53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

Social Network Ad Allocation via Hyperbolic Embedding

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Abstract— **With the increasing popularity and ubiquity of online social networks (SNS), many advertisers choose to post their advertisements (Ads) within SNS. As a central problem for Ad platforms, Ad allocation is to maximize its revenue without overcharging advertisers, and it has received increasing attention from both industry and academia. The offline approach is a high dimensional integer programming problem with constraints incorporating potential allocation requirements from advertisers. In this paper we investigate the SNS Ad allocation problem in a single target group setting, study the connection of SNS advertising and hyperbolic geometry, and propose an approximation using hyperbolic embedding, which not only reduces the dimensionality of SNS Ad allocation problem significantly, but also provides a general framework for designing allocation strategies incorporating business rules. We evaluate the optimality and efficiency of our approach.**

I. INTRODUCTION

In modern social networks (SNS), users expose many personal behaviors and connect to each other based on real world relationships, which makes SNS ideal for target advertising. Advertisers publish their product advertisements within social networks via advertisement platforms (e.g. Facebook), who allocate each advertisement (Ad) to users impressions (i.e. when user is reading a page).

The advertising mechanism used by online Ad platforms, including social network websites, is essentially large auctions where advertisers place bids on user impressions with specified daily or total budgets [1]. In the cost-per-mille model, the advertisement platforms receive the commission for one thousand user impressions displaying the Ad. The Ad allocation problem is a central problem for online Ad platforms: how to maximize revenue while respecting advertisers bid and budget constraints.

Previous research is mainly focusing on search engine settings, where the impression is ad-hoc and associated with search queries. The Ad allocation problem has been formulated as a bipartite matching problem, and several algorithms have been used in practice [1].

Different from search engine setting, in the SNS setting, instead of keywords, each advertiser bids for a target group of users, where the impression of the user is by no means of unit value, and is no longer ad-hoc. Consider a user engaging with the Ad (e.g. "like"in Facebook), her friends in the ego-network can see the Ad and potentially engage with it as well. Without considering the social influence in the Ad allocation, one can easily exceed advertisers' budgets and waste valuable user impressions. Furthermore, the advertising platform needs to define and consider the domain constraints of the Ad allocation to advertisers. For example, there can be a constraint on fairness (i.e. users allocated to advertisers have similar influence distributions), or asking higher price for more influential users.

How to allocate the impressions of users to a set of advertisers with bid constraints while considering the social influence as well as domain constraints at the same time is not well studied. In this paper, we focus on the homogeneous setting, where all advertisers bid on the same group of users (e.g. the whole set of SNS users).

A classic and intuitive approach to tackle the SNS Ad allocation problem is to formulate it as an integer programming (IP) problem, which has two disadvantages. First, the decision variable representing the allocation profile is defined in $N^{|A|\times|V|}$ with A the advertiser set and V the user set, which makes it less efficient in large-scale problems. Second, the domain constraints, such as allocating all advertisers the same user influence distribution, are hard to describe.

In this paper, we propose a novel formulation by mapping the network to a hyperbolic plane [2] to improve the offline running time for the problem (Sec. III). By doing so, we are able to approximate the large scale user-wise calculations with region-wise integrals, changing the discrete domain to a continuous domain, which enables describing the allocation strategy using a 2-D geometry shape and largely reduces the dimensionality (in the order of $O(V)$). On the other hand, region-wise Ad allocation is a convenient framework for representing and visualizing domain constraints. We further develop the optimization process by using impression decomposition that divides the problem into a series of smaller and simpler ones without introducing strong assumptions (Sec. IV-A), and discuss different allocation strategies and their implications to domain constraints (Sec. IV-B). We show the optimality and efficiency with experiments (Sec. V). Finally in Sec. VI we summarize our work and discuss future directions.

II. PRELIMINARIES

A. Advertisement Allocation in Social Networks

AdWords [1], [3], [4] proposed by Mehta et al. solves Ad allocation problem in search engine setting in Google. In its setting, the agent allocates impressions resulting from search queries to advertisers, with each advertiser having a budget constraint on the total spend. Each bidder puts in a

Research partially supported by Maryland Procurement Office contract H98230-14-C-0127, by US Air Force Office of Scientific Research MURI grant FA9550-10-1-0573, and by National Science Foundation grant CNS-1018346

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set of bids for different keywords relevant to the Ad. Once an advertiser's budget is exhausted, it cannot be allocated any more queries. The objective is to maximize the total amount of money (budget) spent by the advertisers, in other words to maximize the total efficiency of the matching. The offline algorithm is formulated as an integer programming (IP) [3]. Due to incomplete information and problem size, AdWords is solved as an online optimization problem in practice, which has achieved near $(1-\frac{1}{e})$ optimality for the worst case [1], [3], [4].

