University of Maryland
CMSC652 — Complexity Theory
Professor Jonathan Katz

Homework 1

Due at the *beginning* of class on Sept. 14

I suggest to use LaTeXwhen typing up your solutions. It will yield more readable solutions, and will provide you with a skill that you will use your entire (graduate) career.

1. Prove that if $\textsc{time}(n^2) = \textsc{ntime}(n^2)$ then $\textsc{time}(n^5) = \textsc{ntime}(n^5)$. (Hint: given a language $L \in \textsc{ntime}(n^5)$, consider the language $L' = \{x0^{|x|^{2.5}} : x \in L\}$.)

2. Assume $L_1, L_2 \in \mathcal{NP}$ and $S_1, S_2 \in \mathsf{co}\mathcal{NP}$. For the purposes of this problem, you should also assume that $\mathcal{NP} \neq \mathsf{co}\mathcal{NP}$. Answer each of the following with proof or with a counterexample:

   (a) Is $L_1 \cup L_2$ necessarily in $\mathcal{NP}$? Is $L_1 \cup S_1$ necessarily in $\mathsf{co}\mathcal{NP}$?

   (b) Is $L_1 \cap L_2$ necessarily in $\mathcal{NP}$? Is $\overline{L}_1 \cap L_2$ necessarily in $\mathsf{co}\mathcal{NP}$?

3. Prove that $\mathcal{P} = \mathsf{co}\mathcal{P}$.

4. Prove that the following language

   $$L = \left\{(M, x, 1^t) : \exists w \in \{0,1\}^t \text{ s.t. } M(x,w) \text{ halts within } t \text{ steps with output 1.}\right\}.$$

   (where M, above, is a deterministic Turing machine) is $\mathcal{NP}$-complete.

5. A language L' is $\mathsf{co}\mathcal{NP}$-hard if for every $L \in \mathsf{co}\mathcal{NP}$ it holds that $L \leq_p L'$; it is $\mathsf{co}\mathcal{NP}$-complete if furthermore $L' \in \mathsf{co}\mathcal{NP}$. Prove or disprove: L is $\mathcal{NP}$-complete iff $\bar{L}$ is $\mathsf{co}\mathcal{NP}$-complete.

6. **Challenge question − optional.** Show a universal *non-deterministic* Turing machine U such that (1) $U(M,x) = M(x)$ for any non-deterministic Turing machine M for which $M(x)$ is defined, and (2) for every M there exists a constant c such that if $M(x)$ runs in time T then $U(M,x)$ runs in time $O(T)$ (once again, the constant in the big-O notation may depend on M).