Due: Feb 21st at the start of class

**Homework #1**

CMSC351H - Spring 2019

PRINT Name: ____________________________ :

- Grades depend on neatness and clarity.
- Write your answers with enough detail about your approach and concepts used, so that the grader will be able to understand it easily. You should ALWAYS prove the correctness of your algorithms either directly or by referring to a proof in the book.
- Write your answers in the spaces provided. If needed, attach other pages.
- The grades would be out of 16. Four problems would be selected and everyone’s grade would be based only on those problems. You will also get 4 bonus points for trying to solve all problems.

1. [Prob 2.3, Pg 31] Find the following sum and prove your claim:

   \[ 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n + 1) \]

Denote the sum we are interested in as \( F(n) = 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n + 1) \). To find \( F(n) \) we subtract the sum \( S(n) = \sum_{i=1}^{n} i \) to find that

\[
F(n) - S(n) = 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n + 1) - 1 - 2 - \cdots - n
= (1 \cdot 2) - (2 \cdot 3) - 2 + \cdots + (n(n + 1) - n)
= 1 \cdot (2 - 1) + 2 \cdot (3 - 1) + \cdots + n \cdot (n + 1 - 1)
= 1^2 + 2^2 + \cdots + n^2.
\]

In class we saw that \( S_2(n) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \), so that

\[
F(n) = S(n) + S_2(n) = \frac{n(n + 1)}{2} + \frac{n(n + 1)(2n + 1)}{6} = \frac{n(n + 1)(n + 2)}{3}
\]

We now prove that by induction. For \( F(1) \) this is trivial. Assume that \( F(n) = \frac{n(n+1)(n+2)}{3} \), we now show that \( F(n + 1) = \frac{(n+1)(n+2)(n+3)}{3} \). We have that

\[
F(n + 1) = F(n) + (n + 1)(n + 2)
= \frac{n(n + 1)(n + 2)}{3} + (n + 1)(n + 2)
= \frac{(n + 1)(n + 2)(n + 3)}{3}.
\]
2. [Prob 2.11, Pg 32] Find an expression for the sum of the $i$th row of the following triangle, and prove the correctness of your claim. Each entry in the triangle is the sum of the three entries directly above it (a nonexisting entry is considered 0)

\[
\begin{array}{cccccc}
1 & 1 & 1 & & & \\
1 & 2 & 3 & 2 & 1 & \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1
\end{array}
\]

The sums of each of the first few rows are 1, 3, 9, 27 which suggests that the sum of the $n$th row is $3^{n-1}$. We prove this by induction. For $n=1$ this is trivial. Assume the sum of the $n$th row is $3^{n-1}$, we now show that the sum of the $(n+1)$-th row is $3^n$. Let $a_{i(n+1)}$ denote the $i$th entry in the $(n+1)$-th row and note that, by definition, $a_{i(n+1)} = \sum_{j=i-1}^{i+1} a_{jn}$. We need to show that $\sum_{i=1}^{2n+1} a_{i(n+1)} = \sum_{i=1}^{2n+1} \sum_{j=i-1}^{i+1} a_{jn} = 3^n$. Each entry from the $n$-th row appears three times in this expression, so $\sum_{i=1}^{2n+1} a_{i(n+1)} = 3 \sum_{j=0}^{2n+2} a_{jn} = 3 \cdot 3^{n-1}$ by the inductive hypothesis which proves the result.
3. [Prob 2.12, Pg 32] Prove that, for all \( n > 1 \), \( \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{13}{24} \).

Let \( P(n) \) be the above inequality. We prove that \( P(n) \) is correct for all \( n > 1 \) using induction. The base case \( P(2) \) is trivial. Now assuming that \( P(n-1) \) is correct we will show that \( P(n+1) \) is correct:

\[
\frac{1}{n+1} + \cdots + \frac{1}{2n} = \frac{1}{(n-1) + 1} + \cdots + \frac{1}{2(n-1) + 1} + \frac{1}{2n - 1} + \frac{1}{2n} - \frac{1}{(n-1) + 1} > \text{by } P(n-1) \frac{13}{24} + \frac{1}{2n - 1} + \frac{1}{2n} - \frac{1}{n}
\]

Now we can easily show that \( \frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{n} \) is non-negative for \( n > 1 \) and the proof is complete.
4. [Prob 2.20, Pg 32] Prove that the regions formed by \( n \) circles with one chord (see Figure) can be colored with three colors such that any neighboring regions are colored differently.

