QUADTREES AND MEDIAL AXIS TRANSFORMS

Hanan Samet

Computer Science Department, University of Maryland
College Park, MD 20742 USA

Abstract

The skeleton and medial axis transform concepts used in traditional image processing representations are adapted to the quadtree representation. A new data structure termed the Quadtree Medial Axis Transform (QMAT) is defined. A QMAT results in a partition of the image into a set of non-disjoint squares having sides whose lengths are sums of powers of two rather than, as is the case with quadtrees, a set of disjoint squares having sides of lengths which are powers of two. Some of the interesting properties of the QMAT vis a vis the quadtree are its compactness and a decreased sensitivity to shift.

1. Introduction

There are a number of methods of representing images [4] among which are borders, arrays, and skeletons. The quadtree [2,3] is interchangeable [1,7] with these representations and it can also be used to compute a number of diverse geometrical properties [8,9] (see the overview in [6]). In this paper we demonstrate the usefulness of the Chessboard distance transform of [10] in computing the skeleton and medial axis transform [4] of an image represented by a quadtree.

2. The quadtree medial axis transform

We are given an image where the set of points in a certain region are labeled S and the set of points outside of the region are labeled $\overline{S}$ (analogous to BLACK and WHITE respectively). We say that for a point x and a set V, the distance according to a suitably defined distance metric, d, from x to a nearest point of V is $d(x,V)=\min\{d(x,y)|y\varepsilon V\}$. Two points x and y are said to be neighbors if $d(x,y)=1$. We are interested in a subset of S, say T, such that all elements of T have a distance from S which is a local maximum. In other words, for each point in T, no neighboring point in S but not in T has a greater distance from $\overline{S}$. The set of points comprising T is said to constitute a skeletal description of S. As an example, consider the rectangle in Figure 1 whose skeleton consists of line segments labeled a,b,c,d, and e. If we know the points of the skeleton and their associated distance values, then we can reconstruct S exactly. The set of points comprising the skeleton and their associated values is termed the medial axis transform (MAT). The MAT of S provides a concise method of defining and representing S.

Clearly, the definition of the distance metric plays an important rôle in determining the form of the MAT. The most commonly known distance metric is the Euclidean distance

$$d_E(p,q)=\mathrm{SQRT}((p_x-q_x)^2+(p_y-q_y))^2$$

whose maximal blocks are discs. Two other metrics which are more commonly known in digital picture processing are the Absolute Value distance (also known as the City Block distance)

$$d_A(p,q)=|p_x-q_x|+|p_y-q_y|$$

whose maximal blocks are diamonds, and the Maximal Value distance (also known as the Chessboard distance)

$$d_M(p,q)=\max\{|p_x-q_x|,|p_y-q_y|\}$$

whose maximal blocks are squares. Note that in any case, the MAT determines the entire image although it is true that a point in the image may lie in more than one maximal block.

Maximal blocks can be of any size and at any position. Thus they are somewhat unwieldy as primitive elements for representation purposes since the process of determining them may be complex. The quadtree approach to image representation is an attempt to exploit the maximal block concept in a more systematic manner. Given a $2^n \times 2^n$ array of unit pixels, we repeatedly subdivide the array into quadrants, subquadrants, ... until we obtain blocks

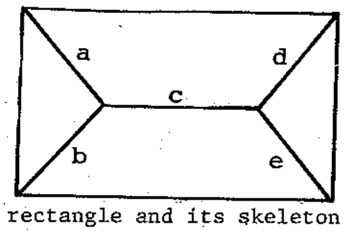
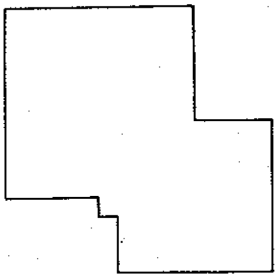


Figure 1. A rectangle and its skeleton using d_E.