In SNS setting, users' daily impression can be derived from usage history, which makes offline optimization approachable. The offline optimization result hints how to leverage social connections to improve the revenue, and gives guidance in designing online optimization algorithms. Consider the single target group setting, where the target group is the same among advertisers, each advertiser $a_j \in A = \{a_1,...,a_{|A|}\}$ bids p_j for all users in the social network. The agent assigns user impressions to a_j before exhausting its budget b_j. The allocation problem here is to maximize the total amount of money (revenue) spent by advertisers. Comparing with AdWords, there are three major differences in SNS setting. First, single user may have multiple impressions that can be assigned to different Ads, comparing to the ad-hoc query-Ad matching. Second, user engagement (for example 'like' a sponsored Ad) incurs influence over other users connected in the network, due to the interconnection nature of social networks. Third, there are domain constraints like fairness within the system. Thus the optimization formulation needs to be adjusted to accommodate the issue of multiple impressions, social influence, and domain constraints.

The offline advertisement allocation problem in this setting can be formulated as an integer programming (IP) problem:

$$\begin{aligned}
\max_{S,I} \quad & \sum_{j=1}^{|A|} p_j \sum_{u_i \in S_j} I_{i,j}\, g(u_i) \\
\text{subject to} \quad & p_j \sum_{u_i \in S_j} I_{i,j}\, g(u_i) \le b_j, \quad \forall a_j \in A \quad \text{(budget)} \\
& \sum_{u_i \in S_j} I_{i,j} \le I_i, \quad \forall u_i \in S \quad \text{(impression)} \\
& I_{i,j} \in \mathbb{N}^+, \quad \forall u_i \in S, a_j \in A \\
& (S,I) \in R_D \quad \text{(domain)}
\end{aligned} \tag{1}$$

where $a_j \in A$ is an Ad with bid p_j, u_i is a user (node) in the network with impression $I_i = \sum_{j=1}^{|A|} I_{i,j}$ sum of impressions assigned to all Ads. $g(u_i)$ is the function that describes the social influence of u_i. S and I are optimization variables, where $I = \{I_{i,j} | u_i \in V, a_j \in A\}$ is the impression allocation strategy for users (nodes) in the network, with V the whole user set. $S = \{S_1,...,S_{|A|}\}$ is the Ad allocation profile, with $S_j = \{u_i | I_{i,j} > 0\} \subseteq V$ the set of users assigned to a_j. R_D is the feasible set determined by domain constraints. All possible solution (S,I) must be compatible with domain constraints.

We discuss fairness as an important example for domain constraints, since it appears as a common requirement and business model in SNS Ad platforms, however our method can be extended to other constraints as well.

According to the differences of allocated user influence demography among Ads, we classify the fairness related domain constraints into three major categories:

1) *Fairness model:* User influences (degree) demography among advertisers are required to be similar. Correspondingly, the fairness constraint over the allocation strategy S in the optimization problem can be analytically expressed as:
$$var(\phi(S)) \le \eta \tag{2}$$
where $\phi(S) = (\phi(S_1), \cdots, \phi(S_{|A|}))$ is the fairness measure of user demography over the vector of optimal allocation; greater $\phi(\cdot)$ corresponds to higher ratio of influential users. Here we use variance to reflect the demography difference, with η as the threshold.
2) *Priority model:* Contrary to the fairness model, the priority model requires more influential users allocated to advertisers of higher priority (e.g. with higher bids). The allocation constraint for the priority model can be described as:
$$\phi(S_j) \le \phi(S_l) \qquad \forall a_j, a_l \in A, \rho_j \le \rho_l \tag{3}$$
where ρ_j is Ad a_j's priority, and greater value represents higher priority.
3) *Partial Fairness model (Hybrid model):* If we want both fairness and priority to co-exist in advertisement allocation (i.e. low bid advertiser is allowed to have some higher influence users), then the allocation strategy should consider both sides:
$$\begin{aligned}
& var(\phi(S)) \in [\underline{\eta}, \overline{\eta}] \\
& \phi(S_j) \le \phi(S_l) \qquad \forall a_j, a_l \in A, \rho_j \le \rho_l
\end{aligned} \tag{4}$$
where $\underline{\eta}$ and $\overline{\eta}$ are the lower and upper bounds for the variance.