We prove this by induction on the number of circles. The base case of \( n=1 \) is trivial (remember the region outside the circle). The following algorithm produces a valid coloring: 1) assume the \( n \)th circle is not drawn; 2) color the regions formed by the first \( n-1 \) circles with three colors (which you can do from the induction hypothesis); 3) now place the \( n \)th circle without its chord and change the colors of regions “in” the \( n \)th circle according to some order (e.g. red to green, green to blue and blue to red); 4) now add the chord to the \( n \)th circle and change the colors of regions “in” the circle again, but on only one side of the chord in the same order. This is the new valid coloring. Why?
5. [Prob 2.27, Pg 34] Put \( n \) points on the boundary of a circle, and connect each point to all the others by a line segment. Assume that no three line segments meet at a point. Calculate the number of regions formed by these line segments inside the circle, and prove your claim.

If we remove a line segment with \( k \) intersections with other line segments and keep all the line segments except that one, the number of regions decreases by \( k + 1 \). That is one for the line segment itself, and one for each intersection the line segment has with other line segments. This suggests that the number of regions is the number of line segments plus the number of intersections. The number of line segments is \( \binom{n}{2} \) since there is a line segment for each pair of points. It is easy to see that since no three line segments meet at one point, there is exactly one intersection for each combination of 4 points out of \( n \) points. Thus, the number of intersections equals to \( \binom{n}{4} \). Therefore, the number of regions formed by connecting \( n \) points on the boundary of a circle is \( \binom{n}{4} + \binom{n}{2} + 1 \).

We can also prove our claim by induction on the number of points. For \( n = 2 \), the number of regions is 2 = \( \binom{2}{2} + \binom{2}{2} + 1 \). Assume that \( \binom{r}{4} = 0 \) for \( r > n \). We wish to prove the number of regions formed by \( n \) points is \( \binom{n}{4} + \binom{n}{2} + 1 \) given that we know the number of regions formed by \( n - 1 \) points is \( \binom{n-1}{4} + \binom{n-1}{2} + 1 \). If we remove point \( n \) along with all \( n - 1 \) line segments connected to it, the number of regions is decreased by \( n - 1 \) for each line segment plus \( \sum_{i=1}^{n-1}(i-1)(n-i-1) \) for the intersections occurred to these \( n - 1 \) line segments. Let \( a_0, a_1, ..., a_{n-2} \) show the number of intersections of the \( i^{th} \) line segment, starting from 0. The following equation holds for each \( a_i \) since another line segment which intersects the \( i^{th} \) line segment can be chosen from the opposite sides of the circle relative to the \( i^{th} \) line segment.

\[
a_i = (i - 1)(n - i - 1)
\]

Thus, we want to show

\[
\binom{n}{4} + \binom{n}{2} + 1 - \binom{n-1}{2} - 1 = \sum_{i=1}^{n-1}(i-1)(n-i-1) + (n-1).
\]

If we simplify the left hand side, we get the following

\[
4(n-1)(n-2)(n-3) + \frac{2(n-1)}{2} = \frac{(n-1)(n-2)(n-3)}{6} + (n-1).
\]

We can show \( \sum_{i=1}^{n-1}(i-1)(n-i-1) = (n-1)(n-2)(n-3)/6 \) by a similar induction. (How?) Therefore, the number of decreased regions confirms the induction hypothesis.
6. [Prob 3.2, Pg 56] Prove that, if \( f(n) = o(g(n)) \) then \( f(n) = O(g(n)) \). Is the opposite true?

\[
f(n) = o(g(n)) \text{ implies that } \lim_{n \to \infty} \frac{f(n)}{c \cdot g(n)} = 0 \text{ for any constant } c. \text{ We can make the ratio arbitrarily close to 0 for any } N \text{ large enough, which implies } f(n) < c \cdot g(n) \text{ for all } n > N \text{ which shows the result. Setting } g(n) = f(n) \text{ is a trivial counter example of the opposite.}
\]
7. Are the following pairs of functions in terms of order of magnitude. In each case, briefly explain whether \( f(n) = O(g(n)) \), \( f(n) = \Omega(g(n)) \), and/or \( f(n) = \Theta(g(n)) \).

a) \( f(n) = 100n + \log n \quad g(n) = n + (\log n)^2 \)

Using a similar approach to part (c) below you can show that \( f(n) = \Theta(n) \) and \( g(n) = \Theta(n) \), and thus \( f(n) = \Theta(g(n)) \).

b) \( f(n) = \log n \quad g(n) = \log(n^2) \)

Since \( \log(n^2) = 2\log n \), we have \( f(n) = \Theta(g(n)) \).