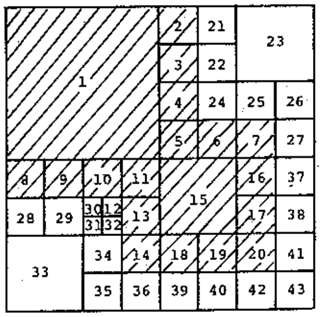
*The support of the U.S. Army Engineer Topographic Laboratory under contract DAAK 70-81-C-0059 is gratefully acknowledged, as is the help of Janet Salzman and Pat Young. The author has benefitted greatly from discussions with Charles R. Dyer, Shmuel Peleg, and Azriel Rosenfeld.

CH1801-0/82/0000/0184 $00.75 © 1982 IEEE

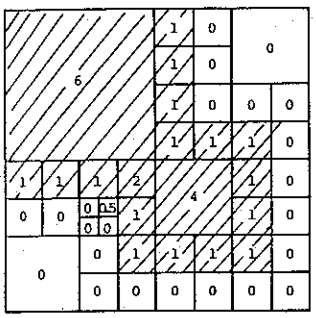
(possibly single pixels) which consist entirely of a single vlaue (e.g., gray level). This process is represented by a tree of out degree 4 in which the root node represents the entire array, the four sons of the root node represent the quadrants, and the terminal nodes correspond to those blocks of the array for which no further subdivision is necessary. The nodes at level k (if any) represent blocks of size $2^k \times 2^k$ and are often referred to as nodes of size 2^k. For example, Figure 2b is a block decomposition of the region in Figure 2a while Figure 2c is the corresponding quadtree. In general, we will be dealing with two values 1 and 0 where BLACK and WHITE square nodes in the tree represent blocks consisting entirely of 1's and 0's respectively. Circular nodes, also termed GRAY nodes, denote non-terminal nodes.



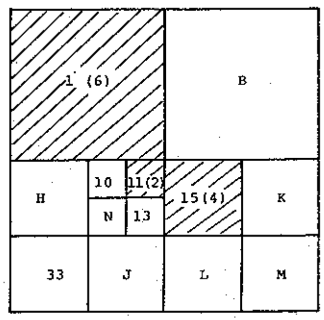
2a. Sample image.



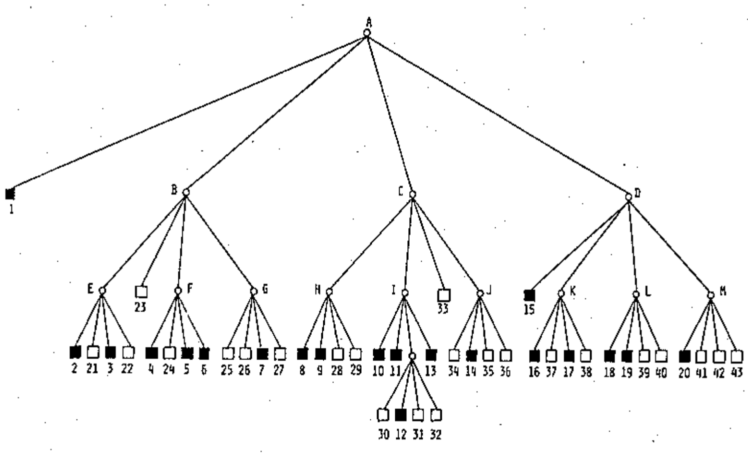
2b. Block decomposition of the image in (a).



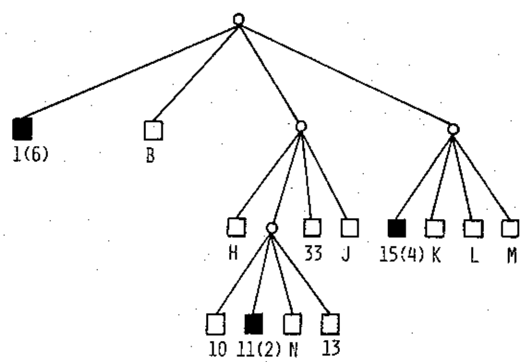
2d. Chessboard distance transform of (b).



