Session 6

Räcke from FRT:

Note that in FRT after removing steiner nodes, we can put $d_T(u,v) = d_G(u,v)$ to have dominance property and possibly decrease the distortion.

Similarly we can use $c(e_T) = \sum_{e' \in \delta(e_T)} c(e')$ [$e' = (u',v') \in \delta(e_T)$ means u' and v' belong to two different component of T after removing e_T]

So in both cases we really care about the tree structure and not really the weights on trees.

In FRT, we want to minimize the distortion $\frac{d_T(e)}{d_G(e)}$ $\forall e \in E(G)$ where $d_T(e) = \sum_{e' \in \text{uniquepath } e \text{ in } T} d_G(e')$

In Räcke, we want to minimize the congestion $\frac{load_T(e)}{c_G(e)}$ $\forall e \in E(G)$ where $= \sum T_{e'e}\, c_{e'}$

$$load_T(e) = \sum_{e_T \in E_T : e \in M_E(e_T)} c(e_T) = \sum_{e_T \in E_T : e \in M_E(e_T)} \sum_{e' \in E(G) : e' \in \delta(e_T)} c_{e'}$$

$$= \sum_{e' \in E(G)} \sum_{\substack{e_T \in \text{uniquepath}(e') \\ e \in M_E(e_T)}} c(e') = \sum_{e' \in E(G)} T_{e'e}\, c(e')$$ where $T_{e'e}$ is the number of times e appears in $m(m'_E(e'))$.

<u>Lm1</u>: For every $\rho \geq 1$ and every family of trees, there is a probabilistic mapping with distortion at most ρ if and only if for every nonnegative coefficient α_e, where $\sum_{e \in E(G)} \alpha_e = 1$, there is a tree T such that $\sum_{e \in E(G)} \alpha_e \frac{d_T(e)}{d_G(e)} \leq \rho$

[or equivalently $\sum_{e \in E(G)} \alpha_e \frac{d_T(e)}{d_G(e)} \leq \rho \sum_{e \in E(G)} \alpha_e$ if $\sum_{e \in E(G)} \alpha_e \neq 1$ because of scaling].

<u>Proof</u>: uses the min max theorem with two player Tree who choose the tree and player Edge who choose an edge. $M_{Te} = \frac{d_T(e)}{d_G(e)}$, where M_{Te} is the payoff vector. (note that since it is a zero-sum game payoff for the tree player is $-$ payoff for the edge player.

A mixed strategy for the tree player is a probability distribution λ_T over all trees while a mixed strategy for the edge player is a probability distribution α_e over all edges. The min-max theorem says $\min_\lambda \max_\alpha \sum_T \sum_e \lambda_T \alpha_e M_{Te} = \max_\alpha \min_\lambda \sum_T \sum_e \lambda_T \alpha_e M_{Te}$

The lemma follows from this. ∎

Lm 2: For every $\rho \geq 1$ and every family of trees, there is a probabilistic mapping with congestion at most ρ if and only if for every non-negative coefficient β_e, there is a tree T such that $\sum_{e \in E(G)} \beta_e \frac{load_T(e)}{c_G(e)} \leq \rho \sum_{e \in E(G)} \beta_e$.

Pf: similar to pf of lm 1 and thus omitted.

Main Thm: For every $\rho \geq 1$ and family of trees (1) and (2) are equivalent

(1): For every collection of lengths $d_G(e)$ there is a probabilistic mapping with distortion at most ρ

(2) For every collection of capacities $c_G(e)$, there is a probabilistic mapping with congestion at most ρ.

Pf: We prove (2) from (1) and the reverse is similar.

Assume (1). Thus by lm 1: for every non-negative coefficient α_e, there is a tree T such that $\sum_{e \in E(G)} \alpha_e \frac{d_T(e)}{d_G(e)} \leq \rho$. Thus $\sum_{e, e' \in E(G)} T_{ee'} \alpha_e \frac{d_G(e')}{d_G(e)} \overset{(\star)}{\leq} \rho \sum_{e \in E(G)} \alpha_e$ for every coefficient α_e

$\boxed{\text{Note that } \sum_{e' \in \text{unique path } e \text{ in } T} d_G(e') = \sum T_{ee'} d_G(e')}$

Now we need to prove there is a probabilistic mapping from E using trees with congestion at most ρ. by lm 2: it suffices to prove that for every non-negative coefficient $\beta_{e'}$ there is a tree T such that $\sum_{e' \in E(G)} \beta_{e'} \frac{load_T(e')}{c_G(e')} \leq \rho \sum_{e' \in E(G)} \beta_{e'}$. Thus $\sum_{e \in E(G), e' \in E(G)} \beta_{e'} T_{ee'} \frac{c_G(e)}{c_G(e')}$ ~~choosing~~ $\overset{(\star\star)}{\leq} \rho \sum_{e \in E(G)} \beta_{e'}$ should be correct for every non-negative coefficient $\beta_{e'}$.

Now choosing $\alpha_e = \beta_e$ and $d_G(e') = \frac{\beta_{e'}}{c_G(e')}$ (and likewise $d_G(e) = \frac{\beta_e}{c_G(e)}$) and substituting in inequality ($\star$), we obtain ($\star\star$). □

[Note that in general $T: E \to \mathbb{P}$ maps to every edge $e \in E$ a path $p \in \mathbb{P}$ and $T_{ee'}$ counts the number of times the edge e' appears on the path $T(e)$.]

Since there $O(\log n)$ for trees and $\tilde{O}(\log n)$ for subtrees, we obtain the same results for congestion. We can obtain a polynomial algorithm by duality (min max).

$\boxed{\text{Note that if we consider embedding into subtrees then } T_{ee'} \text{ is one if } e' \text{ appears in the path unique to } e \text{ in the tree and zero otherwise.}}$

Application to Min Bisection:

Min Bisection: given a graph G with $n=2k$ vertices partition the vertices in two parts with k vertices with minimum width (total capacity of edges with end-points in different sides of bipartization)

algo: find the best solution in each of the spanning trees of the distribution and take the best. (in trees we can solve it with dynamic program).

First note that $\text{load}_T(e)^{e \in T}$, in this case is sum of capacities of edges of G with one end-point in each path $\delta(e)$ in T (for $e \notin T$ the load is zero)

Note that if opt_i is the mapping of opt in tree T_i (delete the edges of opt if they exist in T_i), in this case we pay $\text{load}_T(e)$. Due to probabilistic embedding

$\tilde{O}(\log n)\, opt \geq \sum_T P_T\, opt_T \geq \sum_T P_T\, C_T \geq \sum_T P_T\, C^* \geq C^*$ Note that by ** $\boxed{\sum_i \lambda_i \text{load}_{T_i}(e) \underbrace{\leq}_{B} \tilde{O}(\log n) c_e}$

where C_T is the cost of best solution for tree T and C^* is the cost of best overall solution in the T^*

Note that now if we map C^* back into graph G, and we cut all edges in $\text{load}_{T^*}(e)$ then we have a solution of cost at most C^*, since the load terms count every edge of G cut by the solution corresponding to C^* at least once and perhaps multiple times and also possibly count edges of G not in the bipartization (this is the domination property) $\square$

We can obtain $O(\log n)$ approximation instead of $\tilde{O}(\log n)$ if we use embedding into probabilistic trees and not subtrees.

Note that roughly speaking distance $= \frac{1}{\text{capacity}}$ in these arguments and this the main intuition.