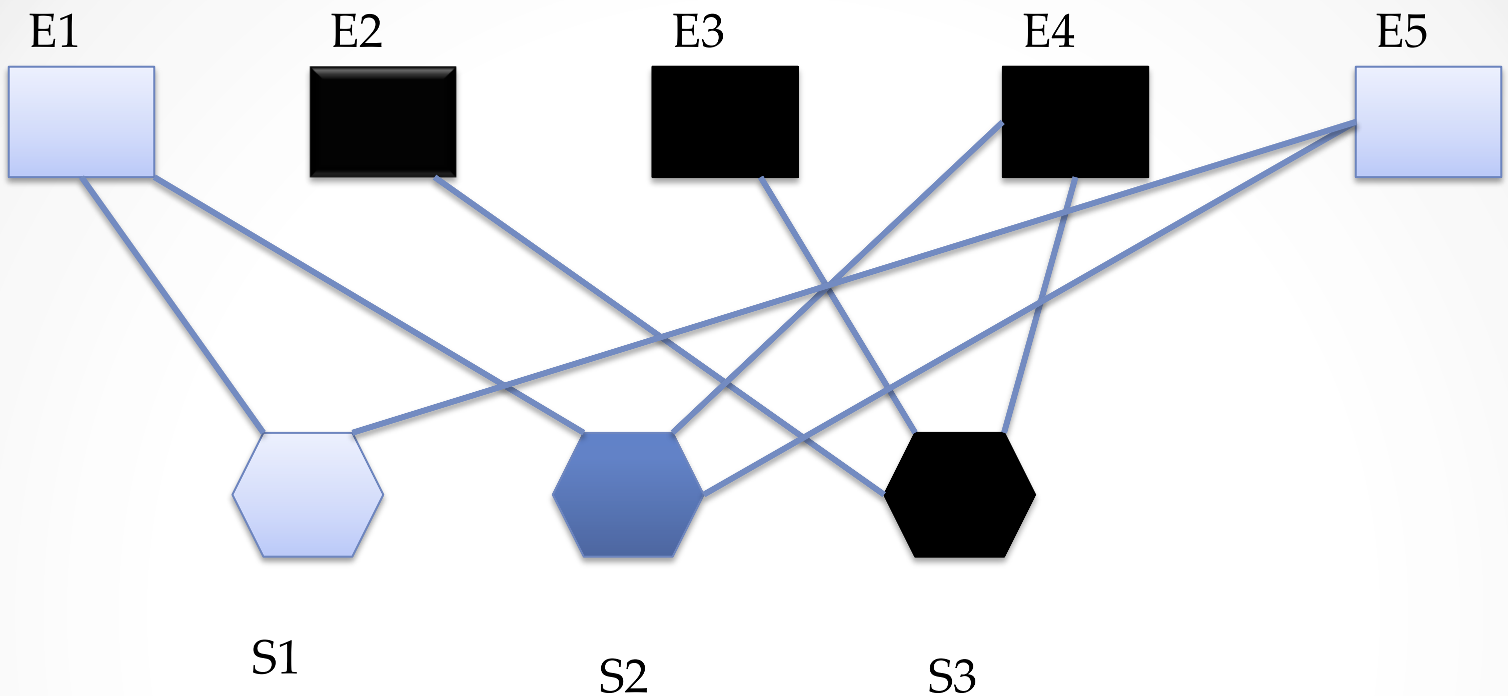
- Reduction from MAX COVER*
  - *where OPT covers all elements
- Sets-> Facilities Elements->Clients



