# Denset k-subgraph Approximation on Intersection Graphs

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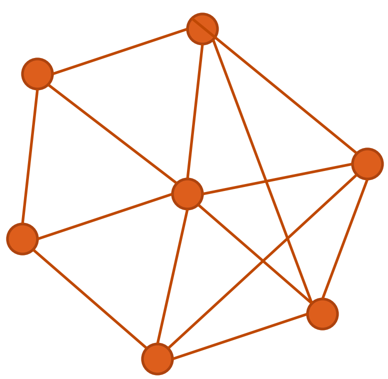
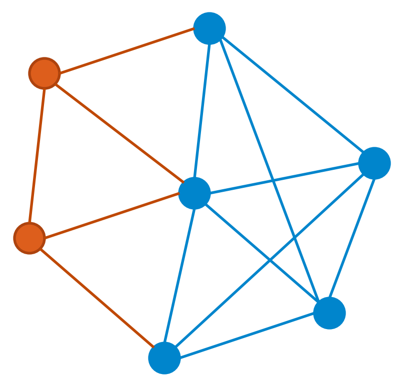
# Problem Definition

- Given:
  - An undirected graph *G(V,E)*
  - An integer *k*
- Goal:
  - Find a subset *S* of *k* vertices such that *|E(G[S])|* is maximized (*G[S]* is the subgraph of *G* induced by *S* )

G:





$k=5$ $|E(G[S])|=9$

# Prior Work

- NP-hard: an immediate reduction from MAX-CLIQUE
- Approximability: despite a lot of efforts, it is less understood (in contrast to MAX-CLIQUE)
  - Lower bounds:
    - No PTAS under certain complexity assumption(Khot '04)
  - Upper bounds for general graphs:
    - $O(n^{0.3885})$-approximation (Kortsarz & Peleg '93)
    - $O(n^{1/3-eps})$-approximation for some eps>0 (Feige et al. '01)
    - $O(n^{1/4-eps})$-approximation for any eps>0 (Bhaskara et al. '10)
    - O(n/k)-approximation (Asahiro et al. '96)
    - n/k –approximation (Feige et al. '97)
    - Many improved bounds for special k values….

# Prior Work

- Upper bounds on special graph classes :
  - PTAS for dense instances (Arora et al. '95)
  - PTAS for H-minor free graphs for any fixed H (Demaine et al '05)
  - PTIME on trees (Maffioli '91), co-graphs (Corneil & Pearl '84)
  - 3-approximation on Chordal graphs by greedy (Liazi et al. '08)
    - NP-hardness on chordal graph (Corneil & Pearl '84)
  - Many others….

# Our Result

- We show that there is an O(σ)-approximation for the class of graphs admitting **σ-elimination orders**

  The class (with constant σ) includes:
  - chordal graphs,
  - circular-arc graphs,
  - claw-free graphs,
  - line graphs of l-hypergraphs,
  - planar graphs,
  - disk graphs,
  - and the intersection graphs of fat objects
- This implies constant approximations for **ALL** above graph classes.

# σ-elimination order

- The notion was introduced by K. Akcoglu et al. '02 (under name sequentially k-independent graphs)
- Its algorithmic aspect has drawn researchers' attention recently. (Ye & Borodin '09)
- A significant generalization of perfect elimination order for Chordal graphs.

# σ-elimination order

- Some Notations:
  - *$pred_L(v)$:* the set of neighbors of *v* that appear before *v* in the vertex ordering *L*
  - *$\alpha(G)$:* the independence number of *G,* i.e., the cardinality of the max independent set of *G*
- perfect elimination order *L*:
  - An ordering of all vertices such that $pred_L(v)$ is a clique
  - A graph has a perfect elimination order iff it is chordal
- σ-elimination order *L*:
  - An ordering of all vertices such that $\alpha(G[pred_L(v)])\leq\sigma$
  - A perfect elimination order is a 1-elimination order

