# Subgraph and Supergraph Problems in r-tournaments

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# Motivation

- Directed feedback vertex problem is fixed parameter tractable in general directed graphs but only tournaments have known $O^*(c^k)$ algorithms ($c$ is a constant and $k$ is the maximum solution size allowed).

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- Directed feedback vertex problem is fixed parameter tractable in general directed graphs but only tournaments have known $O^*(c^k)$ algorithms ($c$ is a constant and $k$ is the maximum solution size allowed).
- We study a class of graphs, named r-tournaments, which naturally bridges the gap between tournaments and general graphs.

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- Clearly by this definition, a 1-tournament is a tournament and a connected directed graph on $n$ vertices is an n-tournament.

# Part I

## Feedback vertex set and c-dominating set in r-tournaments

# Feedback vertex set

**Theorem**

*An algorithm to test if a 2-tournament has a FVS of size atmost $k$ in $O^*(c^k)$ time can be used to test if a directed graph has a FVS of size atmost $k$ in $O^*(c^k)$ for some constant $c \in \mathbb{R}$*

Thus the feedback vertex set has, in the parameterized sense, equivalent complexity in general directed graphs as tournaments.

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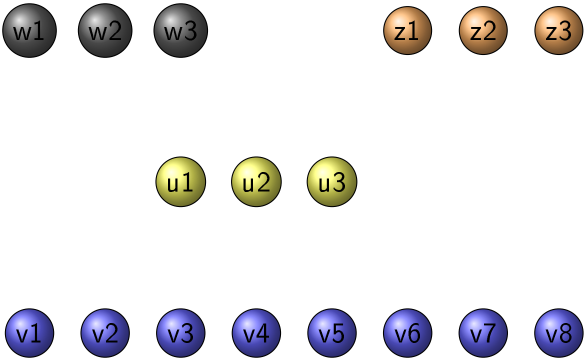
- For every element $u_i$, we add an edge from $u_i$ a vertex $v$ of G if the $i^{th}$ element of the latter's binary representation is 0. Otherwise we add an edge from $v$ to $u_i$. Remaining connections are as shown in following example.

# Example of a graph on 8 vertices

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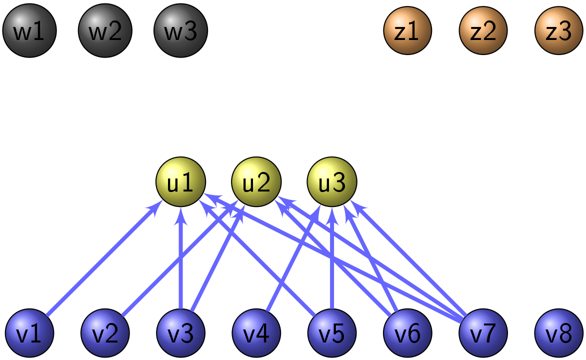


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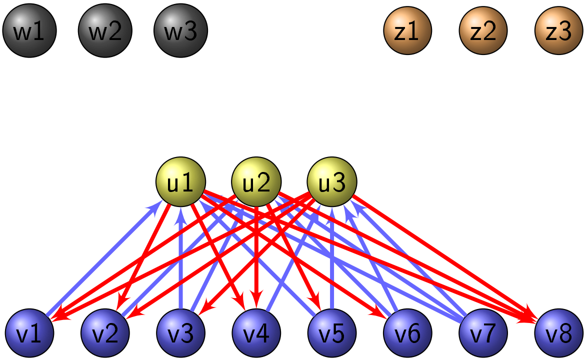


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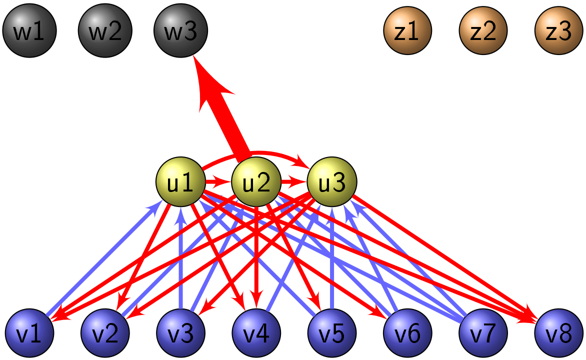