B. Hyperbolic Embedding

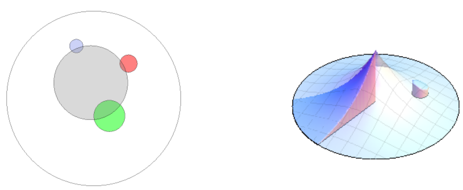
Hyperbolic embedding was developed as a geometric framework to study the structure and function of complex networks by Kleinberg et. al [5] and Krioukov et. al [2] respectively. With the assumption that hyperbolic geometry underlies complex networks like the Internet and social networks, it has been shown [2], [6] that heterogeneous degree distributions and strong clustering in complex networks emerge naturally as simple reactions of the negative curvature and metric property of the underlying hyperbolic geometry. Meanwhile, a network would have an effective hyperbolic geometry underneath if the network is scale-free with heterogeneous degree distribution. Currently hyperbolic embedding [7] has been applied in several areas, including navigability analysis [8], routing algorithms design [9], [10] and link prediction [6], [7] in complex networks.

By utilizing the relation between hyperbolic geometry and properties of complex networks, Krioukov et. al [7] designed HyperMap, a mapping scheme between hyperbolic geometry and statistical mechanics of complex networks. For a social network, it can be mapped into a 2-dimensional disc of hyperbolic space $\mathbb{H}^2$ in 2-D Poincaré model, with each node assigned a virtual coordinate. This mapping captures the most important features of social network, which are small-world effect, power-law degree distribution (scale-free effect) and community structure [7], [11]. After mapping, both expected degree and node density are well-defined within the network radius $R \in [0,1)$; expected degree $E[deg(r)] = p_d(r) \propto e^{-r/2}$, and node density $p_n(r) \propto e^r$, where $r \in [0,R]$ is the distance towards the center of the Poincaré disc.

Due to well-defined degree distribution and node density, this geometric framework can be applied in formulating the optimization problem. It can capture the influence factor in SNS and approximate sum over nodes by the integral over a certain area, which could largely reduce the dimensionality and simplify the optimization problem.

III. PROBLEM DEFINITION

As we mentioned in Sec. II, using hyperbolic embedding, we can map the induced subgraph of bidding category users to a Poincaré disc C into 2-D hyperbolic space. Under single target group setting, we can use uniformly random angle assignment instead, which runs as a linear time embedding algorithm. In this disc, it has uniform node density and exponentially distributed degree along the radius, which is important to formulate the influence effect in the SNS. As shown in Fig. 1, the base circle is the Poincaré disc.



(a) Ad Allocation on Poincaré disc (b) Influence in SNS

Fig. 1. Hyperbolic embedding based SNS Ad allocation

The Ad allocation problem, after the hyperbolic embedding, is to assign each Ad a region of population according to an allocation strategy to maximize the revenue. In Fig. 1(a), the three circles represent the three sets of users assigned to three different advertisers. In Fig. 1(b), we give an example of assigning two advertisers using different geometric shape on C, where the surface on top of C is the degree distribution which describes the influence capability of the user at her position. The allocation is shown as a column corresponding to an area in the circle of the network in the 3-D plot. When considering the impression, the actual profit that the Ad agent can get from allocating the Ad at a user u_i is proportional to

$$I_i \cdot (1 + w \cdot \delta_i) \tag{5}$$

where I_i is the impression of the user u_i, δ_i is the degree of the user. w is a constant presenting click through rate (CTR), which is around 0.003 [12] and a property of the Ad. Without loss of generality, we treat it as a constant. The influence $w \cdot \delta_i$ expresses the influence that shows how many neighbors of u_i are able to see the allocated Ad if u_i clicks it. Since the agent charges the same price for the influence, it should be considered in the optimization. Note that social influence should be related to multi-hop neighbors [13], however in our application scenario, as the CTR is very small, the influence over multi-hop neighbors is negligible.

With the influence definition, we give our formulation of the SNS Ad allocation in single target group setting.

