# Improved Approximation Bounds for Planar Point Pattern Matching*

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**Abstract**

We analyze the performance of simple algorithms for matching two planar point sets under rigid transformations so as to minimize the directed Hausdorff distance between the sets. This is a well studied problem in computational geometry. Goodrich, Mitchell, and Orletsky presented a very simple approximation algorithm for this problem, which computes transformations based on aligning pairs of points. They showed that their algorithm achieves an approximation ratio of 4. We introduce a modification to their algorithm, which is based on aligning midpoints rather than endpoints. This modification has the same simplicity and running time as theirs, and we show that it achieves a better approximation ratio of roughly 3.14. We also analyze the approximation ratio in terms of a instance-specific parameter that is based on the ratio between diameter of the pattern set to the optimum Hausdorff distance. We show that as this ratio increases (as is common in practical applications) the approximation ratio approaches 3 in the limit. We also investigate the performance of the algorithm by Goodrich *et al.* as a function of this ratio, and present nearly matching lower bounds on the approximation ratios of both algorithms.

**Keywords:** Point pattern matching, approximation algorithms, Hausdorff distance.

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## 1 Introduction

Geometric point pattern matching problem is a fundamental computational problem and has numerous applications in areas such as computer vision [19], image and video compression [4], model-based object recognition [15], and computational chemistry [10]. Two point sets are given from some space, a *pattern set* $P$ and *background set* $Q$. Let $m = |P|$ and $n = |Q|$. The goal is to compute the transformation $E$ from some geometric group acting on this space that minimizes some distance measure from the transformed pattern set $E(P)$ to the background set $Q$. Throughout, we will consider point sets in the Euclidean plane under rigid transformations (translation and rotation). Distances between two point sets $P$ and $Q$ will be measured by the *directed Hausdorff distance*, denoted $h(P, Q)$, which is defined to be

$$h(P, Q) \;=\; \max_{p \in P} \min_{q \in Q} \|pq\|,$$

where $\|pq\|$ denotes the Euclidean distance between points $p$ and $q$.

The directional Hausdorff distance is natural in applications where the pattern $P$ is expected to match some subset of the background $Q$. In applications where it is desired that every point of $P$ matches some point of $Q$ and vice versa, a bidirectional similarity measure may be more appropriate such as the *bidirectional Hausdorff distance*, which is defined to be $\max(h(P, Q), h(Q, P))$. When matching with bidirectional similarity measures it may be possible to identify a global reference point (such as the centroid of each set) about which to anchor the alignment. This is not possible, however, under the directed Hausdorff distance. Throughout, unless otherwise specified, we use the term *Hausdorff distance* to denote the directed Hausdorff distance.

The problem of point pattern matching has been widely studied. We will focus on methods from the field of computational geometry. Excellent surveys can be found in [2] and [21]. There are many variants of this problem, depending on the nature of the inputs, the allowable group of aligning transformations and the distance function employed. Some formulations differ from ours either in that they require exact matches [8,20] or they involve bidirectional similarity measures [1,3,9,14,18].

A number of algorithms have been proposed for computing optimal pattern matching under the Hausdorff distance [6,16]. The computational complexity can be quite high, however. For example, the best-known algorithm for determining the translation and rotation that minimize the directed Hausdorff distance between two planar point sets $P$ and $Q$ of sizes $m$ and $n$, respectively, runs in $O(m^3 n^2 \log^2 mn)$ time [6]. This complexity may be unacceptably high for applications involving hundreds or thousands of points. For this reason approximation algorithms have been considered. In this context the *approximation ratio* of a pattern matching algorithm is the maximum ratio over all input instances between the Hausdorff distance produced by the algorithm and the optimum Hausdorff distance.

A simple and natural approach is to consider transformations induced by aligning a small subset of points from one set to the other. This is arguably the simplest and most easily implemented algorithm for approximate pattern matching, and it is the basis of some of the most popular methods in computer vision such as RANSAC [11]. Goodrich, Mitchell, and Orletsky [13] were the first to prove an upper bound on the approximation ratio of such a simple alignment-based algorithm. They considered point pattern matching under a number of different transformation spaces and in different dimensions. For the case of rigid transformations in the plane, their algorithm computes a diametrical pair for $P$ and then computes for every pair of distinct points of $Q$ a rigid transformation that aligns these pairs. (Details are presented in the next section.) It returns the transformation achieving the minimum Hausdorff distance. It runs in $O(n^2 m \log n)$ time. They prove that it returns an aligning transformation whose Hausdorff distance is at most a factor of 4 larger than the optimum Hausdorff distance.

Other authors have considered more precise approximation algorithms for planar point matching. Indyk, Motwani, and Venkatasubramanian [17] presented an $\varepsilon$-approximation algorithm for this problem, that is, one whose Hausdorff distance is most a factor of $(1+\varepsilon)$ greater than the optimum, where $\varepsilon > 0$ is a user-supplied parameter. Their formulation differs somewhat from ours, since it involves a decision problem that is given an estimate of the optimum Hausdorff distance. Their approach is also based on alignments, but it achieves higher accuracy than that of [13] by generating a number of aligning transformations for each aligning pair. Their algorithm's running time is not purely combinatorial, but depends on the *spread* of the set, denoted $\Delta$, which is defined to be the ratio of the distances between the farthest and closest pairs of points. They showed that, for any fixed $\varepsilon > 0$, a $(1+\varepsilon)$-approximation can be computed in time $O(n^{4/3}\Delta^{1/3} \log n)$, where the hidden constant factor depends on $\varepsilon$. Cardoze and Schulman [5] also presented an efficient randomized $\varepsilon$-approximation algorithm, which achieves its efficiency by reduction to a number of 1-dimensional point pattern matching problems, each of which is then reduced to computing a convolution using the fast Fourier transform. Its running time is $O(n^2 \log n + \log^{O(1)} \Delta)$ for any fixed precision parameter and any fixed success probability. Gavrilov *et al.* [12] present an empirical study of a number of pattern matching algorithms.

The purpose of this paper is not to propose a new algorithm, but rather to tighten the analysis of the approximation factors achievable by alignment-based approaches. Our focus will be on the alignment-based algorithm of Goodrich, Mitchell, and Orletsky [13]. Note that our results can also be used to improve the performance of other algorithms that make use of matches based on point-to-point alignments, such as the $\varepsilon$-approximation algorithm by Indyk, Motwani, and Venkatasubramanian [17]. The algorithm of Goodrich *et al.* is based on aligning points one by one, and henceforth we refer to this as *serial alignment*. (We will describe this algorithm in detail in Section 2.) Given the simplicity of this approach, it is natural to establish good bounds on its performance. This also points toward larger questions such as what are the best approaches to alignment-based matching and what are the inherent limits on the accuracy of alignment-based matching. We show that it is possible to improve on their approximation ratio of 4. Our approach has the same running time as theirs and, like theirs, is very easy to implement. It is based on a minor modification that selects the transformation that best aligns the entire subset of points, which we call *symmetric alignment*. Let $A_{ser}$ and $A_{sym}$ denote the approximation ratios for these respective algorithms.

Rather than just considering the worst-case approximation ratios, we analyze the approximation ratios of these algorithms in a manner that is sensitive to the optimal Hausdorff distance. For each problem instance $P$ and $Q$, we define an geometric parameter $\rho$, called the *distance ratio*, to be half the ratio of the diameter of $P$ to the optimum Hausdorff distance between $P$ and $Q$. (A formal definition is given in Section 3.) We show that as the distance ratio increases, the accuracy of the approximation increases as well. We feel that this analysis is useful because large values of $\rho$ often arise in applications. For example, in document analysis and satellite image analysis, the ratio of the diameter of a typical pattern ranges from tens to hundreds of pixels, while the expected digitization error is on the order of a single pixel. Let $A_{ser}(\rho)$ and $A_{sym}(\rho)$ denote the approximation ratios for serial and symmetric alignment, respectively, as a function $\rho$. Our results are summarized in Table 1. The quantity $c_\infty$ is the approximation ratio $A_{ser}(\rho)$ in the limit as $\rho$ approaches infinity, which we show later to be roughly 3.19.

Table 1: Summary of results: Bounds on the approximation ratios for symmetric and serial alignment algorithms as a function of the distance ratio $\rho$, where $c_\infty \approx 3.19$.

| Algorithm | Approximation Ratio | | |
|---|---|---|---|
| | Upper Bound (all $\rho$) | Upper Bound | Lower Bound |
| Symmetric Alignment | $A_{sym} \leq 3.14$ | $A_{sym}(\rho) \leq 3 + \frac{1}{\sqrt{3}\,\rho}$ | $A_{sym}(\rho) \geq 3 + \frac{1}{10\rho^2}$ |
| Serial Alignment | $A_{ser} \leq 3.44$ | $A_{ser}(\rho) \leq c_\infty + \frac{9}{4\rho}$ | $A_{ser}(\rho) \geq c_\infty + \frac{1}{27\rho^2}$ |

Our results imply that the approximation ratio for symmetric alignment is better than that of serial alignment for almost all but very small values of $\rho$. Further, for typical applications where distance ratio is large, the approximation factor of symmetric alignment is close to 3. As mentioned earlier, our results can be applied to other algorithms that are based on point-to-point alignments. For example, the $\varepsilon$-approximation algorithm of Indyk *et al.* [17] uses the simple alignment algorithm as a subroutine. The running time of their algorithm has a cubic dependence on the approximation ratio of the alignment algorithm. So, improving the approximation ratio bound by a factor of $f$ results in factor of $f^3$ reduction in the running time of their algorithm.