2e. Block decomposition of the QMAT of (b). Radius values are within parentheses.



2c. Quadtree representation of the blocks in (b).



2f. QMAT representation of the blocks in (b). Radius values are within parentheses.

Figure 2. An image, its maximal blocks, the corresponding Chessboard distance transform, the block decomposition of the QMAT, and the QMAT. Blocks in the image and in the QMAT are shaded.

In [10] the concept of distance is applied to a quadtree. It is shown that the Chessboard distance metric is especially suitable for a quadtree since it has the property that given a point P, the set of points q such that $d(p,q) \leq t$ is a square. The Chessboard distance transform for a quadtree, DIST, is defined as a function that yields for each BLACK block in the quadtree the Chessboard distance from the center of the block to the nearest point which is on a BLACK-WHITE border. More formally, letting x be the center of a BLACK block b, z be a point on the border of the WHITE block W, say B(W), we have

$$F(b,W)=\min_{z \varepsilon B(W)} d(x,z)$$
$$DIST(b)=\min_W F(b,W)$$

We also say that DIST of a WHITE block is zero and that the border is BLACK for the purpose of the computation of F and DIST.

We now define the Quadtree Medial Axis Transform (QMAT). We first define the Quadtree Skeleton. Let the set of BLACK blocks in the image be denoted by B. For each BLACK block, b_i, let $S(b_i)$ be the part of the image spanned by a square with side width $2*DIST(b_i)$ centered about b_i. The Quadtree Skeleton consists of the set T of BLACK blocks satisfying the following properties:

(1) $area(B) = UNION(S(t_i))$

(2) for any $t_j \varepsilon T \not\exists b_k \varepsilon B (b_k \neq t_j) \ni S(t_j) \subseteq S(b_k)$

(3) $\forall b_i \varepsilon B \; \exists t_j \varepsilon T \ni S(b_i) \subseteq S(t_j)$

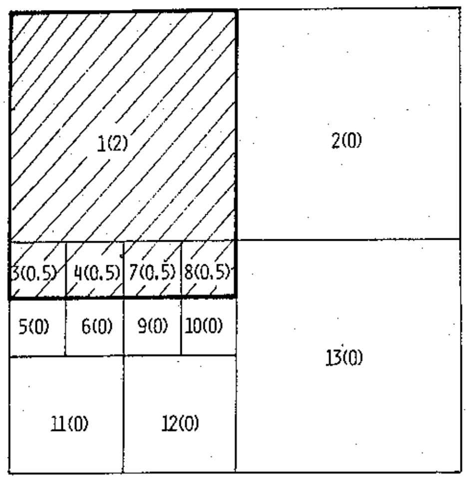
Property (1) insures that the entire image is spanned by the skeleton. Property (2) is termed the subsumption property and we say that b_j is subsumed by b_k when $S(b_j) \subseteq S(b_k)$. Property (2) means that the elements of the Quadtree Skeleton are the blocks with the largest distance transform values. Note that this is not the same as saying that for $t_j \in T \not\exists t_k \in T (k \neq j) \ni S(t_j) \subseteq S(t_k)$ as shown in [5]. Property (3) insures that no block in B and not in T requires more than one element of T for its subsumption - e.g., one half of the block is subsumed by another element of T. Using such a definition it is shown in [5] that the Quadtree Skeleton of an image is unique.

The QMAT of an image is the quadtree whose BLACK nodes correspond to the BLACK blocks comprising the Quadtree Skeleton and their associated Chessboard distance transform values. All remaining nodes in the QMAT are WHITE and GRAY with distance value zero. For example, Figure 2d contains the Chessboard distance transform corresponding to the region given in Figure 2a. Figures 2e and 2f contain the block and tree representations respectively of the QMAT of Figure 2a.