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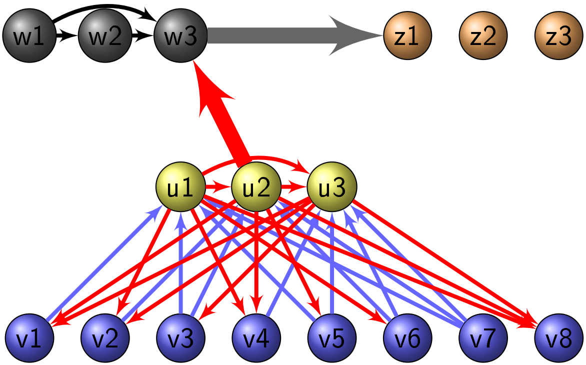


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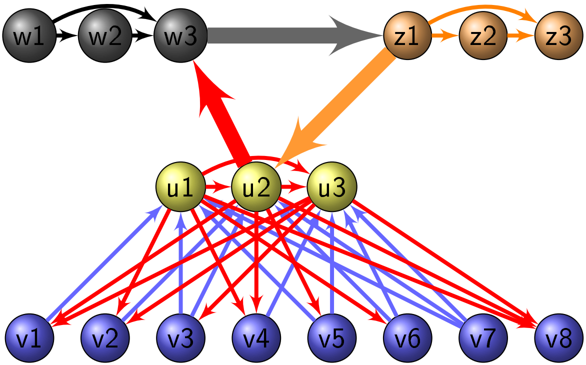


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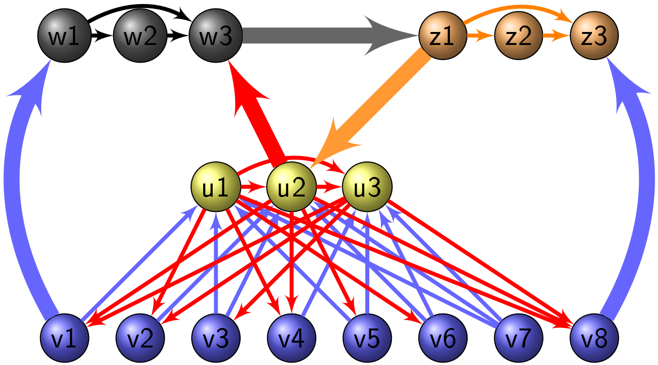


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- If $k$ is the number of vertices to be deleted from $G$ to destroy all directed cycles from it, to make the constructed graph acyclic we must delete exactly $k + \log_2 n$
- The key observation is that at least one of the color groups black, orange, yellow must be completely destroyed.

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- To test if a given graph has a directed FVS of size $k$, we convert the graph into the above 2-tournament.
- If there is an algorithm to solve for FVS of size $k$ in 2-tournament, running in time $O^*(c^k)$ the said procedure will yield an algorithm to test FVS of size $k$ in general digraphs , with running time $O^*(c^{k+\log_2 n}) = O^*(c^k)$

# 2c-dominating set in c-tournaments

**Theorem (Landau)**

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**Theorem (Extension of Landau's theorem)**

*Let $T$ be a c-tournament and $v$ a vertex with a maximum number of vertices at a directed distance at most $c$. Then $\{v\}$ is a $(2c)$-dominating set of $T$.*

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- The above assumption along with the condition that $T$ is c-tournament implies that $N_c(v) \cup \{v\} \subseteq N_c(u)$.
- This is impossible as $v$ has maximum cardinality of $N_c(v)$.

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Also:

- Every c-tournament has a c-dominating set of size $\log_2 n$. This results in an (brute force) algorithm to find out a c-dominating set, running in $O(n^{\log_2 n})$.

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- Given a tournament T, we replace each vertex of $v \in T$ by a path of $c$ vertices $v_i \ni i \in [c]$. If $(u, v) \in T$, we add edges from $u_i \ni i \in [c]$ to $v_1$.

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- The resultant graph CT is a c-tournament and has a c-dominating set of size $k$ iff T has a dominating set of size $k$.

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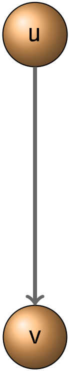
- If D is a dominating set of T then $\{u_1 \ni u \in D\}$ is a c-dominating set of CT, with the same cardinality.
- If CD is a c-dominating set of CT, then the set of of vertices in D obtained by removing subscripts from elements of CD is a dominating set of size atmost $|CD|$.
- To see that D is indeed the dominating set of T, observe that if D does not dominate a vertex $v \in T$ CD does not dominate $v_c$.