The remainder of the paper is organized as follows. In Section 2 we present the serial and symmetric alignment algorithms, and we provide an overview of their approximation ratios. In Sections 3 and 4, we present our upper and lower bounds, respectively, on the approximation ratio for symmetric alignment. In Sections 5 and 6, we present upper and lower bounds for serial alignment. Finally, in Section 7 we summarize and present concluding remarks.

## 2 The Serial and Symmetric Alignment Algorithms

In this section we present a description of the serial alignment algorithm of Goodrich, Mitchell, and Orletsky [13] and review its approximation ratio. We also described our modification of this algorithm, called symmetric alignment. Recall that we are given two planer point sets, a pattern set $P = \{p_1, \ldots, p_m\}$ and a background set $Q = \{q_1, \ldots, q_n\}$. We seek a rigid transformation $E$ that minimizes the (directed) Hausdorff distance from $E(P)$ to $Q$.

The serial alignment algorithm proceeds as follows. Let $(p_1, p_2)$ denote a pair of points having the greatest distance in $P$, that is, a *diametrical pair*. For each distinct pair $(q_1, q_2)$ in $Q$, the algorithm computes a rigid transformation matching $(p_1, p_2)$ with $(q_1, q_2)$ as follows. First, it applies to $P$ a translation that maps $p_1$ to $q_1$. Then it performs a rotation about $p_1$ (after translation) that aligns the directed line segment $\overrightarrow{p_1p_2}$ with $\overrightarrow{q_1q_2}$. The rigid transformation $E$ resulting from the composition of this translation and rotation is then applied to the entire pattern set $P$ and the Hausdorff distance $h(E(P), Q)$ is computed. After repeating this for all pairs $(q_1, q_2)$, the transformation with the smallest Hausdorff distance is returned. The running time of this algorithm is $O(n^2 m \log n)$ because, for each of the $n(n-1)$ distinct pairs of $Q$, we compute the aligning transformation $E$ in $O(1)$ time, and then, for each point $p \in P$, we compute the distance from $E(p)$ to its nearest neighbor in $Q$. (As in [17], if some *a priori* bound is provided for the optimum Hausdorff distance, it is possible to reduce the number of pairs $(q_1, q_2)$ that need to be considered.) Nearest neighbors queries in a planar set $Q$ can be answered in time $O(\log n)$ after $O(n \log n)$ preprocessing [7]. Because the analysis of our improved algorithm is closely related, let us review the proof of the approximation ratio given by Goodrich *et al.* [13]. Throughout, we let $h_{opt}$ denote the optimum Hausdorff distance from $P$ to $Q$.

**Lemma 2.1** (Goodrich, Mitchell, and Orletsky) *The approximation ratio for serial alignment algorithm satisfies* $A_{ser} \leq 4$.

**Proof**: It is easy to see that the serial alignment algorithm is invariant to the initial placement of $P$ in the sense that if $P$ and $P'$ are equal up to a rigid transformation, then (assuming general position) the algorithm will map both these point sets to the same final positions with respect to $Q$ and with the same Hausdorff distance. Thus to simplify matters, we may assume that $P$ has been presented to the algorithm in its optimal position with respect to $Q$. Thus, for each $p_i \in P$, if we let $q_i$ denote its closest point of $Q$ then $\|p_i q_i\| \leq h_{opt}$. Now, run the serial alignment algorithm. We will bound the maximum distance by which any point of $P$ has been displaced relative to its original optimal position, and this will provide the approximation ratio.

First, consider the effect of translation. Since $p_1$ is within distance $h_{opt}$ of $q_1$, each point of $P$ is translated by a distance of at most $h_{opt}$, and so is now within distance $h_{opt} + h_{opt} = 2h_{opt}$ of its closest point of $Q$. Next, consider the effect of rotation. After translation we have $\|p_2 q_2\| \leq 2h_{opt}$, from which it follows that the rotation displaces $p_2$ by an additional distance of at most $2h_{opt}$. Since $p_2$ is the farthest point of $P$ from $p_1$, every other point of $P$ is displaced through rotation by at most this distance. We see that after translation and rotation each point of $P$ is at distance at most $2h_{opt} + 2h_{opt} = 4h_{opt}$ from its closest point of $Q$. Therefore the approximation bound is at most $4h_{opt}/h_{opt} = 4$.

(Observe further that after running the algorithm $p_1$ coincides with $q_1$. Also, since rotation can only decrease the distance between $p_2$ and $q_2$, after serial alignment has completed the distance from $p_2$ to $q_2$ is at most $2h_{opt}$. These observations will be used below in Lemma 2.3.) □

There are two obvious shortcomings with this algorithm and its analysis. The first is that the algorithm does not optimally align the pair $(p_1, p_2)$ with the pair $(q_1, q_2)$ with respect to Hausdorff distance. One would expect such an optimal alignment to provide a better approximation ratio. This observation is the basis of our algorithm. The second shortcoming is that the use of the triangle inequality in summing the two displacements neglects the presence of any geometrical dependence between these two vector quantities.

We present a new approach called *symmetric alignment*. The algorithm differs only in how the aligning transformation is computed. Let the pairs $(p_1, p_2)$ in $P$ and $(q_1, q_2)$ in $Q$ be chosen in the same way as they are in the serial alignment algorithm. Let $m_p$ and $m_q$ denote the respective midpoints of line segments $\overline{p_1p_2}$ and $\overline{q_1q_2}$. First, translate $P$ to map $m_p$ to $m_q$ and then rotate $P$ about $m_p$ to align the directed segments $\overrightarrow{p_1p_2}$ with $\overrightarrow{q_1q_2}$. Thus, the only difference is that we align and rotate around the midpoints. It is easy to see that this alignment transformation minimizes the Hausdorff distance between the pairs $(p_1, p_2)$ and $(q_1, q_2)$.

There is a pathological setting in which symmetric alignment can perform arbitrarily poorly compared to the optimum. This happens when, in the optimal Hausdorff placement of $P$, both $p_1$ and $p_2$ share the same closest point of $Q$. The problem is that symmetric alignment assumes that $q_1$ and $q_2$ are distinct (for otherwise the rotation angle is undefined). For this to occur the distance between $p_1$ and $p_2$ (that is, the diameter of $P$) must be less than the distance between the closest pair of points of $Q$. In such a case, the optimal placement results by simply aligning the center of the smallest enclosing disk for $P$ with any point of $Q$. By standard methods in computational geometry (see [7]) this situation an be detected and handled in $O(m + n \log n)$ time, and so henceforth we will assume that it is not an issue.

Before giving our detailed analysis of the approximation ratio we establish a crude bound on the approximation ratio for symmetric alignment, which justifies the benefit of this approach over serial alignment.

**Lemma 2.2** $A_{sym} \;\leq\; 2 + \sqrt{3} \;\approx\; 3.732.$

**Proof**: As in the proof of Lemma 2.1, let us assume that $P$ has been presented to the algorithm in its optimal position with respect to $Q$. Thus, each point of $P$ is within distance $h_{opt}$ of its closest point of $Q$. Recall that $q_1$ and $q_2$ denote the closest points of $Q$ to $p_1$ and $p_2$, respectively. For $i \in \{1, 2\}$ let $\vec{v}_i$ denote the vector $\overrightarrow{p_iq_i}$.

First, let us consider translation. To match the midpoints $m_p$ and $m_q$, we translate the pattern set $P$ by the vector $\vec{t} = (\vec{v}_1 + \vec{v}_2)/2$. Since both of these points are within distance $h_{opt}$ of their closest point of $Q$, we have $\|\vec{t}\| \leq h_{opt}$. Next we consider rotation. The distance between the translated point $p_1 + \vec{t}$ and $q_1$ is

$$\left\| \vec{v}_1 - \frac{\vec{v}_1 + \vec{v}_2}{2} \right\| \;=\; \left\| \frac{\vec{v}_1 - \vec{v}_2}{2} \right\| \;\leq\; h_{opt}.$$

Because $(p_1, p_2)$ is a diametrical pair, it follows that all the points of $P$ lie in a lune defined by the intersection of two discs both of radius $\|p_1p_2\|$ centered at each of these points. Thus, no point of $P$ is farther from the center of rotation $(m_p)$ than the apex of this lune, which is at distance $(\sqrt{3}/2)\|p_1p_2\|$. Since this rotation moves $p_1$ by a distance of at most $h_{opt}$, and $p_1$ is within distance $(1/2)\|p_1p_2\|$ of $m_p$, it follows that every point $p \in P$ is moved through rotation by an additional distance of at most $\sqrt{3}h_{opt}$.

