An Improved Approximation for k-median, and Positive Correlation in Budgeted Optimization

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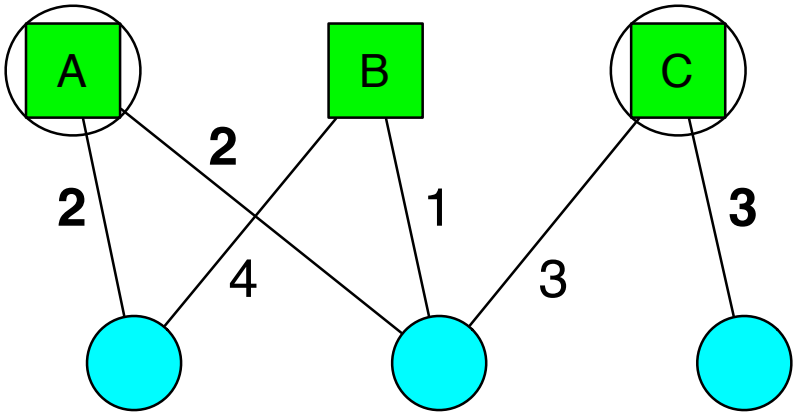
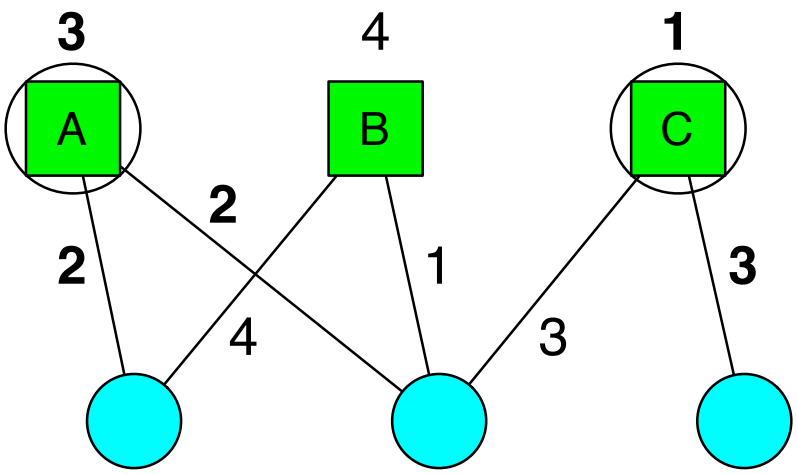
k-median

- Input:
  - set of facilities $\mathcal{F}$
  - set of clients $\mathcal{C}$
  - symmetric distance metric d on $\mathcal{C} \cup \mathcal{F}$
  - an integer $k > 0$
- Goal: pick k facilities which minimize total connection cost of clients.
- Example: $k = 2$, open facilities A, C and pay cost $2 + 2 + 3 = 7$.



Uncapcitated Facility Location

- Input:
  - set of facilities $\mathcal{F}$
  - set of clients $\mathcal{C}$
  - symmetric distance metric d on $\mathcal{C} \cup \mathcal{F}$
  - each facility has an opening cost
- Goal: pick some facilities which minimize total opening cost and connection cost.
- Example: open facilities A, C and pay cost $2 + 2 + 3 + 3 + 1 = 11$.



Bi-point solution as middle step

Bi-point solution: a feasible solution which is a convex combination of two integral solutions.

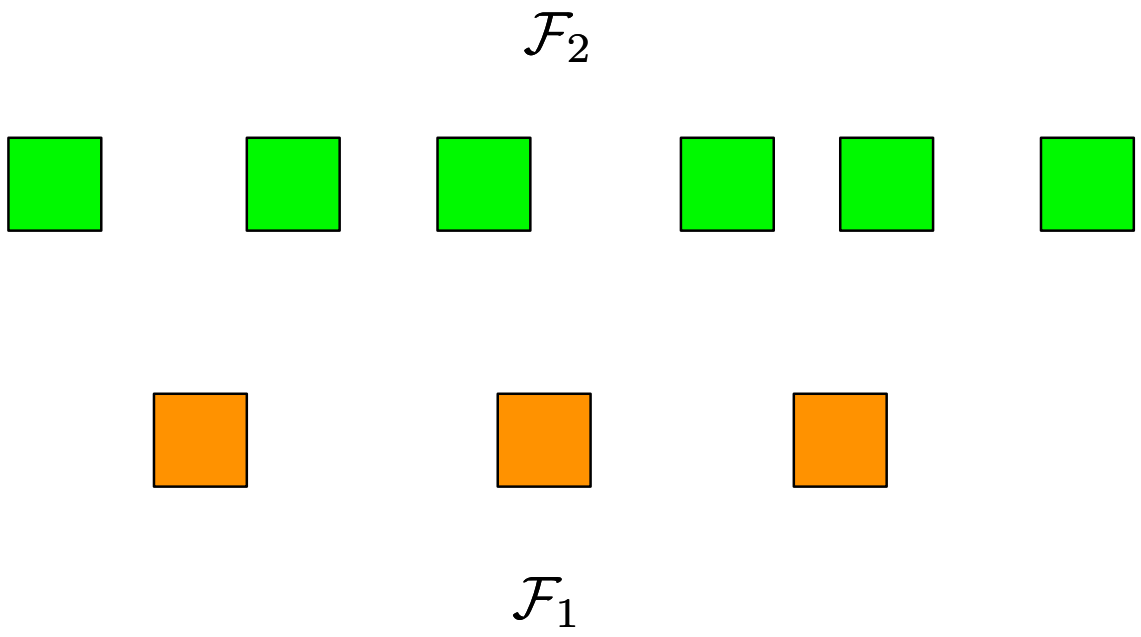
- Step I: Construct a bi-point solution.
  - Use some special UFL algorithms to get a “cheap” bi-point solution.
- Step II: Round bi-point solution to integral one.