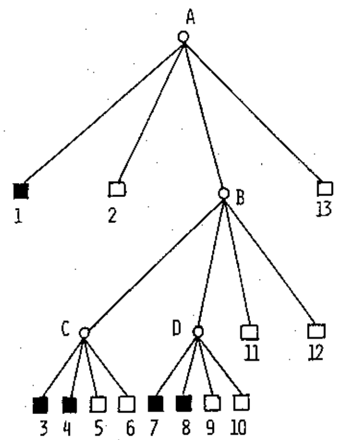
3. Properties of the QMAT

We make the following observations with the aid of Figure 2. The squares spanned by the Chessboard distance transform of the blocks of the QMAT have sides whose lengths are sums of powers of two and they are not ncecessarily disjoint. This is in contrast with the quadtree which is a partition of an image into a set of disjoint squares having sides whose lengths are powers of two.

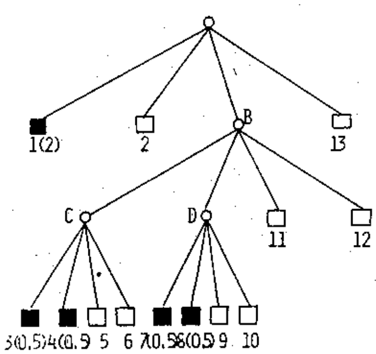
Our interest in the QMAT is more as an alternative data structure for the representation of an image rather than as a skeleton serving to approximate the image. In particular, it has the property that for any image it requires at most as many nodes as the quadtree. This is obvious when we re-



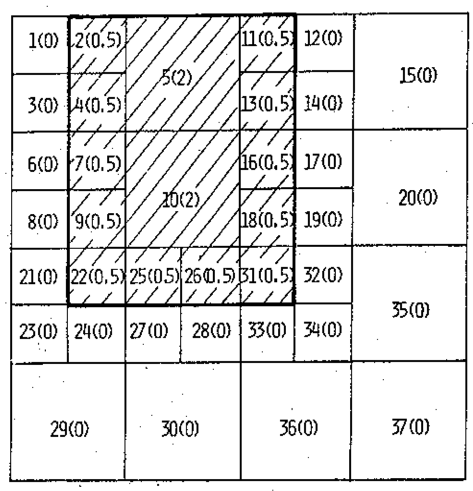
3a. Image. The value of the Chessboard distance transform is within parentheses.



b. Quadtree representation of the image in (a).



3c. QMAT representation of the image in (a). Radius values are within parentheses.



3d. The image in (a) shifted by one unit to the right. The values of the Chessboard distance transform are within parentheses.

Figure 3. An image and its corresponding quadtree and QMAT, and the result of shifting it by one unit to the 'right'. Blocks in the image are shaded.

call that each node in the QMAT corresponds to one or more nodes of the quadtree and that each member of the Quadtree Skeleton is a node in the quadtree. Of course, the QMAT does require that the DIST value be stored with each node. As an example of the savings in storage, consider the image in Figure 2a. The QMAT, shown in Figure 2f, requires 17 nodes while the quadtree, shown in Figure 2c, requires 57 nodes.

The QMAT representation also has the property that the number of nodes necessary to represent an image is not as shift-sensitive as is the quadtree. For example, when the image of Figure 3a is shifted by one unit to the right yielding Figure 3d, its quadtree is considerably larger. In particular, Figure 3b contains 17 nodes while Figure 3e, the quadtree corresponding to the shifted image, contains 49 nodes. However, the QMAT is not as sensitive to shifts since it always requires a number of nodes less than or equal to those contained in the quadtree. In Figure 3, the QMAT of Figure 3a, given in Figure 3c, is identical to the quadtree. However, the QMAT of the shifted image, given in Figure 3f, is considerably smaller than its corresponding quadtree as well as the QMAT of the image prior to the shift (i.e., 9 nodes vs. 17 nodes).

