

## Problem Set 4

Due at the *beginning* of class on Nov. 4

1. Compute  $7^{120007} \bmod 143$  *by hand*. (Note that  $143 = 11 * 13$ .) Use the Chinese remainder theorem, Fermat's little theorem, and the fast modular exponentiation algorithm discussed in class, as needed.
2. Show that  $\mathbb{Z}_{56}$  is not a group under multiplication by showing an element of  $\mathbb{Z}_{56}$  which does not have a multiplicative inverse. Can you *prove* that this element does not have an inverse?
3. How many elements are in  $\mathbb{Z}_{17}^*$ ? Is this group cyclic? If so, determine how many generators it has and list them. If not, find the subgroup of  $\mathbb{Z}_{17}^*$  of largest order.
4. Since 101 is prime,  $\mathbb{Z}_{101}^*$  is a cyclic group. What is its order? Use Prop. 7.13 of Bellare-Rogaway to determine whether 8 is a generator of this group. How many generators does  $\mathbb{Z}_{101}^*$  have? If you chose an element of  $\mathbb{Z}_{101}^*$  at random, what is the probability that it will be a generator?
5. Prove that if  $G$  is a commutative group, then the set of quadratic residues in  $G$  forms a subgroup of  $G$ .
6. Consider the group  $\mathbb{Z}_{35}^*$  (of course,  $35 = 5 * 7$ ). Answer the following questions about this group:
  - How many elements are in this group? List them.
  - (Note: The Chinese Remainder Theorem will make the next two problems much less tedious.) For each element of the group, determine whether it has Jacobi symbol  $+1$  or  $-1$ . How many elements have Jacobi symbol  $+1$ ?
  - For each element which has Jacobi symbol  $+1$ , state whether it is a quadratic residue or not. How many of the elements with Jacobi symbol  $+1$  are quadratic residues?
  - For each element which is a quadratic residue, find all of its square roots.
  - Say we fix our RSA public exponent  $e$  to 5. What is the value of  $d$ , the private exponent?
7. Fix an RSA modulus  $N$  and public exponent  $e$ . Say we have an efficient algorithm  $A$  that can invert RSA for these parameters but which works only 1% of the time. Specifically, let  $S \subset \mathbb{Z}_N^*$  be the set of elements such that  $C \in S \Rightarrow A(C) = C^{1/e} \bmod N$  (i.e.,  $S$  is the set of elements for which  $A$  can compute the inverse). Then since  $A$  works 1% of the time,  $|S|/|\mathbb{Z}_N^*| = 0.01$ .

- (a) Show that if one can compute the inverses of  $C_1$  and  $C_2$ , then one can also efficiently compute the inverse of their product  $C_1C_2 \bmod N$ .
- (b) Show that if one can compute the inverse of  $C' \stackrel{\text{def}}{=} Cr^e \bmod N$ , then one can also efficiently compute the inverse of  $C$ .
- (c) Suggest how to use  $A$  to compute the inverse of *any* element  $C \in \mathbb{Z}_N^*$  with high probability. *Hint:* you will need to use randomization.