Problem 1: Optimal SNS Ad allocation problem: On a 2-D Pincaré disc C with interior representing user set V, each user $u_i \in V$ has a virtual position (r_i, θ_i), degree $d_i = e^{-r_i/2}$ and impression I_i. Given a set of Ads A, each $a_j \in A$ has a budget b_j, bidding price p_j, we allocate an area S_j (of certain geometry shape) to a_j. To maximize the revenue of the SNS (Ad agent) is to solve the following optimization problem:

$$\begin{aligned}
\max_{(S,I)} \quad & \sum_{j=1}^{|A|} p_j f_j(S,I) && \text{(volume)} \\
\text{s. t.} \quad & p_j f_j(S,I) \leq b_j \qquad \forall j \in \{1,\ldots,|A|\} && \text{(budget)} \\
& \sum_{j=1}^{|A|} \sigma_i(S_j,I) \leq I_i, \qquad \forall u_i \in V && \text{(impression)} \\
& S \in R_D && \text{(domain constraint)}
\end{aligned} \tag{6}$$

where $f_j(S,I)$ is the total impression (volume) assigned to advertiser a_j, and $\sigma_i(S_j,I)$ is u_i's impression assigned to a_j. The allocation profile S lies in the feasible set R_D determined by the given fairness model.

The meta formulation is concretized if one defines an allocation strategy (i.e. shape design) for S, which should be able to cope with business rules. For example, *fan*-shaped S guarantees the fairness model, while *ring*-shaped S represents the priority model. Depending on the actual geometry shape of $S = (S_1, \ldots, S_{|A|})$ chosen, the f_j function varies.

The optimal SNS Ad allocation problem in such formulation is non-trivial to solve. There are three major challenges:

a) **Region Design:** Different geometry shapes have various practical impacts on fairness and complexity. Complex shape will make the volume f_j hard to integrate and loses the advantages of the hyperbolic embedding, as in the worst case we need discretize the region to calculate f_j.

b) **Uncorrelated Impressions:** Although degree distribution is well-defined after hyperbolic embedding, the distribution of impression is still unknown and uncorrelated with degree. Without introducing strong assumptions, this will result in sum over users' impression in S_j to get f_j, which significantly increases the dimensionality of the problem.

c) **Region Overlapping:** Users of multiple impressions could be shown more than 1 Ads, which causes intersections among allocated regions of different Ads, appearing as in the Poincaré circle (e.g. in Fig. 1(a)). Region overlapping makes the overall optimization problem more difficult.

To address these issues, we propose a novel unit-impression based decomposition method (in Sec IV-A) which preserves the advantages of hyperbolic embedding, meanwhile derives an optimal solution without strong assumptions. We further discuss shape design in Sec IV-B. As shown later, some simple shapes such as *Fan* not only give scalable formulation but also have sound practical meanings.

IV. ALGORITHM

A. Unit Impression Decomposition

In order to avoid the issues mentioned above, we introduce the unit impression graph, based on which we discuss our novel decomposition method called *Unit Impression Decomposition* without introducing strong assumptions.

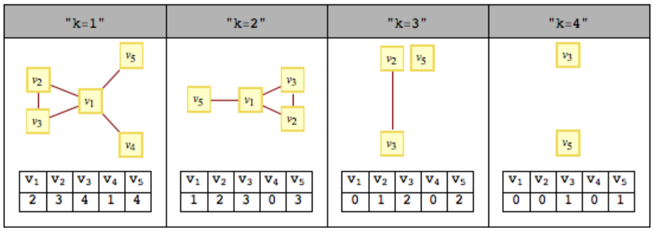


Fig. 2. Illustration of unit impression graph transformation

Definition 1: Unit Impression Graph: For a SNS $G(V,E)$, where V represents users, E represents relationships between user pairs, and each $u_i \in V$ has an impression I_i. G is called a *Unit Impression Graph*, if $\forall u_i \in V$, $I_i = 1$.

The non-overlap constraint emerges naturally with unit impression graphs. Given a SNS, we can induce a set of unit impression graphs. For example, an SNS represented by the first cell in Fig. 2 with impression vector I listed as the table below can be decomposed into a series of *Unit Impression Graphs* $\{G^{(k)}|k=1,2,3,4\}$. $I^{(k)}$ is the impression vector in k^{th} iteration. When $k=1$, $I^{(1)}=I$, and the first cell shows the induced unit impression subgraph $G^{(1)}$. An optimization is performed in $G^{(1)}$. Assuming all users are allocated, the impression vector $I^{(2)}$ is updated by subtracting 1 from each node. Since v_4's impression is 0 when $k=2$, it is not included in the graph $G^{(2)}$ shown in the second cell. In other words, if a user has no impressions anymore, her friends' clicked Ads will not influence her any more that day, thus v_1 is removed in $G^{(3)}$. The decomposition ends at $k=4$, as no one has nonzero impression afterwards.