4. Concluding remarks

The concept of a skeleton and medial axis transform have been adapted to images represented by quadtrees and have resulted in the definition of a new data structure termed the QMAT. Note that any operation that can be performed with a quadtree representation can also be performed with the QMAT. Additional properties of the QMAT are discussed in [5]. An algorithm for the construction of the QMAT given a quadtree representation of an image is found in [5].

Our view of the QMAT as an alternative image representation rather than as a skeletal approximation serves to reinforce the appropriateness of the chessboard distance metric [10] for quadtrees in contrast with the city block distance metric [11]. In particular, the analogy between squares and circles as the basis for the QMAT is noteworthy.

References

1. Dyer, C. R., Rosenfeld, A. & Samet, H. Region representation: boundary codes from quadtrees, Communications of the ACM, March 1980, 171-179.

2. Hunter, G. M. & Steiglitz, K. Operations on images using quadtrees, IEEE Transactions on Pattern Analysis and Machine Intelligence 1, 1979, 145-153.

3. Klinger, A. Patterns and search statistics, in Optimizing Methods in Statistics, 1971, J. S. Rustagi, Ed., Academic Press, New York.

4. Rosenfeld, A. & Kak, A. C. Digital Picture Processing, 1976, Academic Press, New York.

5. Samet, H. A quadtree medial axis transform, Computer Science TR-803, University of Maryland, College Park, MD, August 1979.

6. Samet, H. & Rosenfeld, A. Quadtree representations of binary images, Proceedings of 5ICPR, Miami, Florida, 1980, 815-818.

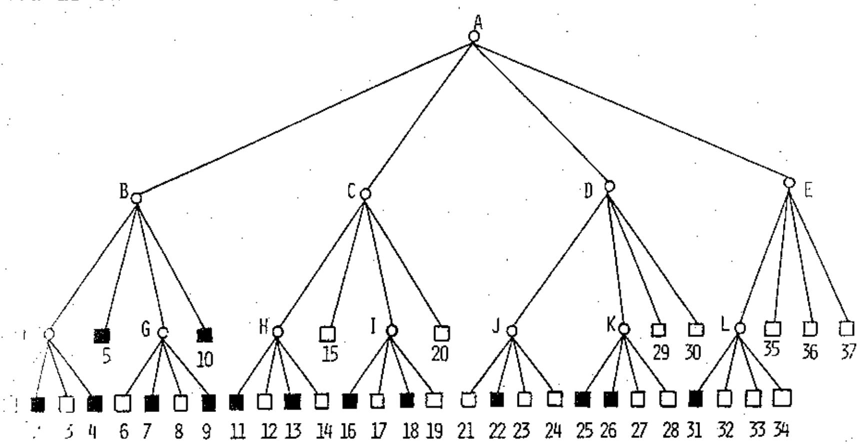
7. Samet, H. An algorithm for converting rasters to quadtrees, IEEE Transactions on Pattern Analysis and Machine Intelligence 3, 1981, 93-95.

8. Samet, H. Connected component labeling using quadtrees, Journal of the ACM, July 1981, 487-501.

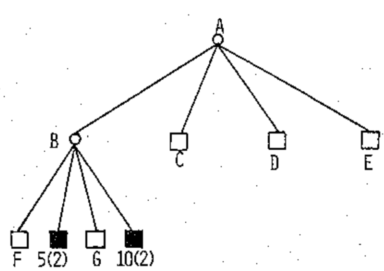
9. Samet, H. Computing perimeters of images represented by quadtrees, IEEE Transactions on Pattern Analysis and Machine Intelligence 3, 1981, 683-687.

10. Samet, H. Distance transform for images represented by quadtrees, IEEE Transactions on Pattern Analysis and Machine Intelligence 4, 1982, 298-303.

11. Shneier, M. Path-length distances for quadtrees, Information Sciences 23, 1981, 49-67.



Quadtree representation of the image in (d).



3f. QMAT representation of the iname in (d). Radius values are within parentheses.

Figure 3, cont'd.