Today, of course not every detail of these problems

1. Set Cover: Applications, e.g., in wireless network: minimum set of transmitters covering every area.


3. Unique Coverage: several applications, e.g., maximize the area set of interferer on a free in wireless network.

1. Set cover: a universe \( U \) of \( n \) elements, a collection \( X \) of subsets of \( U \), \( S_1, \ldots, S_k \), and a cost function \( c: S \rightarrow \mathbb{Q}^+ \).

Find a minimum cost subcollection \( X \) of \( S \) that covers all elements of \( U \).

* Frequency of an element is the number of sets it is in: say \( F \) is the maximum frequency.

* Thm 1: There is a minimum of \( O(\ln n) \) and \( F \) approximation for Set Cover.

1. Greedy \( O(\ln n) \) approximation:

   1. \( C := \emptyset \)
   2. While \( C \neq U \) do

      * Find the sets with the most cost-effectiveness, i.e., \( \frac{\text{cost}(S)}{\text{minima of } 15 - C} \).

      * Pick \( S \) and for each \( e \in S - C \), set \( \text{price}(e) = \alpha \).

      * \( C \leftarrow C \cup S \) only for the analysis (cost(s) is amortized to its new covered elements) thus,

   3. Output the picked sets.

\[ \text{cost(Greedy Alg)} \leq \sum_{e \in C} \text{price}(e) \]
Analysis: OPT covers everything since there should be one set with cost-effectiveness \( \frac{\text{opt}}{|U - C|} \) number the elements of \( U \) in the order in which we add them to \( C \) (break ties arbitrarily).

\[
\text{cost(OPT)} = \text{cost}(S_1) + \text{cost}(S_2) + \cdots + \text{cost}(S_k) = \frac{\text{cost}(S_1)}{|S_1|} + \frac{\text{cost}(S_2)}{|S_2|} + \cdots + \frac{\text{cost}(S_k)}{|S_k|} \\
\geq \alpha \frac{|S_1|}{|S_1|} + \alpha \frac{|S_2|}{|S_2|} + \cdots + \alpha \frac{|S_k|}{|S_k|} \geq \alpha \frac{|U - C|}{|U - C|}. \text{ Thus } \frac{\text{cost(OPT)}}{|U - C|} \geq \alpha.
\]

Thus

\[
\text{cost of alg} = \sum_{e \in V} \text{price}(e) = \text{price}(e_1) + \text{price}(e_2) + \cdots + \text{price}(e_n)
\]

\[
\leq \frac{\text{cost(OPT)}}{n-1} + \frac{\text{cost(OPT)}}{n-2} + \cdots + \frac{\text{cost(OPT)}}{n-1} + \frac{\text{cost(OPT)}}{n-1} + \frac{\text{cost(OPT)}}{n-1} + \cdots + \frac{\text{cost(OPT)}}{n-1}
\]

\[
(\text{why, in the iteration in which element } e_i \text{ is added } |U - C| \geq n-1) \text{ by definition of } e_i.
\]

\[
\leq \text{cost(OPT)} \left( \frac{1}{n-1} + \frac{1}{n-1} + \cdots + \frac{1}{n-1} \right) = \text{cost(OPT)} \ln n = \text{cost(OPT)} \ln n.
\]

The analysis is tight:

\[
\begin{array}{c}
\text{opt choose } s^* \text{ while the greedy picks } \\text{costs: } \frac{1}{n-1} + \frac{1}{n-1} + \cdots + \frac{1}{n-1} \text{ while the cost of greedy is } \ln n.
\end{array}
\]

A Greedy Approximation:

1. Cover \( \emptyset \)
2. While \( E \) contains elements not covered by Cover \( A \) add all sets containing \( e \) to Cover \( A \)

\[
\]
Analysis: \( U \) is an independent set; i.e., no element of our collection can cover both of them. Thus, \( \text{OPT} \) should need at least \( |U| \) sets. We have at most \( F|U| \) sets since each element appears at most \( F \) times. Thus, we have an \( F \)-approximation.

It is easy to construct a tight example (exercise) here.

Does the above analysis work for arbitrary cost of sets?

Answer NO: only when all \( w_i \) are ones.

However, still we can obtain \( F \) approximation by other techniques such as linear programming. How? See below.

\[
\text{obj} = \text{minimize } \sum_{j=1}^{|X|} c_j x_j \\
\text{s.t. } \sum_{j \in X_j} x_j \geq 1 \quad i = 1, \ldots, |U| \\
x_j \geq 0.
\]

Consider an element \( i \), at least one of \( x_j \) where \( i \in X_j \) is at least \( \frac{1}{F} \).

Choose that set and all of its elements and iterate!!!

We pay at most \( F \cdot \text{obj} \leq F \cdot \text{OPT} \) (linear programs, integer program).

Using LP rounding, we can also obtain \( O(\ln n) \)-approximation as before.

However, Feige showed

There is no \( (1 - \epsilon) \ln n \)-approximation to set cover unless \( \text{NP} \subseteq \text{DTIME}(n^{O(\ln^2 n)}) \). Essentially no improvement even by a constant factor.

The problem is equivalent to \text{MINIMUM DOMINATING SET in GRAPHS}.

See section 2 and 13 of \\text{Vazirani} for more details and extensions.