Algorithm 1 Multi-Stage Optimization with Unit Impression

Let K be the max impression of users $\max\{I_i|u_i \in V\}$
Construct first unit graph $G^{(1)} = (V^{(1)}, E^{(1)})$: $V^{(1)} = V, E^{(1)} = E$
Let remaining Ad $A^{(1)} = A$, budget $B^{(1)} = B$
Initial population $P^{(1)} = P$
for $k = 1$ to $K-1$ **do**
  Run optimization over $G^{(k)} = (V^{(k)}, E^{(k)})$ in Eq. 7
  Reach optimal solution $(S^{(k)*}, I^{(k)*})$
  for each $a_j \in A^{(k)}$ update budget **do**
    $b_j^{(k+1)} = b_j^{(k)} - p_j f_j(S_j^{(k)*})$
    $A^{(k+1)} = A^{(k)} \setminus a_j$ (remove a_j in next round) **if** $b_j^{(k+1)} = 0$
  end for
  for each $u_i \in V^{(k)}$ update her impression **do**
    $I_i^{(k+1)} = I_i^{(k)} - 1$
    Remove u_i and her edges from $G^{(k+1)}$ **if** $I_i^{(k+1)} = 0$
  end for
end for
for each advertiser $a_j \in A$ **do**
  Optimal solution $S_j^* = \cup_{k=1}^{K} S_j^{(k)*}$
end for

With the *Unit Impression Decomposition*, we can solve the original problem using a multi-stage optimization process, as shown in Alg. 1. In each iteration we solve the sub-step problem shown in Eq 7. In iteration k, users assigned to Ads in last iteration will subtract theirs impression by 1. If impression is 0, the user is removed from her neighbor's adjacency list and k^{th} unit impression graph $G^{(k)} = (V^{(k)}, E^{(k)})$, with population $P^{(k)}$ updated accordingly. For $\forall a_j \in A^{(k-1)}$, her budget $b_j^{(k)}$ is updated as $b_j^{(k-1)} - p_j f_j(S_j^{(k)*})$. If $b_j^{(k)} > 0$, a_j will participate in the k^{th} iteration. After mapping $G^{(k)}$ onto Poincaré disc, the optimization (Eq. 7) over $G^{(k)}$ is conducted. The whole process finishes when all Ads are allocated or all budgets are used.

$$
\begin{aligned}
\max_{S^{(k)}} \quad & \sum_{j=1}^{|A^{(k)}|} p_j f_j(S^{(k)}, I) = \sum_{j=1}^{|A^{(k)}|} p_j f_j(S_j^{(k)}) \\
\text{s. t.} \quad & p_j f_j(S_j^{(k)}) \le b_j^{(k)} \qquad \forall j \in \{1, \dots, |A^{(k)}|\} \\
& S_i^{(k)} \cap S_j^{(k)} = \emptyset \qquad u_i \in V^{(k)}, \forall j \in \{1, \dots, |A^{(k)}|\} \\
& \cup_{j=1}^{N} S_j^{(k)} \le P^{(k)} \\
& (S, I) \in R_{\mathbb{D}}
\end{aligned} \tag{7}
$$

The maximum number of iterations is bounded by $\max\{I_i|i \in V\}$, i.e. maximum user impression, which is finite. If the optimization has n rounds, the optimal allocation of Ad a_j is an aggregation of assigned areas: $\cup_{k=1}^{n} S_j^{(k)*}$.

The unit impression decomposition process largely simplifies the optimization problem in each stage. The original problem of solving multiple-location Ad allocation with overlapping can be transformed to a multi-stage, single-location Ad allocation problem without overlapping.

B. Allocation Strategies and Impacts

Choosing different geometry shapes defines different optimal SNS Ad allocation problems. In this section, we analyze the impact of different allocation strategies w.r.t. convexity, efficiency, and fairness. We discuss *Fan*, *Ring*, *Circle*, and general shapes. Because of the impression decomposition, we only consider non-intersection case.

