Densest $k$-subgraph Approximation on Intersection Graphs

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Problem Definition

- **Given:**
  - An undirected graph $G(V,E)$
  - An integer $k$

- **Goal:**
  - Find a subset $S$ of $k$ vertices such that $|E(G[S])|$ is maximized ($G[S]$ is the subgraph of $G$ induced by $S$)
Prior Work

- NP-hard: an immediate reduction from MAX-CLIQUE
- Approximability: despite a lot of efforts, it is less understood (in contrast to MAX-CLIQUE)
  - Lower bounds:
    - No PTAS under certain complexity assumption (Khot ’04)
  - Upper bounds for general graphs:
    - $O(n^{0.3885})$-approximation (Kortsarz & Peleg ’93)
    - $O(n^{1/3-\epsilon})$-approximation for some $\epsilon > 0$ (Feige et al. ’01)
    - $O(n^{1/4-\epsilon})$-approximation for any $\epsilon > 0$ (Bhaskara et al. ’10)
    - $O(n/k)$-approximation (Asahiro et al. ’96)
    - $n/k$-approximation (Feige et al. ’97)
    - Many improved bounds for special $k$ values....
Prior Work

• Upper bounds on special graph classes:
  • PTAS for dense instances (Arora et al. ’95)
  • PTAS for H-minor free graphs for any fixed H (Demaine et al ’05)
  • PTIME on trees (Maffioli ’91), co-graphs (Corneil & Pearl ’84)
  • 3-approximation on Chordal graphs by greedy (Liazi et al. ’08)
    • NP-hardness on chordal graph (Corneil & Pearl ’84)
  • Many others....
Our Result

- We show that there is an $O(\sigma)$-approximation for the class of graphs admitting $\sigma$-elimination orders.
- The class (with constant $\sigma$) includes:
  - chordal graphs,
  - circular-arc graphs,
  - claw-free graphs,
  - line graphs of l-hypergraphs,
  - planar graphs,
  - disk graphs,
  - and the intersection graphs of fat objects.
- This implies constant approximations for ALL above graph classes.
$\sigma$-elimination order

- The notion was introduced by K. Akcoglu et al. '02 (under name *sequentially k-independent graphs*).
- Its algorithmic aspect has drawn researchers’ attention recently. (Ye & Borodin ’09)
- A significant generalization of perfect elimination order for Chordal graphs.
σ-elimination order

- Some Notations:
  - $\text{pred}_L(v)$: the set of neighbors of $v$ that appear before $v$ in the vertex ordering $L$
  - $\alpha(G)$: the independence number of $G$, i.e., the cardinality of the max independent set of $G$

- perfect elimination order $L$:
  - An ordering of all vertices such that $\text{pred}_L(v)$ is a clique
  - A graph has a perfect elimination order iff it is chordal

- σ-elimination order $L$:
  - An ordering of all vertices such that $\alpha(G[\text{pred}_L(v)]) \leq \sigma$
  - A perfect elimination order is a 1-elimination order
A little more on $\sigma$-elimination order

- We show that each circular-arc graph has a 2-EO
- Circular-arc graphs:

  The intersection graph of those arcs:
  - each arc is a node
  - there is an edge between two nodes iff two corresponding arcs intersect
A little more on σ-elimination order

- 2-EO: just sort the arcs in an non-increasing order of their lengths.

Proof:

We need to prove $\alpha(\text{pred}(a)) \leq 2$ for all arc $a$.

Consider arc $a$,
- each arc longer than $a$ contains at least one endpoint of $a$
- arcs that share a common point form a clique
- therefore, neighbors of $a$ can be partitioned into two cliques.
Densest k Subgraph Approximation

- The Maximum Density Subgraph Problem (MDSP)
  - Each vertex $v$ has a weight $w(v)$
  - Density of $S$: $\rho(S) = \frac{w(S) + |E[S]|}{|S|}$
  - Find a subset $S$ that maximizes $\rho(S)$
  - PTIME by parametric network flow (Gallo et al. ’89)

- A important property:
  - For any $v$ in $S$, $w(v) + \text{deg}(v,S) \geq \rho(S)$ (since otherwise we can delete $v$ from $S$ to get a subgraph with larger density)
  - $\text{deg}(v,S)$: degree of $v$ in $G[S]$
Densest k Subgraph Approximation

• Outline of the algorithm:
  • Phase 1: Growing
    • Add MDSP solution to $S$ until $|S| > k/2$ (more details soon)
    • We can show that $S$ has a density $\geq O(1)\rho^*$
      ($\rho^*$: the density of OPT)
    • If $|S| \leq k$, we are done.
    • If $|S| > k$:
  • Phase 2: Shrinking
    • We look at the $\sigma$-EO of $S$ and find a subgraph of $S$ s.t. the deg. of each node is $O(1)$ of the deg in $G[S]$.
    • Therefore, it is a $O(1)$-approx.
Densest $k$ Subgraph Approximation

- **Phase 1: Growing**
  - Suppose $S$ is the set of nodes we’ve already chosen
  - Run MDSP on the remaining graph with vertex weight $w(v) = \text{deg}(v, S)$
    - This is used to capture the edges between $S$ and the remaining nodes
  - Repeat until $|S| \geq k$
  - We can show $\text{deg}(v) > O(1) \rho^*$ for all $v$ in $S$
Densest k Subgraph Approximation

- **Phase 2: Shrinking**
  - Suppose we end up with \(|S| > k\) obtained from Phase 1
  - Consider the elimination order of \(G[S]\)
    - If there is a node \(v\) s.t. \(|\text{Pred}(v)| \geq k/2\), then uniformly&randomly pick \(k/2\) vertices from \(\text{Pred}(v)\) is a \(O(1)\)-approximation

**Proof:**
First notice the fact that: if \(\alpha(G) = O(1)\), then \(|E(G)| = O(V^2)\)
Therefore, the expect number of edges in our solution \(SOL\) is
\[
\sum_e \text{Pr}(e \text{ in } SOL) = O(V^2) \times (k/2V) \times (k/2V) = O(k^2)
\]
Densest $k$ Subgraph Approximation

**Phase 2: Shrinking**
- Suppose we end up with $|S| >> k$ obtained from Phase 1
- Consider the elimination order of $G[S]$:
  - If there is no node s.t. $|\text{Pred}(v)| \geq k/2$

**Process EO in the reverse order**
- Either $\text{Pred}(v) = O(1)\rho^*$ or $\text{Succ}(v) = O(1)\rho^*$ (since $\text{deg}(v) = O(1)\rho^*$)
  - If $\text{Pred}(v) = O(1)\rho^*$: $\text{SOL} = \text{SOL} + v + \text{Pred}(v)$
    - We add a graph where each node has deg $O(\rho^*)$
  - If $\text{Succ}(v) = O(1)\rho^*$: $\text{SOL} = \text{SOL} + v$
    - $v$ has $\text{deg} = O(\rho^*)$ in SOL

Therefore, an $O(1)$-approximation
Open Problems

• Finding more graph classes that admit σ-EO (with small σ)
• Finding more applications for σ-EO
  • Ye & Borodin found that a few graph optimization problems are easier if the graph has O(1)-EO
• Closing the huge gap for densest k subgraph

• **OPEN:** Whether densest k subgraph is NP-hard on interval graphs and planar graphs
Thanks