Previous Work

Year	Authors	Bi-point Construction	Bi-point Rounding	*k*-median Approximation
‘99	Charikar et. al.	(LP Rounding)		6.667
‘99	Jain & Vazirani	3 + ε	2	6 + ε
‘02	Jain,Mahdian & Saberi	2 + ε	2	4 + ε
‘01	Arya et. al.	(Local Search)		3 + ε
‘12	Li & Svensson	2 + ε	1.366 + ε	2.732 + ε
‘15	**Our work**	2 + ε	**1.337 + ε**	**2.674 + ε**

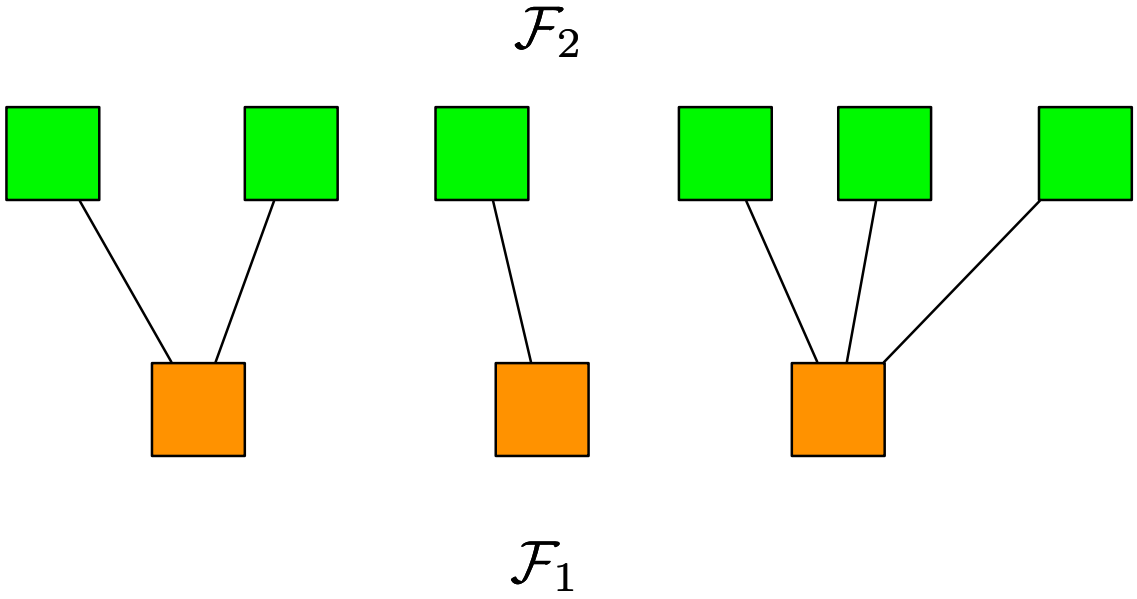
Bi-point Rounding

- Again, bi-point solution is convex combination of two integral solutions, $\mathcal{F}_1$ and $\mathcal{F}_2$.
- $a|\mathcal{F}_1| + b|\mathcal{F}_2| = k$, where $(a, b > 0, a + b = 1)$