Thus, we see that each point of $P$ started within distance $h_{opt}$ of its closest point of $Q$, and it has been displaced through translation and rotation by distances of at most $h_{opt}$ and $\sqrt{3}h_{opt}$, respectively. Therefore, after symmetric alignment each point of $P$ is within distance

$$h_{opt} + h_{opt} + \sqrt{3}h_{opt} \;=\; \left(2 + \sqrt{3}\right) h_{opt} \;\approx\; 3.732 \cdot h_{opt},$$

of its closest point of $Q$. The approximation ratio follows immediately.

(Further observe that the points $p_1$ and $p_2$ started out within distance $h_{opt}$ from their closest points of $Q$, and after alignment this is still true. This fact will be used below in Lemma 2.3.) □

Even though this crude approximation ratio is already an improvement, it suffers from the same problem as the earlier analysis in that it does not consider the geometric relationship between the translation and rotation vectors. Note that this does not imply that symmetric alignment is better than serial alignment for all input instances. In Fig. 1 we show two informal examples. In the first case serial alignment is better and in the second symmetric alignment is better. The following lemma shows, nonetheless, that the two methods achieve comparable approximation ratios on all input instances.

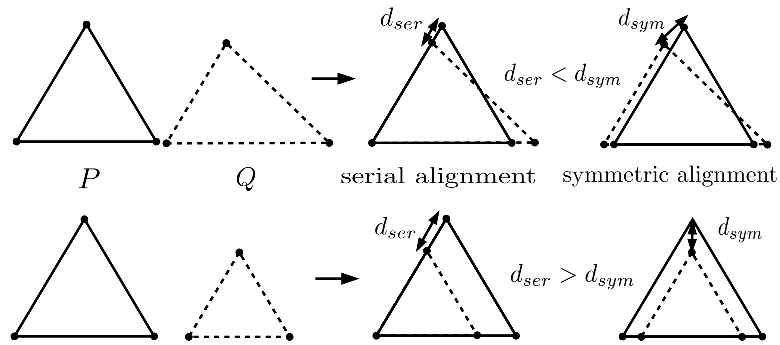


Fig. 1: Comparison of the two algorithms.

**Lemma 2.3** *Given two planar point sets $P$ and $Q$, let $d_{sym}$ and $d_{ser}$ denote the Hausdorff distances between $P$ and $Q$ after running symmetric alignment and serial alignment, respectively. Then*

$$\begin{aligned} d_{sym} &\leq d_{ser} + \min\left(\frac{d_{ser}}{2}, h_{opt}\right) \leq \frac{3}{2} d_{ser} \qquad \textit{and} \\ d_{ser} &\leq d_{sym} + h_{opt} \leq 2 d_{sym}. \end{aligned}$$

**Proof**: Let us assume that $P$ has been presented to the symmetric alignment algorithm in the position output by the serial alignment algorithm. Recall that $(p_1, p_2)$ is a diametrical pair from $P$, and $(q_1, q_2)$ is the corresponding pair from $Q$. From the comments made at the end of the proof of Lemma 2.1 we know that $p_1$ and $q_1$ coincide, and the distance between $p_2$ and $q_2$ is at most $2h_{opt}$. Clearly, the distance between $p_2$ and $q_2$ is also at most $d_{ser}$. Now, we apply symmetric alignment. In order to align the midpoints of $\overline{p_1 p_2}$ with $\overline{q_1 q_2}$, each point of $P$ is translated by the distance $\|p_2 q_2\|/2$, which is at most $\min(d_{ser}, 2h_{opt})/2$. No rotation is needed. Therefore, we have

$$d_{sym} \leq d_{ser} + \frac{\|p_2 q_2\|}{2} \leq d_{ser} + \min\left(\frac{d_{ser}}{2}, h_{opt}\right) \leq \frac{3}{2} d_{ser}.$$

To prove the other bound, let us assume that the point set $P$ has been presented to the serial alignment algorithm in the positions output by the symmetric alignment algorithm. As observed at the end of the proof of Lemma 2.2, $\|p_1 q_1\| \leq h_{opt}$. If we apply serial alignment to these point sets, each point of $P$ will be translated by at most $\|p_1 q_1\|$. No rotation is needed. Therefore,

$$d_{ser} \leq d_{sym} + \|p_1 q_1\| \leq d_{sym} + h_{opt} \leq 2 d_{sym}.$$

□

# 3 Symmetric Alignment: Upper Bound

In this section we derive an upper bound on the approximation ratio for symmetric alignment. As before let $h_{opt}$ denote the optimum Hausdorff distance between $P$ and $Q$ achievable under any rigid transformation of $P$. As mentioned above, our analysis is sensitive to a geometric parameter that is proportional to the ratio of $P$'s diameter to the optimum Hausdorff distance. Define the *distance ratio* to be

$$\rho \;=\; \frac{diam(P)}{2h_{opt}},$$

where $diam(P)$ denotes diameter of $P$. Clearly $h_{opt} \leq diam(P)$ and therefore $\rho \geq \frac{1}{2}$. Recall that $A_{sym}(\rho)$ denotes the approximation ratio for symmetric alignment as a function of the distance ratio. The main result of this section is presented next.

**Theorem 1** *Consider two planar point sets $P$ and $Q$, and let $\rho$ be the distance ratio for $P$ and $Q$. Then*

$$A_{sym}(\rho) \;\leq\; \min\left(3 + \frac{1}{\sqrt{3}\,\rho}, \sqrt{4\rho^2 + 2\rho + 1}\right),$$

*and further $A_{sym}(\rho) \leq 3.14$ for all $\rho$.*

Recall that the distance ratio is large in many applications. This theorem shows that as $\rho$ tends to infinity, the approximation ratio is at most 3. The remainder of this section is devoted to proving this theorem. As usual, it will simplify the presentation to assume that $P$ has been transformed to its optimal placement with respect to $Q$, and we will bound the amount of additional displacement of each point of $P$ relative to this placement. Without loss of generality we may scale space uniformly so that the optimum Hausdorff distance $h_{opt}$ is 1. Recall that $(p_1, p_2)$ is a diametrical pair from $P$, and $(q_1, q_2)$ is the corresponding pair from $Q$. From our scaling it follows that for $i \in \{1, 2\}$, $q_i$ lies within a disc of radius 1 centered at $p_i$. (See Fig. 2.) Let $m_p$ and $m_q$ denote the respective midpoints of the segments $\overline{p_1p_2}$ and $\overline{q_1q_2}$. Note that $\rho$ is just the distance from $m_p$ to either $p_1$ or $p_2$. Let $\alpha \geq 0$ denote the absolute acute angle between the lines supporting the segments $\overline{p_1q_1}$ and $\overline{p_2q_2}$. If $\rho > 1$, then the two unit discs do not intersect. Hence, after aligning the midpoints by translation, the distance from $p_1$ to $q_1$ is still at most $h_{opt}(= 1)$ (see Lemma 2.2), and the orthogonal distance from $p_1$ to the line $\overline{q_1q_2}$, which is $\rho \sin \alpha$, is less than or equal to the distance from $p_1$ to any point on the line. It follows therefore that $0 \leq \rho \sin \alpha \leq 1$.

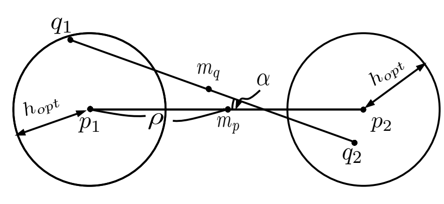


Fig. 2: The positions of the point sets prior to running the algorithm.

Before presenting the details, we give a brief overview of our proof structure. We establish two upper bounds on the approximation ratio, each a function of $\rho$. One bound is better for small $\rho$ and the other better for large $\rho$. We shall show by numerical computations that the two functions cross over one another at a value $\rho^*_{sym} \approx 1.26$ and that $A_{sym}(\rho^*_{sym}) \approx 3.14$. Furthermore, both these functions decrease monotonically as $\rho$ moves away from $\rho^*_{sym}$. We begin by establishing the approximation ratio for low distance ratios.