# Example: 3-dominating set in 3-tournament



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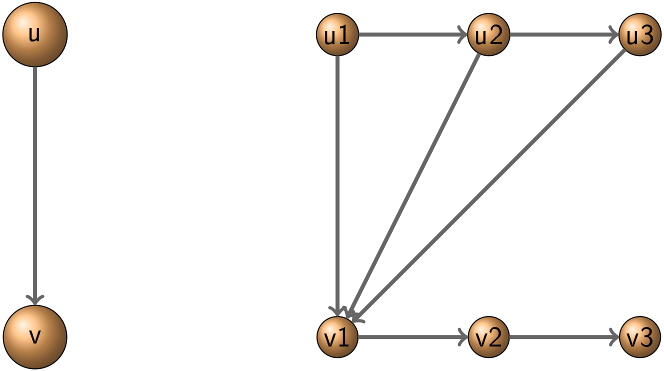

# Part II

## Graph Modification Problems

# Introduction

- A graph modification problem asks for an optimum number of modifications to a graph to obtain another one which satisfies some required property(A property is a class of graphs closed under isomorphism), example: the cluster editing problem .
- We study two problems requiring modifications (edge addition and deletions) to obtain 2-tournaments (a cluster of 2-tournaments in the first problem and a single 2-tournament in the second case).

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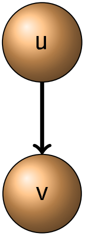
*Given a digraph $G$, add atmost $k$ edges to convert into a 2-tournament.*

We prove that both 2-tournament clustering and 2-tournament completion are NP-Complete. We also prove that while 2-tournament clustering is FPT, 2-tournament completion is W[2]-hard.

# 2-tournament clustering is NPC: Reduction from clique clustering problem




- Given a graph $G = (V, E)$, we construct $G' = (V', E')$ as following:

$$V' = \{u_+, u_- : u \in V\}. \tag{1}$$

$$E' = \{(v_+, v_-) : v \in V\} \cup \{(v_-, u_+), (u_-, v_+)\} \tag{2}$$

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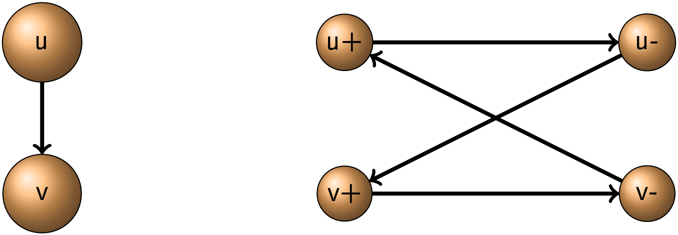


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# Outline of Proof

The following is the series of steps involved in proving the NP-Hardness:

- Any subgraph of $G'$ which is a 2-tournament must have atmost one +ve signed vertex without a pair and atmost one -ve vertex without a pair.
- There is a minimum solution $M$ for the problem of 2-tournament edge clustering such that every vertex of $G'$ has its pair in one component.
- For each $k \geq 0$, an undirected graph G has a clique clustering edge set of size atmost $k$ if and only if $G'$ has a 2-tournament clustering edge set of size $2k$.

## Lemma

*Let G be a directed graph which is not a 2-tournament such that the underlying directed graph is connected. There exist two vertices for which the distance in the undirected graph is at most 3 but are not at directed distance 2 in G.*

## Proof.

- Let S be the set of pairs of vertices not having a directed path of length 2 connecting them. Let u,v be a pair of vertices having least undirected distance among all pairs of S. Let the shortest undirected path connecting them be $P(u,v) = \{u, v_1, v_2, v_3..v\}$.
- Let if possible $|P(u,v)| >= 4$. This means that $v_3 != v$ and $(u, v_3)$ does not belong to S. Hence there is a 2-path connecting $u, v_3$ which would imply P is not the shortest path.

□

# FPT Algorithm

- A simple search tree algorithm is based on the following observation: If there given graph is not a cluster of 2-tournament the earlier lemma gives us a pair of vertices which are not in the at a directed distance 2 but are at an undirected distance atmost 3.
- These vertices cannot be in the same component of the solution graph. Hence atleast one of the edges on the path connecting $u, v$ must be included in the final solution. Branching on each of these solutions yields a $O^*(3^k)$ algorithm.