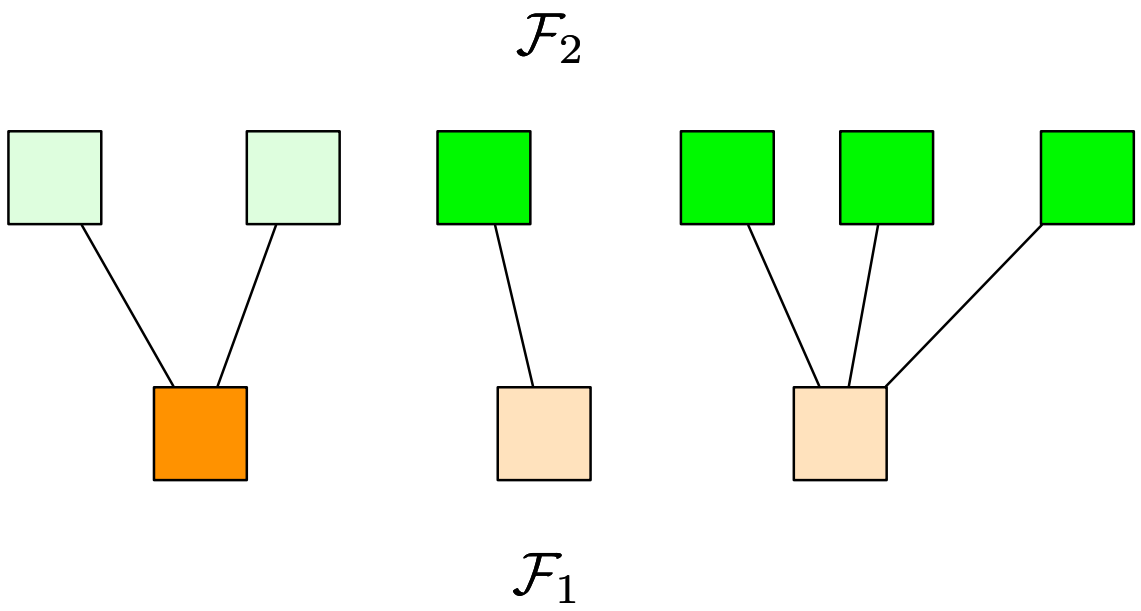
Stars

- Form stars by attaching each facility in $\mathcal{F}_1$ to closest facility in $\mathcal{F}_1$.

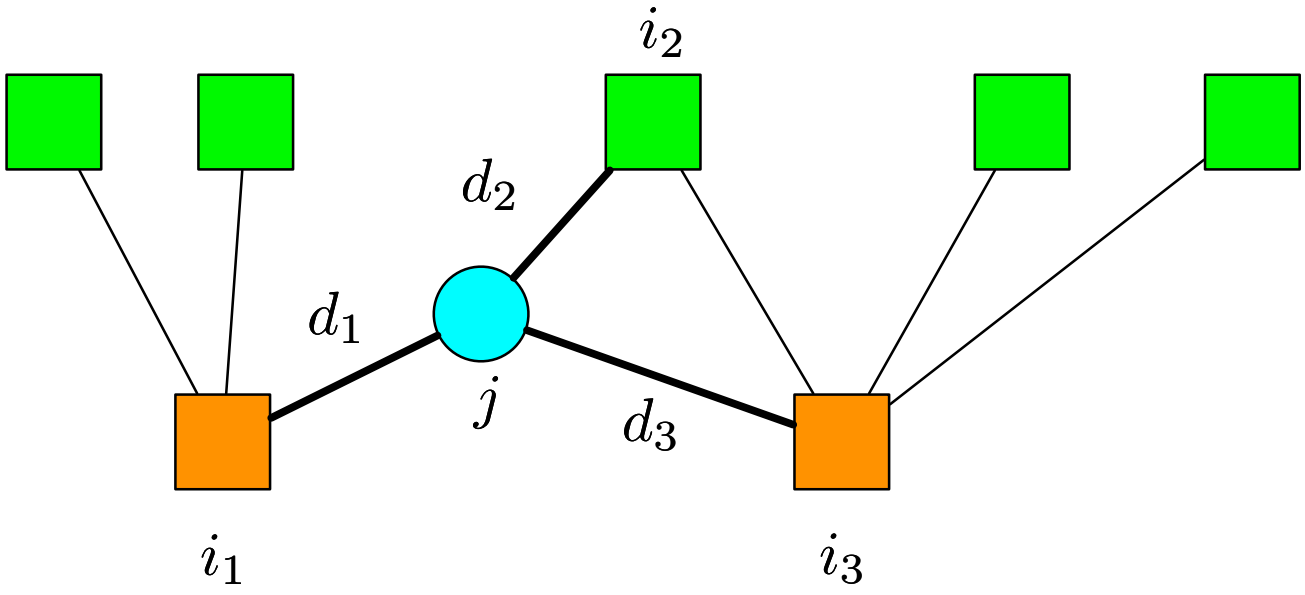


Li-Svensson's rounding algorithm

- Key property: "if some leaf is closed then its root must be open".
- $\Pr[i_1 \text{ is open}] \approx a$ for all $i_1 \in \mathcal{F}_1$
- $\Pr[i_2 \text{ is open}] \approx b$ for all $i_2 \in \mathcal{F}_2$
- approximation factor $= \frac{1+\sqrt{3}}{2} \approx 1.366$
- #open facilities = $a|\mathcal{F}_1| + b|\mathcal{F}_2| + O(1) = k + O(1)$.