**Lemma 3.1** *For all $\rho$, $A_{sym}(\rho) \leq \sqrt{4\rho^2 + 2\rho + 1}$.*

**Proof**: Let $(p_1, p_2)$ and $(q_1, q_2)$ be as defined in the algorithm. As mentioned in the proof of Lemma 2.2, every point of $P$ lies within a lune formed by the intersection of two discs of radius $2\rho$ centered at $p_1$ and $p_2$. Hence, every point of $P$ is within distance $\sqrt{3}\rho$ of the midpoint $m_p$. After applying symmetric alignment, $\overline{p_1p_2}$ and $\overline{q_1q_2}$ are collinear and, for $i \in \{1, 2\}$, the distance $\|p_iq_i\|$ is at most $h_{opt}(= 1)$. Since, $\|p_im_p\| = \rho$, it follows directly that every point of $P$ is within distance $\sqrt{(\rho + 1)^2 + (\sqrt{3}\rho)^2}$ of either $q_1$ or $q_2$. Simplifying yields the desired bound. □

The above approximation ratio is very crude as it considers no points of $Q$ other than $q_1$ and $q_2$. Hence, it is only useful for very small values of $\rho$. To deal with the case of high $\rho$ values we consider the effects of translation and rotation separately. We will characterize the set of possible translations that align $m_p$ with $m_q$ as a function of $\rho$ and the rotation angle $\alpha$. The distance by which an arbitrary point of $p \in P \setminus \{p_1, p_2\}$ can be displaced by the rotation is not only a function of these two parameters, but is also a function of the distance of this point from the center of rotation $m_p$. We will then bound the sum of these two displacement distances. These translation and rotation analyses are presented in Sections 3.1 and 3.2, respectively, and these results are combined in Section 3.3.

## 3.1 Translational Displacement

We begin by considering the space of possible translations that align $m_p$ with $m_q$. It will make matters a bit simpler to think of translating $Q$ to align $m_q$ with $m_p$, but of course any bounds on the distance of translation will apply to case of translating $P$. The translation is given by a vector which we denote by $\vec{t}$. The following lemma bounds the length of this translation vector as a function of $\rho$ and $\alpha$.

**Lemma 3.2** *The symmetric alignment's translation transformation displaces any point of $P$ by a vector $\vec{t}$ of length*

$$\|\vec{t}\| \leq \sqrt{1 - \rho^2 \sin^2 \alpha}.$$

**Proof**: Recall that $(p_1, p_2)$ and $(q_1, q_2)$ denote the respective point pairs from the sets $P$ and $Q$ and that $\alpha$ is the acute angle between the lines $\overline{p_1p_2}$ and $\overline{q_1q_2}$. Since we have assumed that $P$ is placed in the optimal position, $q_1$ and $q_2$ must lie inside the circles of radius $h_{opt}(= 1)$ centered at $p_1$ and $p_2$, respectively. For the sake of illustration let us assume that $\overrightarrow{p_1p_2}$ is directed horizontally from left to right and that $\overline{q_1q_2}$ has a nonpositive slope. (See Fig. 3(a).)

Consider a line passing through $m_p$ that is parallel to $\overline{q_1q_2}$. Let $r_1$ and $r_2$ be the respective orthogonal projections of $p_1$ and $p_2$ onto the line $\overline{q_1q_2}$, and let $s_1$ and $s_2$ denote the respective signed displacements $\|\overrightarrow{r_1q_1}\|$ and $\|\overrightarrow{r_2q_2}\|$ along this line (See Fig. 3(b).) Consider a coordinate system centered at $m_p$ whose positive $u$-axis is located along a line $\ell$ that is directed from left to

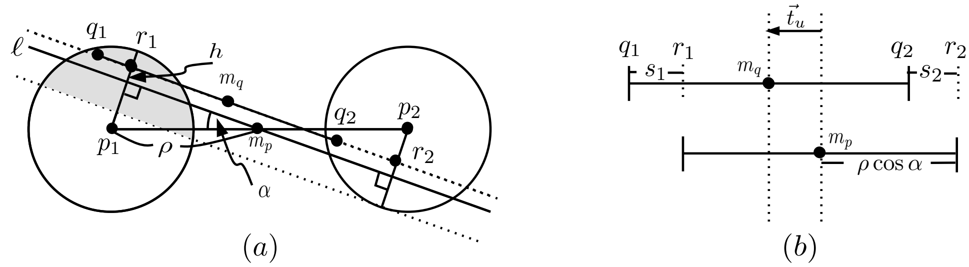


Fig. 3: Analysis of the midpoint translation.

right, and whose positive $v$-axis is perpendicular to this and directed upwards. In this coordinate system we have $m_p = (0,0)$, $q_1 = (s_1 - \rho\cos\alpha, h)$, and $q_2 = (s_2 + \rho\cos\alpha, h)$. Thus, $m_q = ((s_1+s_2)/2, h)$, and the translation vector $\vec{t}$ is $m_q - m_p = m_q$. By straightforward calculations we have $|s_1| \le \sqrt{1-(\rho\sin\alpha+h)^2}$ and $|s_2| \le \sqrt{(1-(\rho\sin\alpha-h)^2}$. Therefore,

$$\begin{aligned}
\|\vec{t}\|^2 &= u^2+v^2 = \left(\frac{s_1+s_2}{2}\right)^2 + h^2 = \frac{1}{2}(s_1^2+s_2^2) - \frac{1}{4}(s_1-s_2)^2 + h^2 \\
&\le \frac{1}{2}(s_1^2+s_2^2) + h^2 \le \frac{1}{2}\left[(1-(\rho\sin\alpha+h)^2) + (1-(\rho\sin\alpha-h)^2)\right] + h^2 \\
&= 1-\rho^2\sin^2\alpha.
\end{aligned}$$

Observe that the translation is maximized when $s_1 = s_2$, and this implies that the line $\overline{q_1q_2}$ passes through $m_p$. (This will be used in our lower bound construction in Section 4.) □

Given $\alpha$ and $\rho$, let us refer to the set of all valid $(u,v)$ translation vectors as the *translation space*, denoted $\mathcal{T}_\rho(\alpha)$. A more detailed analysis of the set of valid $(u,v)$ pairs shows that the set of possible translational displacements lies within a region of the $(u,v)$ plane as illustrated in the shaded region of Fig. 4, but we will not need this for our subsequent analysis.

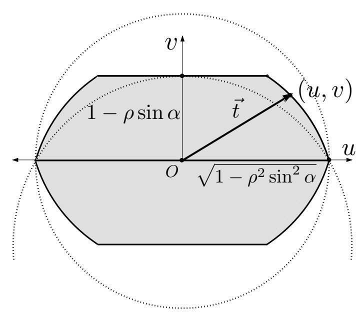


Fig. 4: Translation space, $\mathcal{T}_\rho(\alpha)$, for symmetric alignment.

## 3.2 Rotational Displacement

Next we consider the effect of rotation on the approximation error. Unlike translation we consider the placement of points in $P$ because the distance by which a point is displaced by rotation is determined by the distance to the center of rotation. For each point $p \in P$, let $\vec{r}_p$ denote its displacement due to rotation. As mentioned in the proof of Lemma 2.2 every point of $P$ lies within a lune formed by the intersection of two discs of radius $2\rho$ centered at $p_1$ and $p_2$. (See Fig. 5(a).) The following lemma describes the possible displacements of a point of $P$ under the rotational part of the aligning transformation. This is done relative to a coordinate system centered at $m_p$, whose $x$-axis is directed towards $p_2$.

**Lemma 3.3** *The symmetric alignment's rotation transformation displaces any point $p \in P$ by a vector $\vec{r}_p$ of length*

$$\|\vec{r}_p\| \leq 2\sqrt{3}\rho \sin \frac{\alpha}{2}.$$

**Proof**: Let $\theta$ be the signed angle from $\overrightarrow{m_p p_2}$ to $\overrightarrow{m_p p}$. We may assume $-\pi \leq \theta \leq \pi$. Let $p'$ denote the point after rotating $p$ clockwise about $m_p$ by angle $\alpha$. (See Fig. 5(a).)

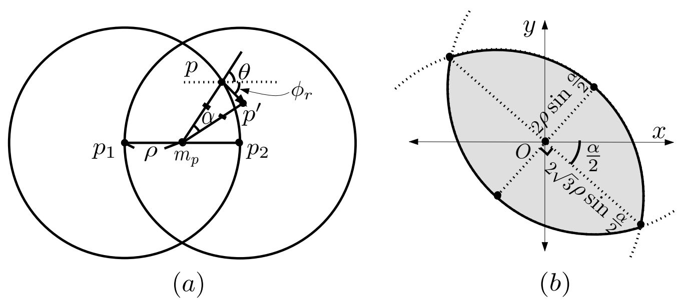


Fig. 5: Analysis of the rotational displacements for symmetric alignment.

By simple trigonometry and the observation that $\triangle m_p p p'$ is isosceles, it follows that the displacement vector $\overrightarrow{pp'}$, length $\|\vec{r}_p\| = \|m_p p\| \cdot 2 \sin \frac{\alpha}{2}$. The farthest point of the lune from $m_p$ is easily seen to be the apex, which is at distance $\sqrt{3}\rho$. Thus, $\|\vec{r}_p\|$ is at most $2\sqrt{3}\rho \sin \frac{\alpha}{2}$.

