University of Maryland
CMSC652 — Complexity Theory
Professor Jonathan Katz

Problem Set 2

Due at the *beginning* of class on Sept. 22

These problems are not particularly difficult, but they will require some thought.

1. Prove that any $\mathcal{NP}$-complete language L (more formally, the natural relation derived from L) is self-reducible in the sense discussed in class.

2. As part of the proof that the language CS (circuit satisfiability) is $\mathcal{NP}$-complete, we used the fact that any boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ can be encoded as a circuit (using "and", "or", and "not" gates). Prove this by giving an explicit description of a circuit that computes a given function f. What is the worst-case number of gates your circuit uses as a function of n, the number of input bits?

3. (**optional—unsolved problem**) Prove that Sudoku (or, more formally, the generalization to $n \times n$ boards) is $\mathcal{NP}$-complete. Note: by using Google you can find a solution, but I am looking for a simpler solution that relies on a reduction from a more well-known $\mathcal{NP}$-complete problem. I don't know how difficult it is to find a "simpler" solution, so work on this problem at your own risk!

4. Answer the following questions regarding $\mathcal{NP}$ and co$\mathcal{NP}$.

   (a) In class we gave two definitions of co$\mathcal{NP}$: first, co$\mathcal{NP} = \left\{L \mid \bar{L} \in \mathcal{NP}\right\}$, and second, $L \in$ co$\mathcal{NP}$ if there exists a poly-time computable (and polynomially-bounded) relation R such that $x \in L \Leftrightarrow \forall w : R(x,w)$. Prove that these definitions of co$\mathcal{NP}$ are equivalent.

   (b) Let L be an $\mathcal{NP}$-complete language. Show a Cook reduction from any language $L' \in$ co$\mathcal{NP}$ to L. Why is your reduction *not* a Karp reduction?

   (c) Show that if there exists a language $L \in \mathcal{NP}$ such that there is a Karp reduction from every language $L' \in$ co$\mathcal{NP}$ to L, then $\mathcal{NP} =$ co$\mathcal{NP}$.

5. For any complexity class $\mathcal{C}$, show that co$\mathcal{C} \subseteq \mathcal{C}$ implies co$\mathcal{C} = \mathcal{C}$.

6. Recall that for any language L, the *Kleene star* of the language is

$$L^* \stackrel{\text{def}}{=} \{x_1 \cdots x_k \mid x_i \in L, k \geq 0\}.$$

   A complexity class $\mathcal{C}$ is *closed* under Kleene star if $L \in \mathcal{C}$ implies $L^* \in \mathcal{C}$. Prove that $\mathcal{P}$, $\mathcal{NP}$, and co$\mathcal{NP}$ are closed under Kleene star. (Note: this problem is more difficult than the rest, but not too difficult to be optional!)