1) **Fan:** We propose the *Fan* shaped Ad allocation strategy for the *fairness model* discussed in Sec. II, where each advertiser is assigned a fan area, as shown in Fig. 3. The allocation area S_j for advertisement j is a fan (or pie) of angle θ_j in the Poincaré circle of network. As expected degree is exponentially distributed along the radius, thus at point (x', θ'), it will be:

$$\delta(x', \theta') = \delta(x') = c \cdot e^{-\frac{x'}{2}} \tag{8}$$

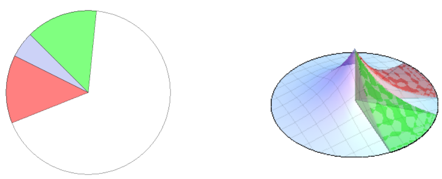
The corresponding volume function $f_j(S_j)$ is:

$$
\begin{aligned}
f_j(S_j) = f_j(\theta_j) &= a \int_0^R e^{\tau} \int_0^{\theta_j} (1 + w \cdot \delta(\tau))) d\alpha d\tau \\
&= a \cdot \theta_j (2wc(e^{\frac{R}{2}} - 1) + e^R - 1) = \alpha \cdot \theta_j
\end{aligned} \tag{9}
$$

with $\alpha = a(2wc(e^{R/2} - 1) + e^R - 1)$ a constant. Integrating the unit impression decomposition and fan-shaped allocation strategy, we re-formulate the optimization problem in Eq. 7:

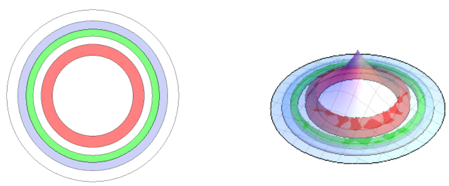
$$
\begin{aligned}
\max_{\Theta} \quad & \sum_{j=1}^{|A^{(k)}|} p_j \alpha^{(k)} \theta_j^{(k)} \\
\text{s.t.} \quad & p_j f_j(\theta_j^{(k)}) \le b_j^{(k)} \qquad \forall j \in \{1, 2, \dots, |A^{(k)}|\} \\
& \theta_j^{(k)} \ge 0 \qquad \forall j \in \{1, 2, \dots, |A^{(k)}|\} \\
& \sum_{j=1}^{|A^{(k)}|} \theta_j^{(k)} \le 2\pi
\end{aligned} \tag{10}
$$

Here $S_j \in R_D$ is eliminated as *Fan* shape enforces the fairness model. With impression decomposition and *Fan* shape Ad allocation strategy, the optimization problem now is a series of linear programming (LP) problems. Comparing with the



(a) *Fan* allocation 2D (b) *Fan* allocation 3D

Fig. 3. Fan allocation strategy



(a) *Ring* allocation 2D (b) *Ring* allocation 3D

Fig. 4. Ring allocation strategy

IP formulation described in Eq. 1, the decision variable $\Theta \in [0, 2\pi]^{|A|}$ in our formulation only has $|A|$ dimensions, which is much lower than $|A| \times |V|$. This improvement is significant as $|A|$ is around one million, but $|V|$ is at billion level [14].

The *Fan* shaped allocation strategy has many advantages:

a) *Convexity:* f_j is a linear function of θ_j (Eq. 9), which makes the optimization problem a linear programming (LP) problem (Eq. 10) and much easier to solve than other shapes.

b) *Efficiency:* Fans of different Ads can be arranged tightly close to each other and impressions can be completely utilized in each round of optimization with enough budgets, thus number of iterations are minimized. Furthermore, residual graphs can be generated independently, thus all iterations can run in parallel with careful budget arrangement.

c) *Fairness:* The *fairness model* is well supported, since all the areas allocated to Ads have similar demography due to well-defined node density and expected degree distribution.

2) **Ring:** When considering the *priority model* by assigning more influential users to higher priority bidders, we propose to use the *Ring* shaped allocation strategy (Fig. 4). Let $r_{j,s}$ and $r_{j,e}$ be the starting and ending radius, and ρ_j be the priority value of a_j, the expression for f_j over the ring $[r_{j,s}, r_{j,e})$ in Poincaré disc is a function of $r_{j,s}$ and $r_{j,e}$:

$$\begin{aligned} f_j(S_j) = f_j(r_{j,s}, r_{j,e}) &= a \int_{r_{j,s}}^{r_{j,e}} e^{\tau} \int_0^{2\pi} (1 + w \cdot \delta(\tau))) d\alpha d\tau \\ &= 2\pi a \; (2wc \cdot e^{\frac{r_{j,e}}{2}} - 2wc \cdot e^{\frac{r_{j,s}}{2}} - e^{r_{j,s}} + e^{r_{j,e}}) \end{aligned} \tag{11}$$