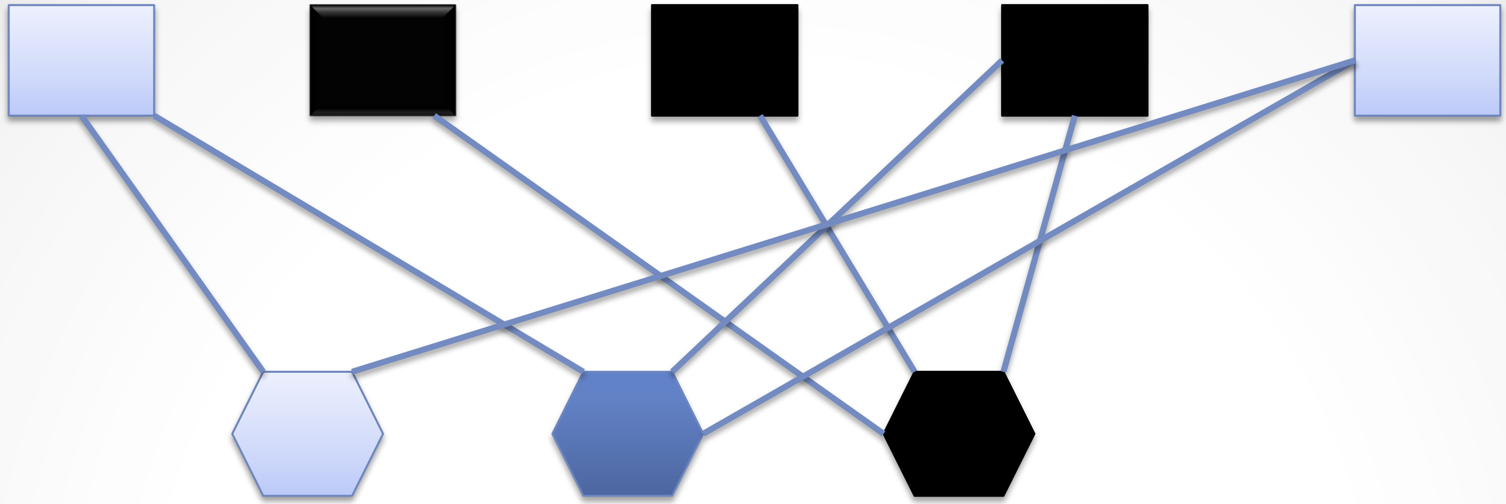


- Facility (set) is distance 1 from all clients (elements) in set.
- By triangle inequality, other distances are 3.
- If facilities "cover" many clients, cost will be small.