# 2-tournament completion is NP Complete and W[2] hard

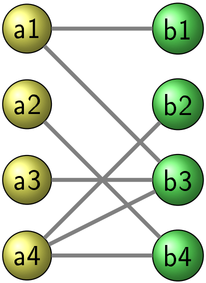
We prove the NPCompleteness and W[2] hardness of the following problem:

**Problem (Single Vertex Satisfaction)**

*Given a directed graph $G = (V, E)$ and a vertex $v \in V$ add atmost $k$ edges to $G$ such that in the resultant graph every vertex of $G$ is either at directed distance at most 2 from $v$ or has it at a directed distance at most 2.*

# SVS is NPC and W[2] hard

- Reduction from dominating set problem which is NPC and W[2]C.
- Let $G = A \cup B$ (partitions $A$, $B$) be a bipartite graph. We add a new vertex $v$ to $G$ and direct the edges from $A$ to $B$ to get $G'$, as shown.



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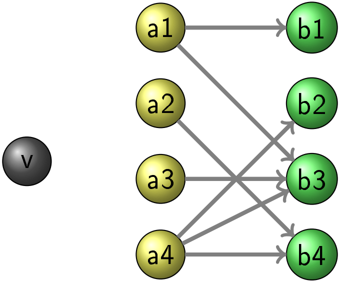
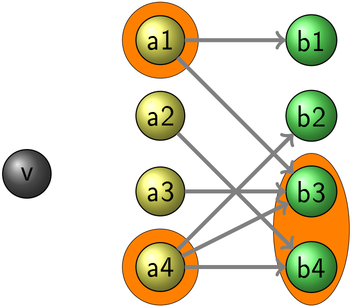


# Proof outline

- If D is a dominating set of $G$, by adding edges edges from $v$ to all vertices of $D \cap A$ and from all vertices of $D \cap B$ to $v$ we get a graph in which $v$ satisfies all the 2-tournament property with all vertices.



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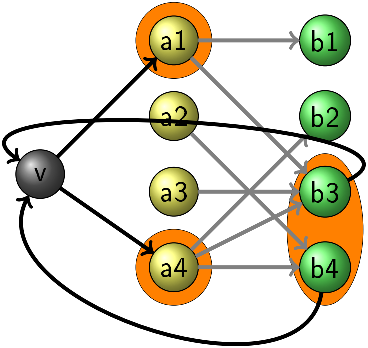

# Reduction: Contd.

- Let M be a (edge set) solution to the SVS instance $(G', v)$. We prove that there is a dominating set of size $|M|$.
- Let M = $M_G \cup M_v$, where $M_G$ edges whose end points are in G and $M_v$ has edges incident on $v$.
- Let $D_{G'}$ be the minimum dominating set of (underlying undirected graph) $G'$ and $D_G$ be the minimum dominating set of $G$.

# Reduction: Contd.

- Adding $k$ edges to graph G reduces the dominating set size by $k$ atmost:

$$D_{G'} \geq DG - |M_G| \tag{3}$$

- Since $v$ is at distance 2 from all the vertices in the underlying undirected graph and $M_v$ is its neighborhood, the latter is a dominating set of $G'$.

$$D_{G'} \leq |M_v| \tag{4}$$

- From the above equations we have:

$$|M_G| + |M_v| = k \geq D(G) \tag{5}$$

.

# 2-tournament completion is NPC and W[2] hard: Reduction from Single Vertex Satisfaction

Given a graph $G = (V, E)$ we construct $G' = (V', E')$ in the following way. $G'$ has an SVS edge set of size $k$ iff $G$ has a 2-tournament completion edge set of size $k$:

- $$\begin{aligned} V' &= V \cup V_1 \cup \{v_{ex}\} \\ V_1 &= \{v_{u,w} : \forall \{u,w\} \in V - \{v\}\} \end{aligned} \quad (6)$$

- $$\begin{aligned} E' = E \quad &\cup \quad \{(u, v_{u,w}), (v_{u,w}, w), \forall v_{u,w} \in V_1\} \\ &\cup \quad \{(u, v_{ex}), \forall u \in V\} \cup \{(v_{ex}, v_{u,w}), \forall v_{u,w} \in V_1\} \\ &\cup \quad \{(u, v) \forall \{u, v\} \in V_1\} \end{aligned} \quad (7)$$

# Example