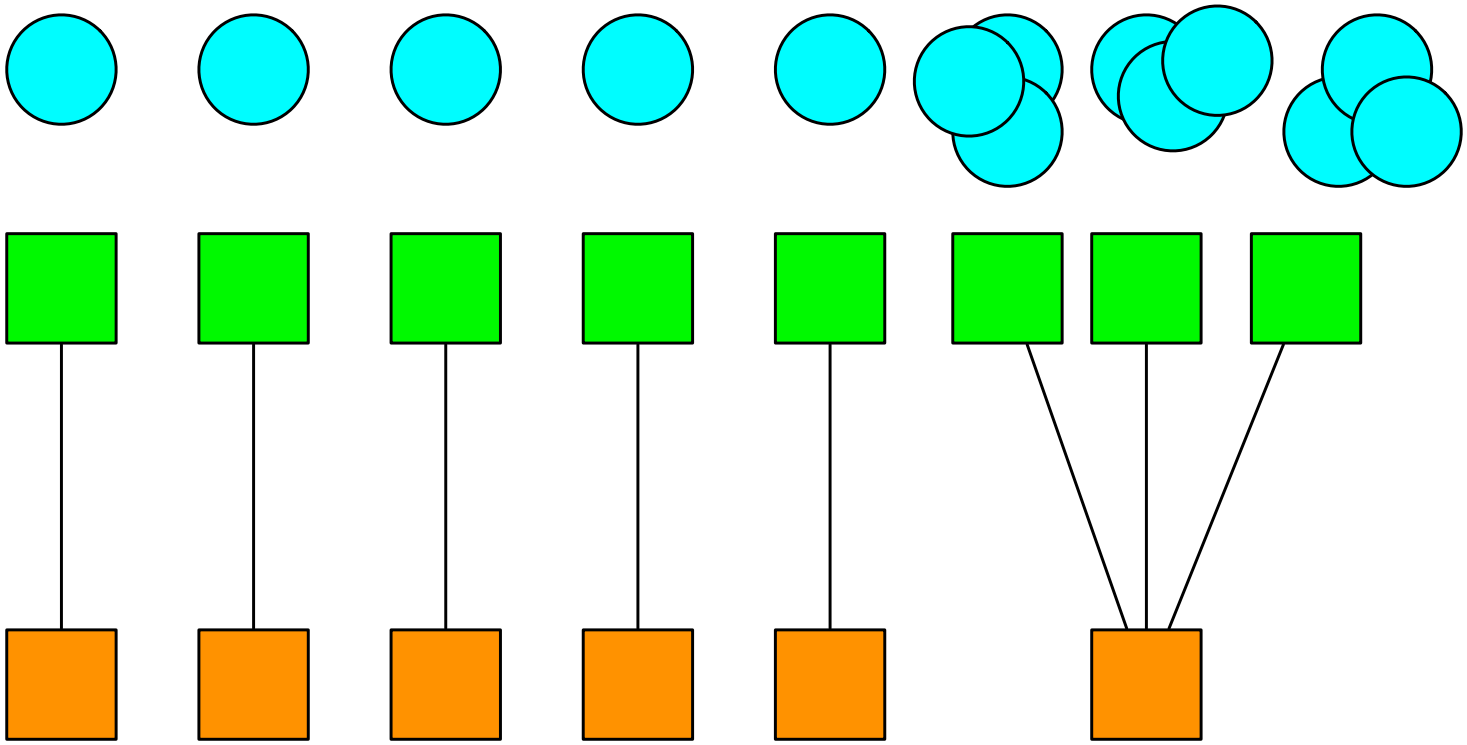
Distance bounds



- There is an open facility in $\{i_1, i_2, i_3\}$
- $d_3 \leq d_2 + d(i_2, i_3) \leq d_1 + 2d_2$
- The total (expected) connection cost is a (non-linear) function of $\text{cost}(\mathcal{F}_1)$, $\text{cost}(\mathcal{F}_2)$, a, b which can be related to OPT.