- small k-median cost ↔ large coverage



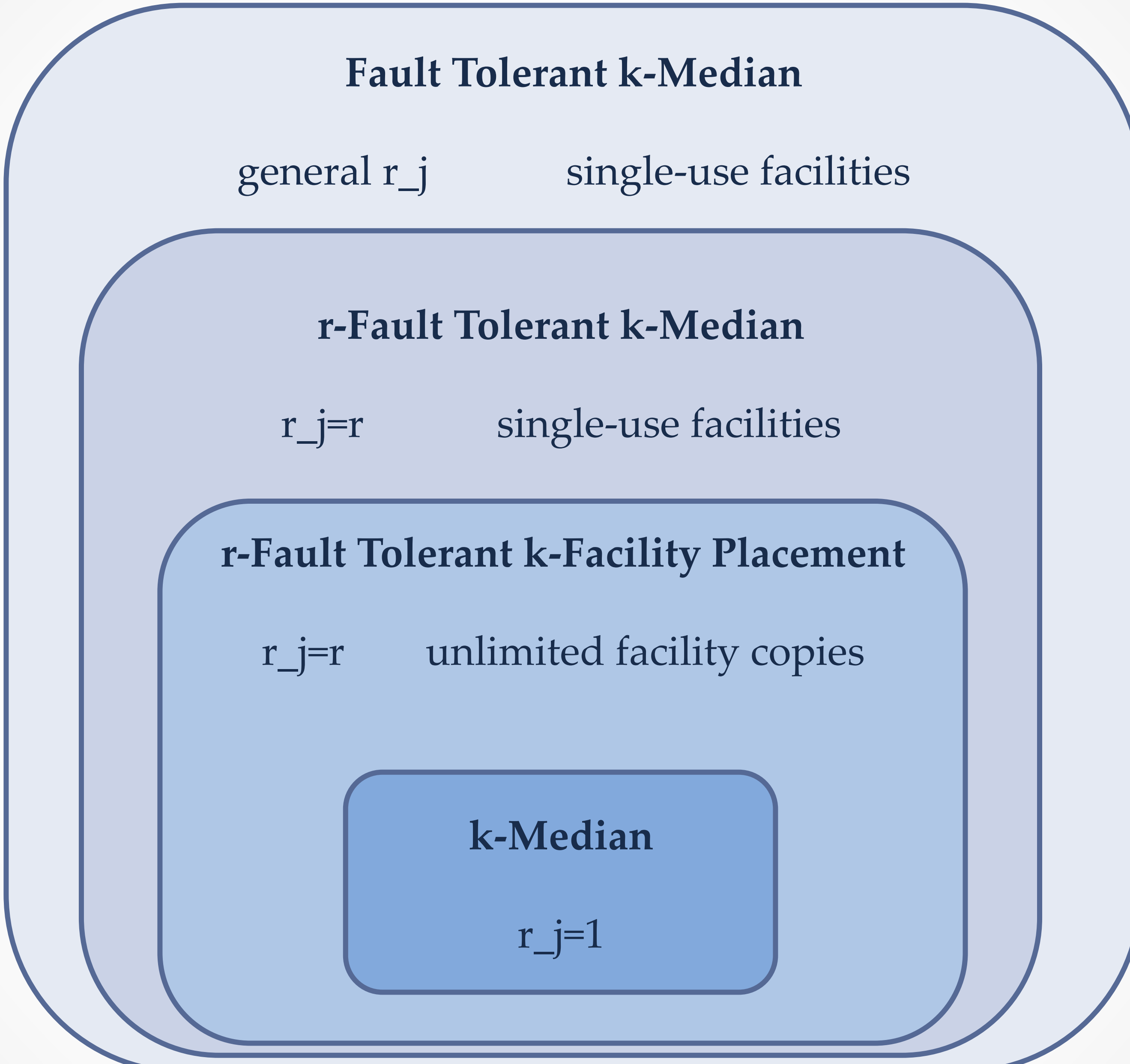
- Suppose sets T cover β of elements
- What is k-median cost of T?
  - Covered clients pay 1
  - Uncovered clients pay 3
- $COST(T)= 1^* \beta +3^*(1- \beta)=3-2 \beta$