□

As we did for translation, for a given pair of parameter values $\alpha$ and $\theta$ we can define the set of possible vector displacements for all points $p \in P$, which we call the *rotation space*, denoted $\mathcal{R}_\rho(\alpha)$. This is illustrated in Fig. 5(b), but is not needed for the rest of our analysis. Henceforth, when $p$ is clear for context, we will refer to $\vec{r}_p$ as $\vec{r}$.

### 3.3 Combining Translation and Rotation

We are now ready to derive the approximation ratio for the symmetric alignment algorithm by combining the bounds on the translational and rotational displacements. Recall that we apply Lemma 3.1 to bound $A_{sym}(\rho)$ for small values of $\rho$, and so here we will assume that $\rho \geq \rho^*_{sym}$.

Recall that at the start of the algorithm, the points are assumed to be placed in the optimal positions and that space has been scaled so that $h_{opt} = 1$. Let us fix any point $p \in P$, let $\vec{t}$ denote its displacement due to translation, and let $\vec{r} = \vec{r}_p$ denote its displacement due to rotation. It follows that $p$ has been displaced from its initial position by a distance of $\|\vec{t} + \vec{r}\|$. Since its initial position was within distance $h_{opt}$ $(= 1)$ of some point of $Q$, it is now within distance $\|\vec{t} + \vec{r}\| + 1$

of some point of $Q$. Recalling that $\mathcal{T}_\rho(\alpha)$ and $\mathcal{R}_\rho(\alpha)$ denote the space of possible translational and rotational displacement vectors and that $0 \leq \sin\alpha \leq 1/\rho$, we have

$$A_{sym}(\rho) \;\leq\; \max_{\alpha}\left(\max_{\substack{\vec{t}\in\mathcal{T}_\rho(\alpha)\\ \vec{r}\in\mathcal{R}_\rho(\alpha)}} \|\vec{t}+\vec{r}\|\right) + 1.$$

Unfortunately, determining this length bound exactly would involve solving a relatively high order equation involving trigonometric functions. Instead, we will simplify matters by applying the triangle inequality to separate the translational and rotational components, which are in turn bounded in Lemmas 3.2 and 3.3.

$$\begin{aligned} A_{sym}(\rho) &\leq \max_{\alpha}\left(\max_{\vec{t}\in\mathcal{T}_\rho(\alpha)} \|\vec{t}\| + \max_{\vec{r}\in\mathcal{R}_\rho(\alpha)} \|\vec{r}\|\right) + 1 \\ &\leq \max_{\alpha}\left(\sqrt{1-\rho^2\sin^2\alpha} + 2\sqrt{3}\rho\sin\frac{\alpha}{2}\right) + 1. \end{aligned}$$

Substituting $x = \cos\alpha$, it follows that the quantity to be maximized is

$$f_\rho(x) \;=\; \sqrt{1-\rho^2(1-x^2)} + \sqrt{6}\rho\sqrt{1-x} + 1, \qquad \text{where } \sqrt{1-\frac{1}{\rho^2}} \leq x \leq 1.$$

To determine the maximum value of $f_\rho$ we take the partial derivative with respect to $x$, yielding

$$\frac{\partial f_\rho}{\partial x} \;=\; -\frac{\sqrt{3}\rho}{\sqrt{2(1-x)}} + \frac{\rho^2 x}{\sqrt{1-\rho^2(1-x^2)}}.$$

By our assumption that $\rho \geq \rho^*_{sym} \approx 1.26$, it follows by symbolic manipulations that $\partial f_\rho/\partial x = 0$ has a single real root for $\rho \geq \rho^*_{sym}$, which is

$$\begin{aligned} x_0(\rho) &= \frac{1}{6}\left(-1 + c_1(\rho) + \frac{1}{c_1(\rho)}\right), \\ &\quad \text{where} \quad c_1(\rho) = \frac{\rho^2}{(161\rho^6 + 18\rho^4\sqrt{c_2(\rho)} - 162\rho^4)^{1/3}} \\ &\quad \text{and} \quad c_2(\rho) = 80\rho^4 - 161\rho^2 + 81. \end{aligned}$$

By an analysis of this derivative it follows that the function $f_\rho$ achieves its maximum value when $x = x_0(\rho)$. Thus, we have the following.

**Lemma 3.4** *For $\rho \geq \rho^*_{sym} \approx 1.26$ we have*

$$A_{sym}(\rho) \;\leq\; f_\rho(x_0(\rho)) \;=\; \sqrt{1-\rho^2(1-x_0(\rho)^2)} + \sqrt{6}\rho\sqrt{1-x_0(\rho)} + 1,$$

*where $x_0(\rho)$ is defined above.*

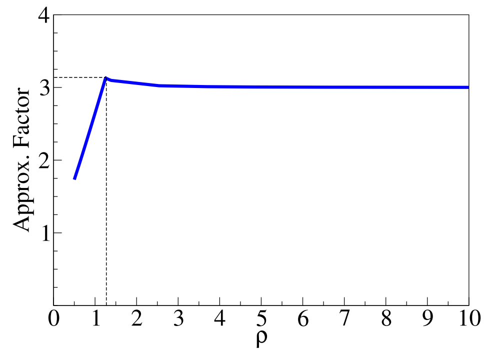


Fig. 6: The approximation ratio for the symmetric alignment algorithm as a function of $\rho$.

Unfortunately, this function is too complex to reason about easily. Nonetheless, we can evaluate it numerically for any fixed value of $\rho$. The resulting approximation ratio (together with the alternate approximation ratio of Lemma 3.1) is plotted as a function of $\rho$ in Fig. 6 below. A numerical analysis shows that the two functions cross over at $\rho^*_{sym} \approx 1.26$, and at this point the function achieves its maximum value of roughly 3.14. The figure also indicates that the approximation ratio approaches 3 as $\rho$ tends to $\infty$. The following result establishes this asymptotic convergence by proving a somewhat weaker approximation ratio.

**Lemma 3.5** *If $\rho > \rho^*_{sym}$ then $A_{sym}(\rho) \leq 3 + \frac{1}{\sqrt{3}\rho}$.*

**Proof**: Before presenting the general case, we consider the simpler limiting case when $\rho$ tends to $\infty$. Since $0 \leq \sin\alpha \leq 1/\rho$, in the limit $\alpha$ approaches 0. Using the fact of $\lim_{\alpha\to 0} \frac{\sin\alpha}{\alpha} = 1$ we have

$$\begin{aligned} A_{sym}(\infty) &= \lim_{\rho\to\infty} A_{sym}(\rho) \leq \lim_{\rho\to\infty} \sqrt{1-\rho^2\sin^2\alpha} + 2\sqrt{3}\rho\sin\frac{\alpha}{2} + 1 \\ &= \lim_{\rho\to\infty} \sqrt{1-\rho^2\alpha^2} + \sqrt{3}\rho\alpha + 1. \end{aligned}$$

Let $x = \rho\alpha$. In the limit we have $0 \leq x \leq 1$ and so

$$A_{sym}(\infty) \leq \max_{0\leq x\leq 1} \left(\sqrt{1-x^2} + \sqrt{3}x + 1\right).$$

It is easy to verify that $A_{sym}(\infty)$ achieves a maximum value of 3 when $x = \sqrt{3}/2$.

Next, we consider the general case. To simplify $A_{sym}(\rho)$, we apply a Taylor's series expansion.

$$\begin{aligned} A_{sym}(\rho) &\leq \max_{\alpha} \left(\sqrt{1-\rho^2\sin^2\alpha} + 2\sqrt{3}\rho\sin\frac{\alpha}{2} + 1\right) \qquad (1) \\ &\leq \max_{\alpha} \left(\sqrt{1-\rho^2\left(\alpha - \frac{\alpha^3}{6}\right)^2} + \sqrt{3}\rho\alpha + 1\right). \end{aligned}$$

Since $0 \leq \alpha \leq \arcsin(1/\rho)$, we have $0 \leq \rho\alpha \leq \rho \arcsin(1/\rho)$. For all $\rho > \rho^*_{sym}$, observe that $\rho \arcsin(1/\rho)$ is at most $\rho^*_{sym} \arcsin(1/\rho^*_{sym})$. Let us denote the value by $c_0$, which by numerical evaluation is approximately 1.16. Thus, we have $0 \leq \rho\alpha < c_0$. The rest of the analysis is broken into two cases: $0 \leq \rho\alpha \leq 1$ and $1 < \rho\alpha < c_0$.