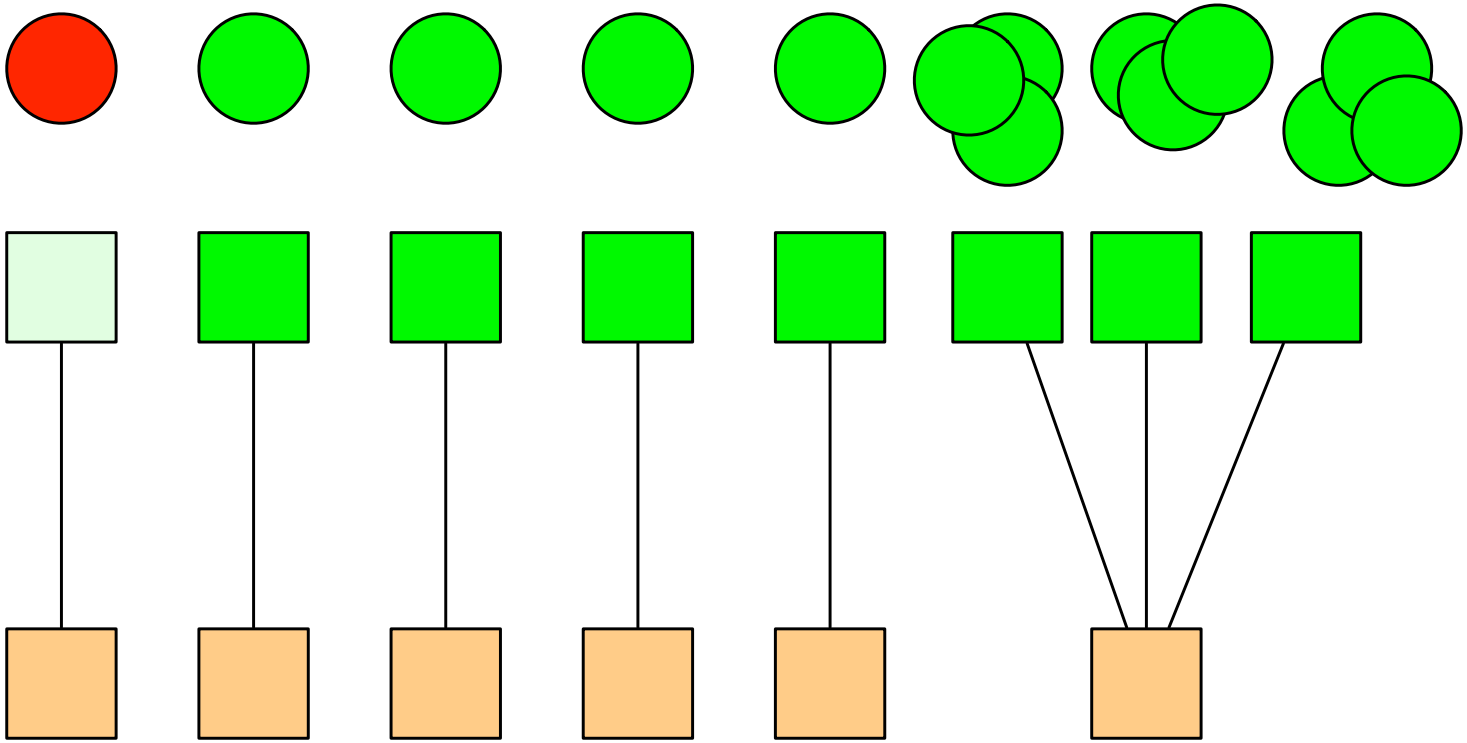
Tight Instance

- In tight case, almost all stars have exactly 1 leaf.
- BUT still many clients near 2-leafed stars.