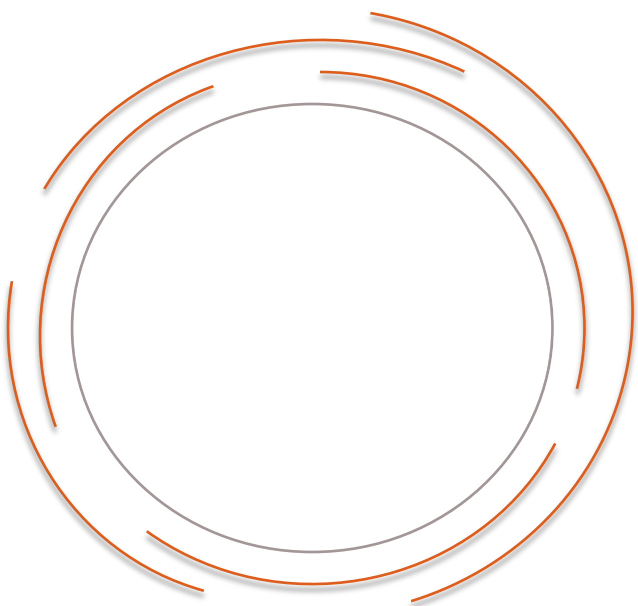
# A little more on σ-elimination order

- We show that each circular-arc graph has a 2-EO
- Circular-arc graphs:

The intersection graph of those arcs:

-each arc is a node

-there is an edge between two nodes iff two corresponding arcs intersect



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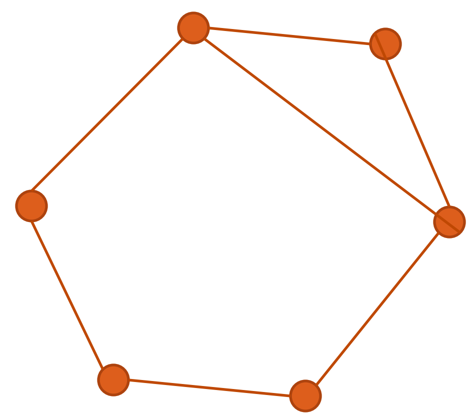


# A little more on σ-elimination order

- 2-EO: just sort the arcs in an non-increasing order of their lengths.

**Proof:**

We need to prove $\alpha(pred(a)) \leq 2$ for all arc *a*.

Consider arc *a*,

- each arc longer than *a* contains at least one endpoint of *a*

- arcs that share a common point form a clique

- therefore, neighbors of *a* can be partitioned into two cliques.

# Densest k Subgraph Approximation

- The Maximum Density Subgraph Problem (MDSP)
  - Each vertex $v$ has a weight $w(v)$
  - Density of $S$: $\rho(S)=(w(S)+|E[S]|)/|S|$
  - Find a subset $S$ that maximizes $\rho(S)$
  - PTIME by parametric network flow (Gallo et al. ’89)

  - A important property:
    - For any $v$ in $S$, $w(v)+deg(v,S)\geq\rho(S)$ (since otherwise we can delete $v$ from $S$ to get a subgraph with larger density)
      - $deg(v,S)$: degree of $v$ in $G[S]$