In the first case ($0 \leq \rho\alpha \leq 1$) observe that $1 - \rho^2\alpha^2 \geq 0$. We argued above that $(\sqrt{1-\rho^2\alpha^2} + \sqrt{3}\rho\alpha + 1) \leq 3$ and so

$$\begin{aligned} A_{sym}(\rho) &\leq \max_{\alpha} \left( \sqrt{1 - \rho^2 \left(\alpha - \alpha^3/6\right)^2} + \sqrt{3}\rho\alpha + 1 + \left[ 3 - \left( \sqrt{1-\rho^2\alpha^2} + \sqrt{3}\rho\alpha + 1 \right) \right] \right) \\ &= \max_{\alpha} \left( \sqrt{1 - \rho^2 \left(\alpha - \alpha^3/6\right)^2} - \sqrt{1-\rho^2\alpha^2} + 3 \right). \end{aligned}$$

To simplify this let $x = \rho\alpha$. We see that $A_{sym}(\rho) \leq \max_{0 \leq x \leq 1} g(x)$ where,

$$g(x) = \sqrt{1 - x^2 \left(1 - \frac{x^2}{6\rho^2}\right)^2} - \sqrt{1-x^2} + 3.$$

For all fixed $\rho$ this is a monotonically increasing function in $x$. Since $x \leq 1$, it is easy to verify that this function achieves its maximum value at $x = 1$. Therefore,

$$A_{sym}(\rho) \leq g(1) = 3 + \sqrt{1 - \left(1 - \frac{1}{6\rho^2}\right)^2} = 3 + \sqrt{\frac{1}{3\rho^2} - \frac{1}{36\rho^4}} \leq 3 + \frac{1}{\sqrt{3}\rho},$$

as desired.

For the second case ($1 < \rho\alpha < c_0$) we cannot use the above approach because $1 - \rho^2\alpha^2 < 0$. Nonetheless, by expanding the first term of Eq. (1) we have

$$\begin{aligned} \sqrt{1 - \rho^2 \sin^2 \alpha} &\leq \sqrt{1 - \rho^2 \left(\alpha - \alpha^3/6\right)^2} = \sqrt{(1-\rho^2\alpha^2) + \frac{1}{3}\rho^2\alpha^4 - \frac{1}{36}\rho^2\alpha^6} \\ &< \sqrt{\frac{1}{3}\rho^2\alpha^4} = \frac{1}{\sqrt{3}}\rho\alpha^2. \end{aligned}$$

Therefore,

$$A_{sym}(\rho) \leq \max_{\alpha} \left( \frac{1}{\sqrt{3}}\rho\alpha^2 + \sqrt{3}\rho\alpha + 1 \right).$$

It is straightforward to verify that the above function achieves its maximum value when $\rho\alpha = c_0$, and this value is at most $3 + 1/(\sqrt{3}\rho)$. □

Combining this with Lemma 3.1 and the previous comments about the cross-over point completes the proof of Theorem 1.

## 4 Symmetric Alignment: Lower Bound

It is natural to wonder whether the upper bound on the approximation ratio $A_{sym}$ proved in Theorem 1 is tight. We show that this is nearly the case. In particular, we show that for all sufficiently large $\rho$ the approximation factor is strictly greater than 3, and it approaches 3 in the limit.

**Theorem 2** *For all sufficiently large $\rho$, there exists an input on which symmetric alignment achieves an approximation ratio of at least* $3 + \frac{1}{10\rho^2}$.

The remainder of this section is devoted to proving this. Our approach is to consider a configuration of points that generates the worst case (that is, the largest translation and rotational displacements) for Lemmas 3.2 and 3.3. Consider a fixed value of $\rho$. We define the pattern point set $P = \{p_1, p_2, p_3, p_4\}$ as follows. The first three points form an equilateral triangle of side length $2\rho$ oriented in counterclockwise order. We place $p_4$ at the midpoint of $\overline{p_1p_2}$. (See Fig. 7(a)) By an infinitesimal perturbation of the points, we may assume that the pair $(p_1, p_2)$ is the unique diametrical pair for $P$.

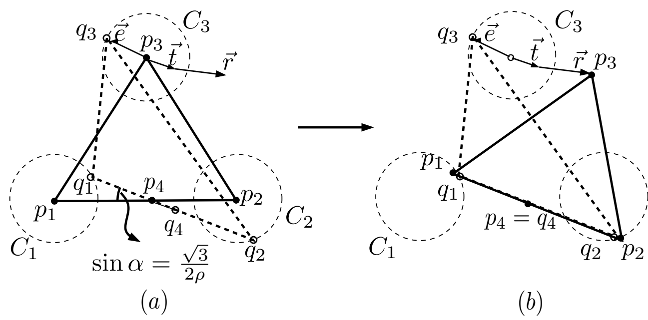


Fig. 7: The lower bound on $A_{sym}$.

We define the background set $Q = \{q_1, q_2, q_3, q_4\}$ so that the optimum Hausdorff distance will be at most 1. For $1 \leq i \leq 4$, let $C_i$ be a circle of unit radius centered $p_i$. Consider an angle $\alpha > 0$ such that $\sin\alpha = \frac{\sqrt{3}}{2\rho}$ and construct a line passing through $p_4$ (the midpoint of $\overline{p_1p_2}$) forming an angle $\alpha$ with line $\overline{p_1p_2}$. Place $q_1$ and $q_2$ at the rightmost intersection points of this line and $C_1$ and $C_2$, respectively. (See Fig. 7(a).) Running symmetric alignment so that $(p_1, p_2)$ is aligned with $(q_1, q_2)$ results in a translation and rotation. Let $\vec{t}$ and $\vec{r}$ denote the translation and rotational displacement vectors for $p_3$, respectively. Let $q_3 = p_3 + \vec{e}$, where $\vec{e}$ is a vector of unit length whose direction is chosen to be directly opposite that of $\vec{t} + \vec{r}$. Intuitively, $q_3$ has been chosen to be as far away as possible from $p_3$ after alignment. Finally, place $q_4$ at the midpoint of $\overline{q_1q_2}$. Observe that prior to alignment, each $p_i$ is within distance 1 from $q_i$, and therefore $h_{opt} \leq 1$.

Consider the placement of $P$ and $Q$ after symmetric alignment. (See Fig. 7(b).) We will analyze the displacement distance of $p_3$ relative to $q_3$. Because of the asymmetry introduced by $p_4$ and $q_4$, it is intuitively clear that if the distance ratio is high enough, symmetric alignment will achieve the best match by aligning the pair $(q_1, q_2)$ with $(p_1, p_2)$. We establish this formally in the next lemma.

**Lemma 4.1** *For all sufficiently large $\rho$, symmetric alignment aligns $(p_1, p_2)$ with $(q_1, q_2)$.*

**Proof**: Assume that $\rho > 2A_{sym}$. We will show that the choice of any other alignment pair will result in a Hausdorff distance greater than $A_{sym}h_{opt}$, a contradiction. Suppose that the algorithm aligns $(p_1, p_2)$ with some pair $(q_i, q_j)$, where $(q_i, q_j) \neq (q_1, q_2)$. For $1 \leq i \leq 4$, let $C_i$ denote the circle of radius $A_{sym}$ centered at point $p_i$, after alignment. (These are different from the circles $C_i$ used in the construction.) Since $\rho > 2A_{sym}$, all these circles are disjoint. In order for the Hausdorff distance to be less than $A_{sym}$, after alignment each of these circles must contain exactly one point of $Q$.

First, we will show that $q_4$ should be in $C_4$. Suppose that to the contrary that $q_4$ is in circle $C_k$, $k \neq 4$. Then $q_1$ must lie in some other circle. Since $q_1$ and $q_2$ are symmetric with respect to the point $q_4$, the point $q_2$ cannot be in any circle, a contradiction.

From the hypothesis that $(p_1, p_2) \neq (q_1, q_2)$, either $q_1$ or $q_2$ should be in $C_3$. Let us assume that $q_1$ is in $C_3$. (The other case is similar.) Then $q_2$ may be in either $C_1$ or $C_2$. However, if $q_2$ lies in $C_2$, then clearly $q_3$ lies in none of the circles. (Note that we do not allow for reflection.) Thus, $q_2$ must lie within $C_1$ and $q_3$ lies within $C_2$, implying that $(q_2, q_3)$ is the chosen aligning pair for $(p_1, p_2)$. Thus, (after alignment) the midpoint of $\overline{q_2q_3}$, called it $m_{23}$, coincides with $p_4$. Since $\triangle q_4q_2m_{23}$ and $\triangle q_1q_2q_3$ are similar up to a scale factor of 2, we have

$$\|p_4q_4\| \;=\; \frac{\|q_1q_3\|}{2}.$$

Also, observe that prior to alignment, each point of $Q$ is within distance $h_{opt}$ of its corresponding point of $P$, and so by the triangle inequality the distance between any two points $\|q_iq_j\|$ of $Q$ is within $2h_{opt}$ of the distance $\|p_ip_j\|$. So $\|q_1q_3\| \geq \|p_1p_3\| - 2h_{opt}$. Since the transformation is rigid, this is true after alignment. Since $\|p_1p_3\| = 2\rho$, we have

$$\|p_4q_4\| \;\geq\; \frac{\|p_1p_3\| - 2h_{opt}}{2} \;=\; \frac{2\rho - 2h_{opt}}{2} \;=\; \rho - h_{opt}.$$