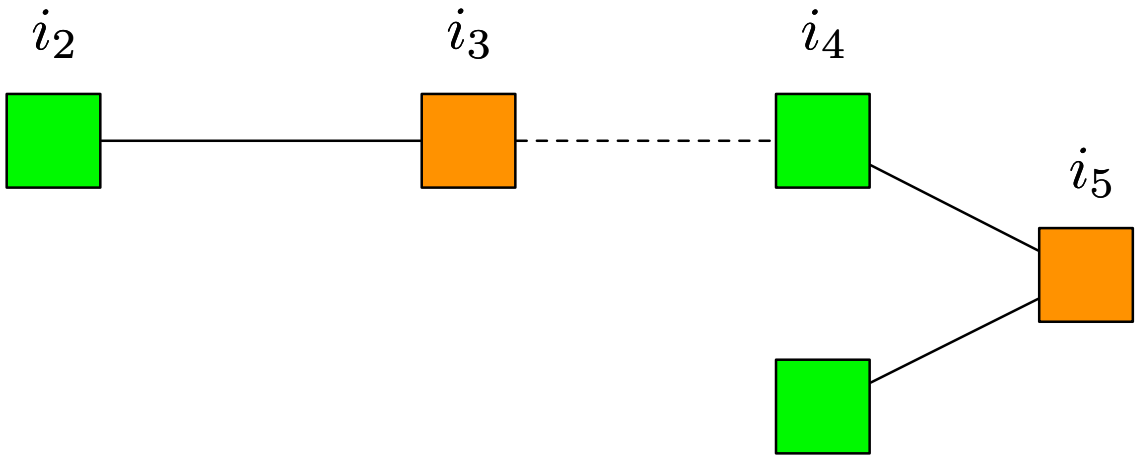


Tight Instance

- Why not close some 1-leafed stars to open more leaves of the 2-leafed stars?



Long 1-leafed stars

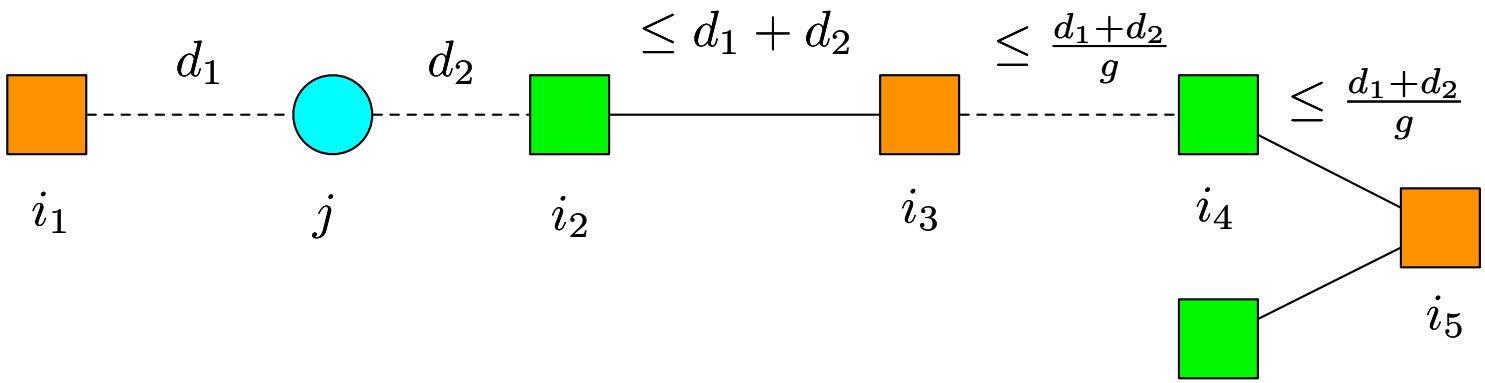


Consider any 1-leafed star (i_2, i_3) and let i_4 be the closest leaf of a 2-leafed star to i_3. Then the star (i_2, i_3) is *long* if

$$d(i_2, i_3) \geq g d(i_3, i_4),$$

for some constant $g > 0$.

Long 1-leafed stars can be closed



Bi-point Rounding

Improved Bi-point rounding algorithm:

- Consists of 8 different rounding strategies.
- Analysis uses a non-linear factor revealing program.
- Approximation ratio: 1.337.

Dependent Rounding

Given a vector $P = (p_1, \ldots, p_n)$ with $0 \leq p_i \leq 1$, Dependent Rounding algorithm rounds P into a 0-1 vector X such that

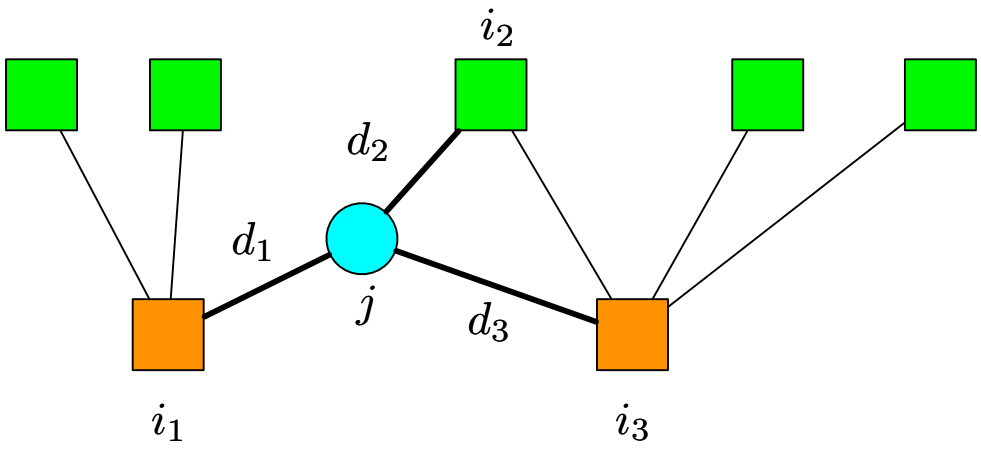
- the sum is preserved: $\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} p_i$,
- the marginals are preserved: $\Pr[X_i = 1] = p_i$,
- *negative correlation* between X_i's: e.g.
  - $\Pr[X_i = 1 \wedge X_j = 1] \leq p_i p_j$
  - $\Pr[X_i = 0 \wedge X_j = 0] \leq (1 - p_i)(1 - p_j)$