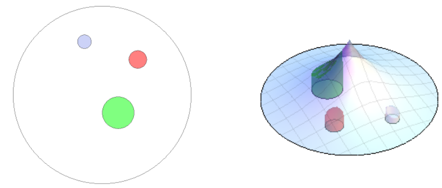
The optimization can be formulated as:

$$\begin{aligned} \max_{S^{(k)}} \quad & \sum_{j=1}^{|A^{(k)}|} p_j f_j(r_{j,s}^{(k)}, r_{j,e}^{(k)}) \\ \text{s.t.} \quad & p_j f_j(r_{j,s}^{(k)}, r_{j,e}^{(k)}) \le b_j^{(k)} \qquad \forall j \in \{1, 2, ..., |A^{(k)}|\} \\ & 0 \le r_{j,s}^{(k)} \le r_{j,e}^{(k)} \le R \qquad \forall j \in \{1, 2, ..., |A^{(k)}|\} \\ & r_{j,e}^{(k)} \le r_{l,s}^{(k)} \qquad \forall \rho_l \le \rho_j \end{aligned} \tag{12}$$

where the last constraint abstracts the priority model, that Ads of higher priority are arranged in inner area. Decision variable $((r_{1,s}, r_{1,e}), \cdots, (r_{|A|,s}, r_{|A|,e})) \in \mathbb{R}^{2|A|}$.

The features of the *Ring* shaped allocation strategy are:

a) *Convexity:* As we can see in Eq. 11, the volume function $f_j(r_{j,s}, r_{j,e})$ is nonlinear in $r_{j,e}$ and $r_{j,s}$.



(a) *Circle* allocation 2D (b) *Circle* allocation 3D

Fig. 5. Circle Ad allocation strategy

b) *Efficiency:* Similar to fan-shape, rings of different Ads can be arranged tightly and impressions can be completely utilized in each round of optimization. Sub-step iterations are minimized and parallel iteration could be supported.

c) *Fairness:* Ring shaped allocation strategy represents *priority model*, in the sense that Ads of different priorities have different demographical population.

3) **Circle:** Another natural idea to allocate Ads within hyperbolic embedded SNS is to use circle as the allocation region, as shown in Fig. 5. It's a potential solution to incorporate the *partial fairness model* mentioned in Sec. II. The *Circle* allocation strategy for a_j can be represented using center position and radius (x_j, θ_j, r_j). $f_j(S_j)$ therefore is:

$$f_j(S_j) = f_j(x_j, \theta_j, r_j) = a \int_0^{r_j} e^{\tau} \int_0^{2\pi} (1 + w \cdot \delta(dis(x_j, \tau, \alpha))) d\alpha d\tau \tag{13}$$

where $dis(x_j, \tau, \alpha) = \sqrt{x_j{}^2 + \tau^2 - 2x_j \tau cos(\alpha)}$ is the distance between a point (τ, α) from x_j and the disc center.

Using such allocation strategy, the k^{th} round of the sequential optimization problem can be written as

$$\begin{aligned} \max_{x^{(k)}, \theta^{(k)}, r^{(k)}} \quad & \sum_{j=1}^{|A|} f_j(x_j^{(k)}, \theta_j^{(k)}, r_j^{(k)}) \\ \text{s.t.} \quad & f_j(x_j^{(k)}, \theta_j^{(k)}, r_j^{(k)}) \le b_j^{(k)}, \qquad \forall j \in \{1, 2, ..., |A^{(k)}|\} \\ & 0 \le x_j^{(k)}, \; r_j^{(k)} \le R^{(k)}, \qquad \forall j \in \{1, 2, ..., |A^{(k)}|\} \\ & x_j^{(k)} + r_j^{(k)} \le R^{(k)}, \qquad \forall j \in \{1, 2, ..., |A^{(k)}|\} \\ & (x_l^{(k)})^2 + (x_j^{(k)})^2 - 2x_l^{(k)} x_j^{(k)} cos(\theta_l^{(k)} - \theta_j^{(k)}) \\ & \ge (r_l^{(k)} + r_j^{(k)})^2 \qquad \forall j, l \in \{1, 2, ..., |A^{(k)}|\} \end{aligned} \tag{14}$$

Features corresponding to *Circle* allocation strategy are:

a) *Convexity:* The f_j (Eq. (13)) is not convex in $\theta_j^{(k)}$.

b) *Efficiency:* Impressions cannot be fully utilized in each iterations thus more iterations are needed.

c) *Fairness*: The *Circle* allocation reflects the *partial fairness model*, since circles of similar sizes and centers have similar influence demography, while circles at different positions with different radii have different demography. It can be tuned by adding size and position constraints.