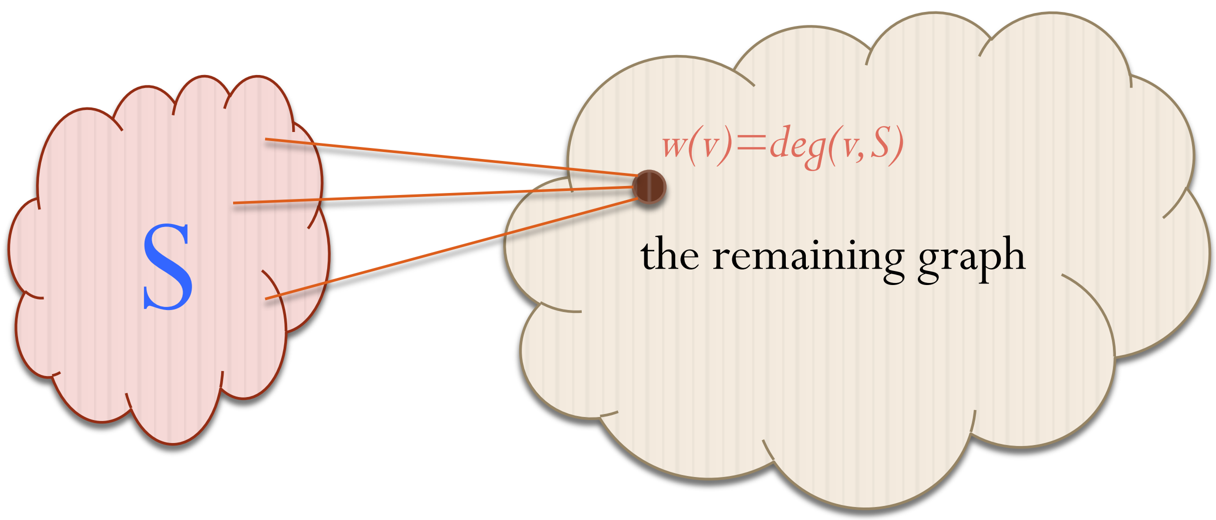
# Densest k Subgraph Approximation

- Outline of the algorithm:
- **Phase1: Growing**
  - Add MDSP solution to *S* until $|S|>k/2$ (more details soon)
  - We can show that *S* has a density $\geq O(1)\rho^*$

    ($\rho^*$: the density of OPT)
  - If $|S|\leq k$, we are done.
  - If $|S|>k$ :
- **Phase 2: Shrinking**
  - We look at the σ-EO of *S* and find a subgraph of *S* s.t. the deg. of each node is $O(1)$ of the deg in $G[S]$.
  - Therefore, it is a $O(1)$-approx.

# Densest k Subgraph Approximation

- **Phase 1: Growing**
  - Suppose S is the set of nodes we've already chosen
  - Run MDSP on the remaining graph with vertex weight *w(v)=deg(v,S)*
    - This is used to capture the edges between *S* and the remaining nodes
  - Repeat until *|S|≥k*
  - We can show *deg(v)>O(1) ρ** for all *v* in *S*



# Densest k Subgraph Approximation

- **Phase 2: Shrinking**
  - Suppose we end up with $|S|>>k$ obtained from Phase 1
  - Consider the elimination order of $G[S]$
    - If there is a node v s.t. $|Pred(v)|\geq k/2$, then uniformly&randomly pick $k/2$ vertices from $Pred(v)$ is a $O(1)$-approximation

**Proof:**

First notice the fact that: if $\alpha(G)=O(1)$, then $|E(G)|=O(V^2)$

Therefore, the expect number of edges in our solution *SOL* is

$$\Sigma_e Pr(e \text{ in } SOL)=O(V^2)*(k/2V)*(k/2V)=O(k^2)$$

# Densest k Subgraph Approximation

- **Phase 2: Shrinking**
  - Suppose we end up with *|S|>>k* obtained from Phase 1
  - Consider the elimination order of *G[S]*
    - If there is no node s.t. $|Pred(v)| \geq k/2$

    - Process EO in the *reverse order*

      Either $Pred(v)=O(1)\rho^*$ or $Succ(v)=O(1)\rho^*$ (since $deg(v)=O(1)\rho^*$)

      - If $Pred(v)=O(1)\rho^*$ : $SOL = SOL + v + Pred(v)$
        - We add a graph where each node has deg $O(\rho^*)$
      - If $Succ(v)=O(1)\rho^*$ : $SOL = SOL + v$
        - $v$ has $deg=O(\rho^*)$ in SOL

**KEY:** How EO is used to help us

Therefore, an O(1)-approximation

# Open Problems

- Finding more graph classes that admit σ-EO (with small σ)
- Finding more applications for σ-EO
  - Ye & Borodin found that a few graph optimization problems are easier if the graph has O(1)-EO
- Closing the huge gap for densest k subgraph

- **OPEN:** Whether densest k subgraph is NP-hard on interval graphs and planar graphs

- Thanks