A Typical Example of Negative Correlation in Facility Location Problems

Assume that we have a client j and a set $\mathcal{S}$ of facilities that are close to j. Each facility $i \in \mathcal{S}$ has an *opening variable* y_i. Applying dependent rounding to opening variables in $\mathcal{S}$ gives

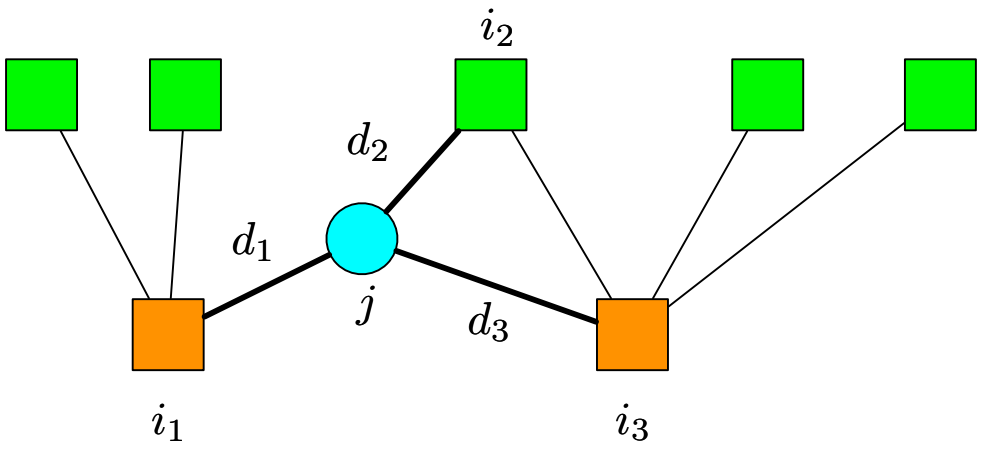
$$\begin{aligned}
\Pr[\text{all facilities in } \mathcal{S} \text{ are closed}] &= \Pr\left[\bigwedge_{i \in \mathcal{S}} \{i \text{ is closed}\}\right] \\
&\leq \prod_{i \in \mathcal{S}} (1 - \Pr[i \text{ is open}]) \\
&\leq \exp\left(-\sum_{i \in \mathcal{S}} y_i\right).
\end{aligned}$$

Positive Correlation



- For each root $i \in \mathcal{F}_1$, let X_i be the indicator whether i or its leaves is opened.

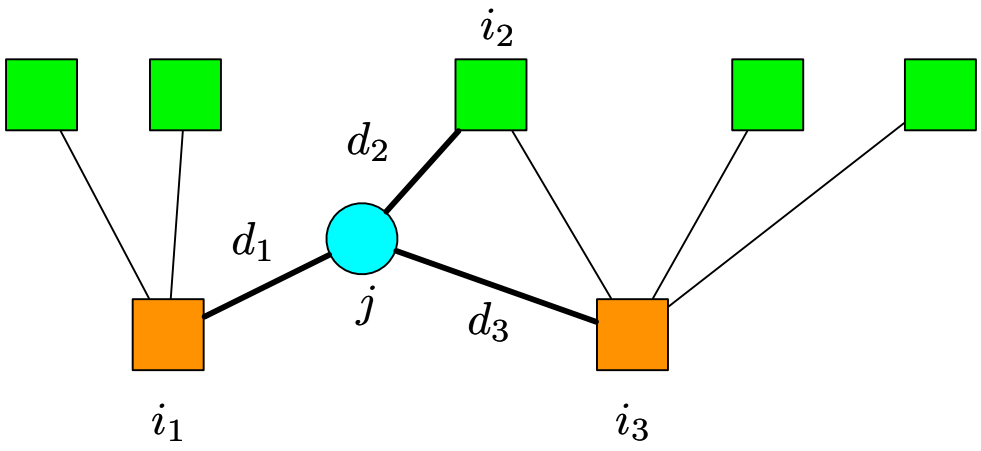
Positive Correlation



- For each root $i \in \mathcal{F}_1$, let X_i be the indicator whether i or its leaves is opened.
- We want to upper-bound $\mathrm{E}\left[\,\mathrm{cost}\,(j)\right]$:

$$\mathrm{E}\left[\,\mathrm{cost}\,(j)\right] \leq \Pr[X_{i_3} = 0]d_2 + \Pr[X_{i_3} = 1 \wedge X_{i_1} = 1]d_1 + \Pr[X_{i_3} = 1 \wedge X_{i_1} = 0]d_3$$