4) **General Allocation Strategies:** As we showed earlier, shape design is a powerful and intuitive way to represent domain constraints, such as fairness. It's worth discussing the general allocation strategy to incorporate with other domain constraints and show the limitation of our method.

a) *Convexity:* Convex problems have prominent advantages in solvability, reliability and efficiency. To have convexity, we can design shapes of convex volumes about radial coordinate r and angular coordinate θ. Non-convex volume require to apply numerical or combinatorial methods.

TABLE I

OPTIMAL VALUE OF DIFFERENT APPROACHES

	Network size	1,000	10,000	100,000
Revenue	Baseline IP	108	1157	11703
	Baseline IP (Priority)	108	1157	11700
	Baseline IP (fairness)	108	1157	11703
	Fan shape allocation	108	1156	11669

TABLE II

ALGORITHM EFFICIENCY OF DIFFERENT APPROACHES

	Network size	1,000	10,000	100,000
Runtime (sec.)	IP (fairness)	19.5157	132.9285	500.2050
	Fan shape allocation	0.1092	0.0763	0.0780

b) *Efficiency:* Another important factor of runtime is the number of unit impression graphs. The less unallocated area in one iteration, the fewer iterations needed. Parallel execution could be supported with careful budget arrangement if no residual space left in each iteration.

c) *Domain constraint:* Fairness constraints are well-defined by user influence demography, which allows us to use fan, ring and circle to specify different fairness models. Other business rules that have well-defined metrics over the graph have the potential to apply in our framework.

V. EXPERIMENTS

We conducted experiments to show advantages of our formulation over the original IP formulation. We implemented the hyperbolic embedding algorithm mentioned in Sec. II and the unit graph impression optimization routine in Alg. 1. To model the fairness constraints in the IP formulation, we added the following constraints as R_D in Eq. 1:

1) *Fairness* model: We define the linear constraint as:

$$|\frac{\sum_{u_i \in S_j} d_i}{|S_j|} - d_V| \leq \eta, \qquad \forall j \in \{1,...,|A|\} \qquad (15)$$

where d_i is the degree of u_i, d_V is the average degree of the whole network graph, η is the threshold to measure the deviation of the user influence demography.

2) *Priority* model: We define the linear constraint as:

$$\sum_{u_i \in S_j} d_i \leq \sum_{u_i \in S_l} d_i, \qquad \forall j,l \in \{1,...,|A|\}, p_l \leq p_j \qquad (16)$$

where the constraint enforces advertisers with higher bid have the users with higher influence in the model.

3) Other models can formulate constraints accordingly.

We used the Stanford Network Analysis Platform (SNAP) [15] to generate networks of power-law degree distribution with $\alpha = 2.2$. We generated graphs of size $1,000$, $10,000$ and $100,000$ of which the impressions following a Poisson distribution with mean $\lambda = 10$. We fixed the number of Ads to be 10, each a_j bids $p_j \sim \mathcal{N}(0.1,\ 0.01)$. For $1,000$, $10,000$ and $100,000$ graphs, we generated the budgets of advertisers from normal distributions $\mathcal{N}(15,\ 25)$, $\mathcal{N}(150,\ 2500)$, $\mathcal{N}(1,500,\ 2.5 \times 10^5)$ accordingly. Both baseline IP and our novel approach are based on the same impression decomposition procedure. Without loss of generality, we compare both models via the optimization over the first graph $G^{(1)}$. For the optimization, we used IBM ILOG CPLEX 12.6 for both the baseline IP and *Fan* shape LP formulation.

For optimality, we compared the optimal values of our approach against the baseline IP under various network sizes. As shown in Table I, we notice that (1) different fairness constraints lead to different optimal solutions, but they reach similar optimal values. (2) the approximation approach has impressive performance in approaching the optimal value. In terms of efficiency, we listed the runtime for our *Fan* shape allocation and the baseline IP formulation under fairness model in Table II. Our approach is 2-6 orders of magnitudes faster than IP whose runtime grows linearly with size of the graph, while our approach has constant runtime when number of Ads is fixed. Our algorithm has prominent advantage in efficiency while close to the optimal value.

VI. CONCLUSION

In this paper, we developed a novel formulation of the SNS Ad allocation problem via hyperbolic embedding. We proposed an impression decomposition method to make it highly scalable and tractable. We also discussed fairness constraints and corresponding allocation strategies. Our formulation successfully reduces the dimensionality and complexity of the optimization problem while respecting fairness as domain constraint, and enables application in large-scale social networks, without introducing strong assumptions.

We will extend our work to incorporate more real-world domain constraints as well as multiple target groups. The possibility of parallel execution of different iterations, and the feasibility of our approach in online setting would be significant parts of our future work as well.

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