This exceeds $A_{sym}h_{opt}$ for all sufficiently large $\rho$, yielding the desired contradiction. □

To complete the analysis of the lower bound, let us consider the translational and rotational displacement distances of $p_3$ that result from symmetric alignment. The line segment $\overline{q_1q_2}$ passes through $p_4$ (the midpoint of $\overline{p_1p_2}$). Let $\overline{q_2'q_2}$ denote the intersection of the line segment $\overline{q_1q_2}$ with the circle $C_2$. By simple trigonometry $\|q_2'q_2\| = 2\sqrt{1 - \rho^2 \sin^2 \alpha}$, which equals 1 by our choice of $\alpha$. It is easy to see that the translation distance is half the chord length $\|q_2'q_2\|$ along a direction at angle $-\alpha$. Thus the translational displacement vector is

$$\vec{t} \;=\; \frac{1}{2}\|q_2q_2'\| \left(\cos(-\alpha), \sin(-\alpha)\right) \;=\; \frac{1}{2}\left(\cos(-\alpha), \sin(-\alpha)\right).$$

Next, let us consider the displacement of $p_3$ due to rotation. The rotation is about $p_4$, the midpoint of $p_1$ and $p_2$, and the angle of rotation is $\alpha$. Since $\triangle p_1p_2p_3$ is an equilateral triangle of side length $2\rho$, the distance from the center of rotation to $p_3$ is $\sqrt{3}\rho$. From simple trigonometry, the displacement distance due to rotation is $\sqrt{3}\rho \cdot 2\sin\frac{\alpha}{2}$ and the direction of the displacement is $-\frac{\alpha}{2}$. Thus the rotational displacement vector is

$$\vec{r} \;=\; 2\sqrt{3}\rho \sin\frac{\alpha}{2}\left(\cos\left(-\frac{\alpha}{2}\right), \sin\left(-\frac{\alpha}{2}\right)\right).$$

Observe that the angle between $\vec{t}$ and $\vec{r}$ is $\alpha/2$.

To complete the analysis, we decompose the rotational displacement vector $\vec{r}$ into two components, $\vec{r}_1$ is parallel to $\vec{t}$ and $\vec{r}_2$ is perpendicular to it. we have

$$\begin{aligned}
\|\vec{r}_1\| &= 2\sqrt{3}\rho \sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \;=\; \sqrt{3}\rho\sin\alpha \;=\; \frac{3}{2} \\
\|\vec{r}_2\| &= 2\sqrt{3}\rho \sin^2\frac{\alpha}{2} \;=\; \sqrt{3}\rho(1 - \cos\alpha) \;=\; \sqrt{3}\rho\left(1 - \sqrt{1 - \frac{3}{4\rho^2}}\right).
\end{aligned}$$

The squared length of the displacement vector is $D = \|\vec{t}+\vec{r}_1+\vec{r}_2\|^2 = \left(\|\vec{t}\| + \|\vec{r}_1\|\right)^2 + \|\vec{r}_2\|^2$. Thus, we have

$$D \;=\; \left(\frac{1}{2}+\frac{3}{2}\right)^2 + \|\vec{r}_2\|^2 \;=\; 4+3\rho^2\left(1-\sqrt{1-\frac{3}{4\rho^2}}\right)^2.$$

Using the fact that $\sqrt{1-x} \leq 1-\frac{x}{2}$ we have

$$D \;\geq\; 4+3\rho^2\left(1-\left(1-\frac{3}{8\rho^2}\right)\right)^2 \;=\; 4+\frac{27}{64\rho^2}.$$

We may assume that $\rho \geq 2$, from which it follows that

$$\frac{27}{64\rho^2} \;\geq\; \frac{4}{10\rho^2}+\frac{1}{400\rho^2} \;\geq\; \frac{4}{10\rho^2}+\frac{1}{100\rho^4}.$$

Thus, we obtain

$$D \;\geq\; 4+\frac{4}{10\rho^2}+\frac{1}{100\rho^4} \;=\; \left(2+\frac{1}{10\rho^2}\right)^2 \qquad \text{and so} \qquad \sqrt{D} \;\geq\; 2+\frac{1}{10\rho^2}.$$

By adding the initial distance of 1 from $q_3$ to $p_3$ to the displacement, it follows that the final distance from $p_3$ to $q_3$ is $1+\sqrt{D}$. Combining this with the fact that $h_{opt} \leq 1$, the approximation ratio satisfies

$$A_{sym}(\rho) \;\geq\; \frac{1+\sqrt{D}}{h_{opt}} \;\geq\; 3+\frac{1}{10\rho^2}.$$

This completes the proof of Theorem 2. Clearly this is greater than 3 for all positive $\rho$ and converges to 3 in the limit as $\rho$ tends to $\infty$.

## 5 Serial Alignment: Upper Bound

In this section we derive an upper bound on the approximation ratio $A_{ser}(\rho)$ for serial alignment as a function of the distance ratio $\rho$. (Recall the definitions presented at the start of Section 3.) The main result of this section is presented below.

**Theorem 3** *Consider two planar point sets $P$ and $Q$, and let $\rho$ be the distance ratio for $P$ and $Q$. Then*

$$A_{ser}(\rho) \;\leq\; \min\left(c_\infty+\frac{9}{4\rho},\, 2\rho\right),$$

*where $c_\infty$ is defined to be $\lim_{\rho\to\infty} A_{ser}(\rho) \approx 3.196$. Further $A_{ser}(\rho) \leq 3.44$ for all $\rho$.*

As in the analysis of symmetric alignment we will establish two approximation ratios as functions of $\rho$, one is better for low $\rho$ and the other for high $\rho$. We will show that the two functions cross over at a value $\rho^*_{ser} \approx 1.72$ and that $A_{ser}(\rho^*_{ser}) \approx 3.44$. Furthermore, both these functions decrease monotonically as $\rho$ moves away from $\rho^*_{ser}$. The approximation ratio for low distance ratios is presented in Lemma 5.1 below. As usual, assume that $P$ has been transformed to its optimal placement with respect to $Q$. Let $(p_1, p_2)$ denote the diametrical pair in $P$ and let $(q_1, q_2)$ denote the corresponding pair $Q$. Let $\alpha \geq 0$ denote the absolute acute angle between the lines $\overline{p_1p_2}$ and $\overline{q_1q_2}$.

**Lemma 5.1** *For all $\rho$, $A_{ser}(\rho) \leq 2\rho$.*

**Proof**: After applying serial alignment $p_1$ and $q_1$ coincide. Thus, after alignment the distance between any point $p \in P$ and $q_1$ is at most $2\rho$ since the distance from $p_1$ to $p$ is at most $2\rho$. Therefore, the Hausdorff distance after alignment is at most $2\rho$. □

To deal with the more interesting case of high $\rho$ we consider the effects of translation and rotation separately as functions of $\rho$ and the rotation angle $\alpha$. These results are presented Sections 5.1 and 5.2, respectively. Unlike the corresponding analysis for symmetric alignment of Section 3, the analysis here will require a more detailed understanding of the geometric structure of the translation and rotation spaces.

## 5.1 Translational Displacement

Let us consider the space of possible translation vectors that align $p_1$ with $q_1$. Since $P$ is in its optimal placement, it follows that $q_1$ and $q_2$ lie within two circles $C_1$ and $C_2$ of radius $h_{opt}$ $(= 1)$ centered at $p_1$ and $p_2$, respectively. (See Fig. 8(a).) For the sake of illustration, let us assume that $\overrightarrow{p_1p_2}$ is directed horizontally from left to right as shown in the figure and that the line $\overline{q_1q_2}$ has a nonpositive slope (that is, $\alpha$ is a clockwise angle). Consider a line $\ell$ that is parallel to $\overline{q_1q_2}$ and is tangent to $C_2$ from below. This line forms an angle of $\alpha$ with respect to $\overline{p_1p_2}$ and clearly it intersects $C_1$.

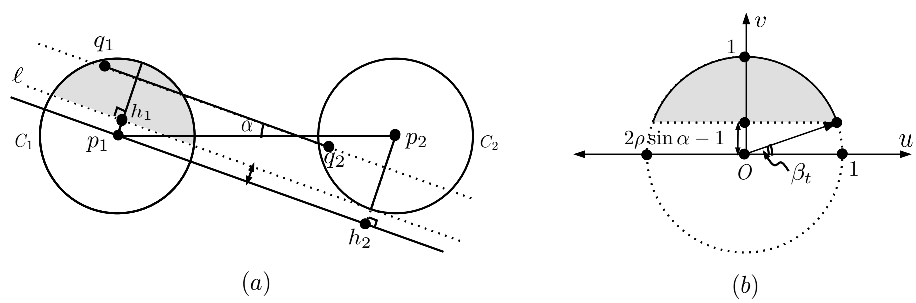


Fig. 8: The translation space for serial alignment.