Positive Correlation



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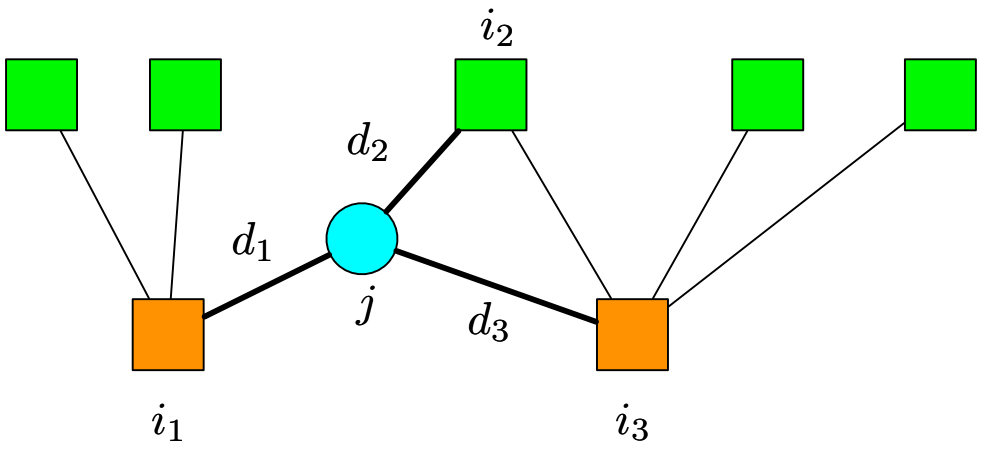

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  - How about $\Pr[X_{i_3} = 1 \wedge X_{i_1} = 0]$?

Near Independence

- Independence:

$$\Pr[A \wedge B \wedge C] = p_A p_B p_C$$

- Near-Independence:

$$(1-\epsilon)p_A p_B p_C \leq \Pr[A \wedge B \wedge C] \leq (1+\epsilon)p_A p_B p_C$$

- Can dependent rounding achieve this property?
  Answer: Yes, but with limits...

Extreme Examples

- Observation 1: if $p_1 = p_2 = \ldots = p_n = 1/2$, we have

$$\Pr[X_1 = 1 \wedge X_2 = 0 \wedge \ldots \wedge X_n = 0] = 0.$$

On the other hand, $\Pr[X_1 = 1] \prod_{i=2}^{n} \Pr[X_i = 0] = 2^{-n} > 0$.
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- Observation 2: if $p_1 = p_2 = \ldots = p_n = 1/n$. Then,

$$\Pr[X_1 = 1 \wedge X_2 = 1] = 0.$$

  On the other hand, $\Pr[X_1 = 1] \Pr[X_2 = 1] = 1/n^2 > 0$.
  Get around this by by requiring that the values p_i are not too close to 0 *or* 1.

Modified Dependent Rounding

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- **Our results:** if there is some $\alpha > 0$ that $\alpha \leq p_i \leq 1 - \alpha$ for all i, then for any t events of the form $X_i = 0$ or $X_i = 1$, their joint probability lies within

$$\left(1 - \left(1 - \frac{t^2}{n\alpha^2}\right)\right) \text{ and } \left(1 + \left(1 + \frac{t}{n\alpha^2}\right)^{t-1}\right),$$

of the value it would if these events were independent.

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  of the value it would if these events were independent.
  - in particular, these factors are $(1 + o(1))$ and $(1 - o(1))$ if $t = o(\alpha\sqrt{n})$.

Modified Dependent Rounding

Application to k-median:

- reduce #extra facilities from $O(1/\epsilon)$ to $O(\log(1/\epsilon))$
- reduce run-time from $n^{O(1/\epsilon^2)}$ to $n^{O(1/\epsilon \log(1/\epsilon))}$.

Open questions

- Closing the gap for k-median?
  - Our algorithm: $(2.674 + \epsilon)$-apx
  - Best known lower bound: $1 + 2/e \approx 1.735$
- Other applications of dependent rounding with near-independence property?

Thank you!