c) \( f(n) = n^{1/2} \quad g(n) = (\log n)^5 \)

We show \( g(n) = O(f(n)) \) which implies that \( f(n) = \Omega(g(n)) \). Let \( h(n) = \frac{\log n}{2} \), so \( g(n) = (2h(n))^5 = \Theta(h(n)^5) \). Now, \( f(n) = e^{\log n^{1/2}} = e^{h(n)} \). By Theorem 3.1 (Theorem 1, Lecture 4 notes), \( h(n)^5 = O(e^{h(n)}) \), which proves the result.

d) \( f(n) = n^{2^n} \quad g(n) = 3^n \)

We have \( \frac{g(n)}{f(n)} = \frac{1.5^n}{n} \). By increasing \( n \) we can make this ratio arbitrary big, thus \( f(n) = O(g(n)) \).
8. [Prob 3.16, Pg 57] Find a counterexample to the following claim: \( f(n) = O(s(n)) \) and \( g(n) = O(r(n)) \) imply \( f(n)/g(n) = O(s(n)/r(n)) \)

Since we don’t need upper bounds to be tight, finding counterexamples in this case is easy. For example \( f(n) = n \log n, g(n) = n \), and \( s(n) = r(n) = 2^n \), where \( \frac{f(n)}{g(n)} = \log n \), but \( \frac{s(n)}{r(n)} = 1 \).
9. [Prob 3.13, Pg 57] Use the following result: \( \sum_{i=1}^{n} f(i) \leq \int_{x=1}^{x=n+1} f(x) \, dx \) (this is Eq. 3.34 in the book) to show that \( \sum_{i=1}^{n} i^k = O(n^{k+1}) \).

Setting \( f(i) = i^k \), and evaluating the integral we get \( \sum_{i=1}^{n} i^k \leq \frac{1}{k+1} (n + 1)^{(k+1)} \) and the result follows.
10. [Prob 2.18, Pg 32] Given a set of \( n \) points in the plane such that any three of them are contained in a unit-size cycle, prove that all \( n \) points are contained in a unit-size cycle.

We use an induction on the number of points. The statement trivially holds for \( n = 3 \), since there is a circle with radius one covering all the points. We remove the \( n^{th} \) point, and there is a circle \( C \) of radius one covering all \( n - 1 \) remaining points according to the induction hypothesis. By adding the \( n^{th} \) point back, we claim there still exists a unit-size circle covering all the points. Consider a circle of radius 2 centered on point \( n \) and call it \( C' \). We know all the points are contained in \( C' \), otherwise there is no circle covering point \( n \) along with a point outside \( C' \) and another arbitrary point. If circle \( C \) doesn’t cover point \( n \), we can move it closer to point \( n \) in the direction of a vector from the center of \( C \) to point \( n \). We can show if circle \( C \) doesn’t reach point \( n \), it is stuck because of two points \( i \) and \( j \) such that there is no circle of radius one covering \( u, v, \) and \( n \) at the same time. Why?
11. [Prob 2.7, Pg 31] Given a set of $n + 1$ numbers out of first $2n$ natural numbers $1, 2, \ldots, 2n$, prove that there are two numbers in the set, one of which divides the other.

Assume there is a partitioning of the first $2n$ numbers into $n$ disjoint subsets $S_1, S_2, \ldots, S_n$ such that for each pair of numbers $a, b \in S_i$ belonging to a same subset either $a$ divides $b$ or vice versa. We call such a partitioning a good partitioning. For any subset $S' \subseteq \{1, 2, \ldots, 2n\}$ of size $n + 1$, there is at least one $S_i$ which has 2 numbers in common with $S'$. Thus, there are two numbers in $S'$ that one of them divides the other.

We show there is always a good partitioning of $\{1, 2, \ldots, 2n\}$ into $n$ subsets by induction on $n$. For the base case of $n = 1$, we have $S_1 = \{1, 2\}$. Given a good partitioning $S_1, S_2, \ldots, S_{n-1}$ of the first $2(n - 1)$ numbers, we look for a good partitioning of the first $2n$ numbers into $n$ subsets. We create a new subset $S_n = \{2n - 1\}$ and add $2n$ to the subset $S_i$ such that $n \in S_i$. The greatest number in $S_i$ before adding $2n$ should be $n$ since $2n$ is the smallest number which is divided by $n$. Also, all other numbers in $S_i$ divide $n$ and therefore $S_1, S_2, \ldots, S_n$ is a good partitioning of the first $2n$ numbers into $n$ subsets.