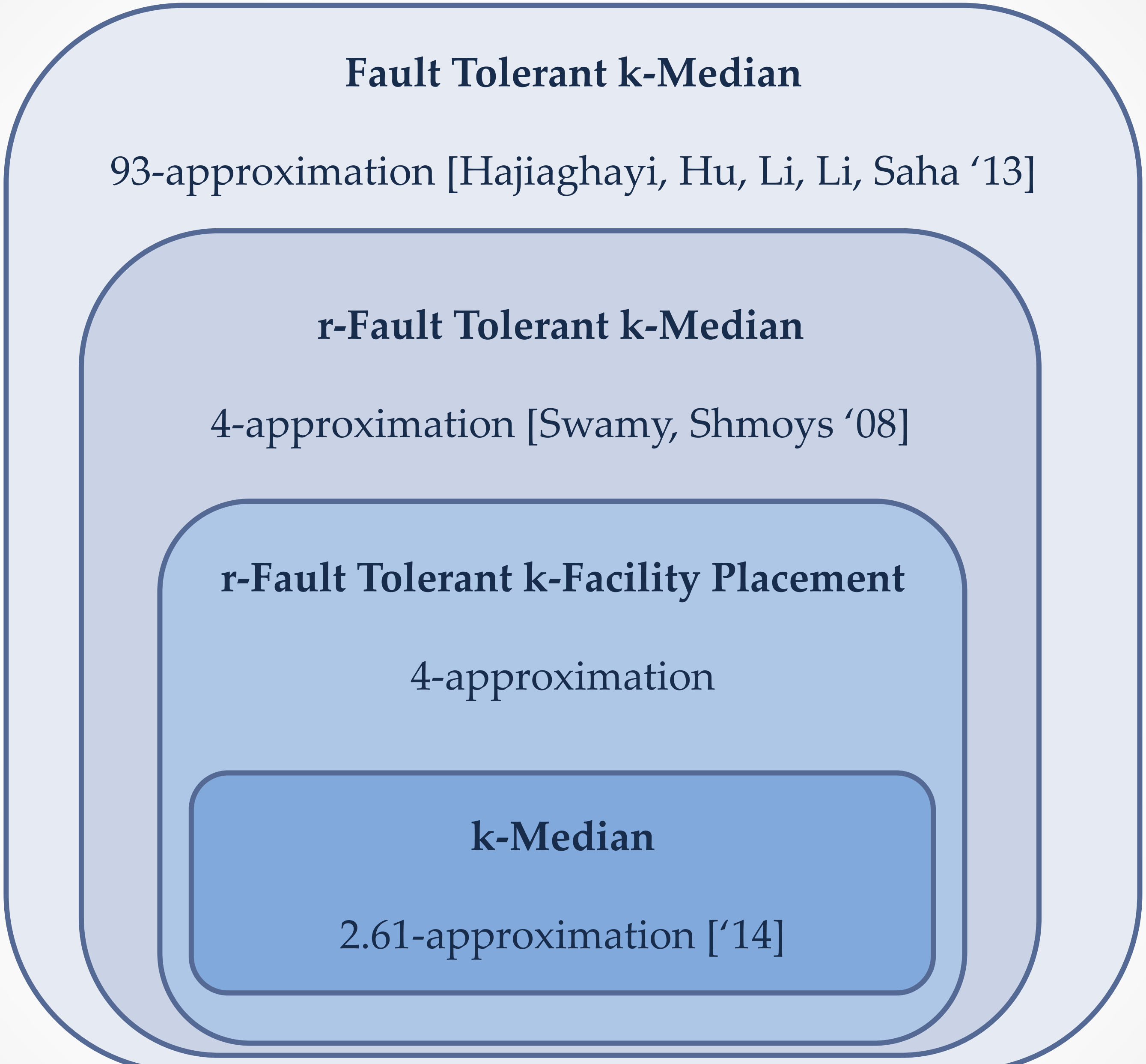


- $COST(T) = 1 * \beta + 3*(1- \beta) = 3-2\beta$

$$COST(T) < 1+2/e \rightarrow \beta > 1-1/e$$

- $OPT_k = 1$
- Conclusion: k-median is 1+2/e hard


Fault Tolerant k-Median
general r_j
single-use facilities
r-Fault Tolerant k-Median
r_j=r
single-use facilities
r-Fault Tolerant k-Facility Placement
r_j=r
unlimited facility copies
k-Median
r_j=1


Fault Tolerant k-Median
93-approximation [Hajiaghayi, Hu, Li, Li, Saha ‘13]
r-Fault Tolerant k-Median
4-approximation [Swamy, Shmoys ‘08]
r-Fault Tolerant k-Facility Placement
4-approximation
k-Median
2.61-approximation [‘14]

r-Fault Tolerant k-Facility Placement

- For r=1, this is k-median, and has hardness 1+2/e

- For uniform r>=2, we adopt reduction

  - use k*r facilities

  - estimate cost as function of coverage (loose)

  → 1+1/e^r hardness

Improved Lower Bound

- From previous section, cheapest solution possible – cover $1-e^{-r}$ elements, r times
- Is this really easy, if it is hard to cover more than $1-e^{-r}$ at least once ?
  - Answer: NO !
- This above observation, leads to better lower bounds

Improved Constraints

- Apply sampling to get sets of multiple sizes.
- Expected fraction of elements uncovered at least $e^{-\gamma r}$
- New constraints gives 1.344 lower bound for r =2

Hardness values for various r

r	1	2	3	4	5	6	7	8
γ_r	1	0.705	0.494	0.379	0.306	0.257	0.222	0.195
new hardness	1.736	1.344	1.224	1.167	1.132	1.110	1.094	1.082
$1+\frac{2}{e^r}$	1.736	1.271	1.010	1.037	1.013	1.005	1.0018	1.0007
integrality gap	1.736	1.541	1.448	1.391	1.351	1.321	1.298	1.279

Integrality Gap

- k-median has known integrality gap 2
- rFTkMP for r>1 also has integrality gap 2
- Gap of 2 is “weak” in k-median
- Stronger gap is 1+2/e
- Question: what is the “stronger” gap for rFTkMP?

Future Directions

- Extend a similar approach to get a lower bound on FTFP.
  - Note, no lower bound known currently

- Large gap between Lower and Upper bound – Improve from either directions

Questions/Comments ?