Let $h_1$ denote the orthogonal projection of $p_1$ onto the line $\ell$, and let $h_2$ denote the orthogonal projection of $p_2$ onto the line passing through $p_1$ and parallel to $\ell$. By simple trigonometry $\|p_2h_2\| = \|p_1p_2\| \sin\alpha$ and since $C_2$'s radius is 1, the signed distance from $p_1$ to $h_1$ is $2\rho\sin\alpha - 1$. The interior of circle $C_1$ lying above $\ell$ defines a region called a *segment* of $C_1$. (This is the shaded region of Fig. 8(a).) Clearly, $q_1$ lies within this segment. It follows that the set of possible translation vectors $\overrightarrow{p_1q_1}$ is a copy of this region translated so that $p_1$ coincides with the origin.

As in Section 3.1, we introduce $(u, v)$ coordinate system by rotating original coordinate system by $\alpha$. In particular, the $u$-axis is parallel to $\overline{q_1q_2}$, and the $v$-axis is orthogonal and directed upwards. This yields the following characterization of the translation space in terms of this coordinate system, which is illustrated in Fig. 8(b). In this section and the next it will simplify notation somewhat to use polar coordinates to represent vectors. The polar coordinates $\langle s, \phi\rangle$ indicate that $s$ is the length of the vector, and $\phi$ is a counterclockwise angle with respect to the positive horizontal axis.

**Lemma 5.2** *For fixed $\alpha$ and $\rho$, the translation space $\mathcal{T}_\rho(\alpha)$ in $(u, v)$ coordinates is the segment of a circle of radius $h_{opt}$ $(= 1)$ centered the origin that lies above the line $v = 2\rho \sin \alpha - 1$. The rightmost vertex of this segment has polar coordinates $\langle 1, \beta_t \rangle$, where $\sin \beta_t = 2\rho \sin \alpha - 1$.*

For now we assume that $\beta_t > 0$. Later in Lemma 5.6 we will show that this assumption is safe, since if $\beta_t \leq 0$ the resulting bound will be weaker than for positive $\beta_t$.

## 5.2 Rotational Displacement

Next we consider the effects of rotation on the approximation error. As in Section 3.2 we need to consider the placement of points in $P$ because the distance by which a point is displaced by rotation depends on the distance from this point to the center of rotation. For given $\rho$ and $\alpha$ values, let $\mathcal{R}_\rho(\alpha)$ denote the set of possible rotational displacement vectors.

Because $(p_1, p_2)$ is a diametrical pair, every point of $P$ lies within a lune formed by the intersection of two discs of radius $2\rho$ centered at $p_1$ and $p_2$, respectively. (See Fig. 9(a).) As before, for the sake of illustration let us assume that $\overrightarrow{p_1 p_2}$ is directed horizontally from left to right as shown in the figure, and that $\alpha$ is a clockwise angle. Along any fixed direction, the farther that a point is from the center of rotation $p_1$ the greater its rotational displacement is. So we will concentrate on the most distant points from the center of rotation, that is, points lying on the boundary of this lune. For any angle $\theta$, where $-\pi/2 \leq \theta \leq \pi/2$, let $p_\theta$ denote the point on this lune such that the signed angle $\angle p_2 p_1 p_\theta$ is $\theta$. (See Fig 9(a).) Let $p'_\theta$ denote this point's position after a clockwise rotation about $p_1$ by angle $\alpha$. Let $\vec{r}(\theta) = \overrightarrow{p_\theta p'_\theta}$ be the corresponding rotational displacement vector.

The following lemma characterizes $\mathcal{R}_\rho(\alpha)$ by describing $\vec{r}(\theta)$ as a function of $\theta$. This region is illustrated in Fig. 9(b).

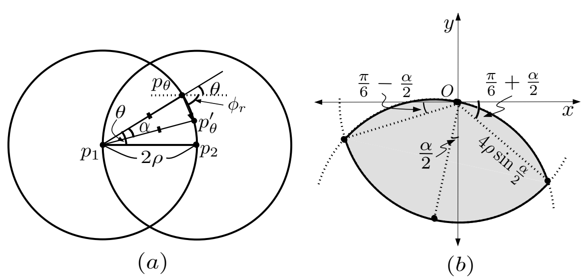


Fig. 9: The rotation space for serial alignment.

**Lemma 5.3** *Consider fixed $\alpha$ and $\rho$, and let*

$$s_0 = 4\rho \sin \frac{\alpha}{2}, \qquad \phi_1 = -\left(\frac{5\pi}{6} + \frac{\alpha}{2}\right), \qquad \textit{and} \qquad \phi_2 = -\left(\frac{\pi}{6} + \frac{\alpha}{2}\right).$$

*The rotational displacement space $\mathcal{R}_\rho(\alpha)$ is a region bounded by two circular arcs:*

(i) *the arc of a circle centered at the origin of radius $s_0$ for $\phi \in [\phi_1, \phi_2]$.*

(ii) *the arc of a circle passing through the origin whose radius is* $s_0$*, and whose center lies at the polar coordinates* $\langle s_0, -\frac{\pi+\alpha}{2} \rangle$*, for* $\phi \in [\phi_2, 0] \cup [-\pi, \phi_1]$*.*

*Furthermore, for all* $\vec{r} \in R_\rho(\alpha)$*, we have* $\|\vec{r}\| \leq s_0$*.*

**Proof**: Throughout we will use the following simple observation. The triangle $\triangle p_1 p_\theta p'_\theta$ is isosceles and its apex angle is $\alpha$. Thus, the length of the displacement vector $\overrightarrow{p_\theta p'_\theta}$ is the length of the triangle's base, which is $2\|p_1 p_\theta\| \sin \frac{\alpha}{2}$, and its angle with respect to $\overrightarrow{p_1 p_2}$ is $-\left(\frac{\pi}{2} - \theta + \frac{\alpha}{2}\right)$. We consider two cases depending on $\theta$, where $-\pi/2 \leq \theta \leq \pi/2$.

(i) If $\theta \in [-\pi/3, \pi/3]$ then $p_\theta$ lies on the opposite side of the lune from $p_1$, that is, on the arc of the circle of radius $2\rho$ centered at $p_1$. Because all such points are at distance $2\rho$ from $p_1$, by the above observation, the length of the displacement vector for all such points is $4\rho \sin \frac{\alpha}{2} = s_0$, and the direction is $\phi = -\left(\frac{\pi}{2} - \theta + \frac{\alpha}{2}\right)$. For the given range of $\theta$ we have

$$\phi \in \left[-\left(\frac{\pi}{2} + \frac{\pi}{3} + \frac{\alpha}{2}\right), -\left(\frac{\pi}{2} - \frac{\pi}{3} + \frac{\alpha}{2}\right)\right] = \left[-\left(\frac{5\pi}{6} + \frac{\alpha}{2}\right), -\left(\frac{\pi}{6} + \frac{\alpha}{2}\right)\right] = [\phi_1, \phi_2].$$

Clearly this is a circular arc centered at the origin of radius $s_0$.

(ii) If $\theta \in [\pi/3, \pi/2] \cup [-\pi/2, -\pi/3]$ then $p_\theta$ lies on the left boundary of the lune passing through $p_1$. This is the arc of the circle of radius $2\rho$ centered at $p_2$. By simple trigonometry we have

$$\|p_1 p_\theta\| = 2\rho \cdot 2 \sin \frac{\pi - 2\theta}{2} = 4\rho \cos \theta.$$

By the above observation, the length of the rotational displacement vector is $8\rho \cos \theta \sin \frac{\alpha}{2}$ and its direction is $\phi = -\left(\frac{\pi}{2} - \theta + \frac{\alpha}{2}\right)$. Therefore, as a function of $\theta$, the polar coordinates of the displacement vector are

$$\left\langle 8\rho \cos \theta \sin \frac{\alpha}{2}, -\left(\frac{\pi}{2} - \theta + \frac{\alpha}{2}\right) \right\rangle.$$

It is easy to verify algebraically the resulting curve is the arc of a circle passing through the origin whose radius is $4\rho \sin \frac{\alpha}{2} = s_0$, and whose center lies at the polar coordinates

$$\left\langle 4\rho \sin \frac{\alpha}{2}, -\frac{\pi + \alpha}{2} \right\rangle = \left\langle s_0, -\frac{\pi + \alpha}{2} \right\rangle.$$

□

## 5.3 Combining Translation and Rotation

Finally we use the results of the prior two sections to derive the approximation ratio for the serial alignment algorithm. The space of possible translational displacement vectors, $\mathcal{T}_\rho(\alpha)$, was shown in Fig. 8 and space of possible rotational displacement vectors, $\mathcal{R}_\rho(\alpha)$, was shown in Fig. 9. The former was shown in $(u, v)$ coordinates (aligned with $\overline{q_1 q_2}$) and the latter was shown in $(x, y)$ coordinates

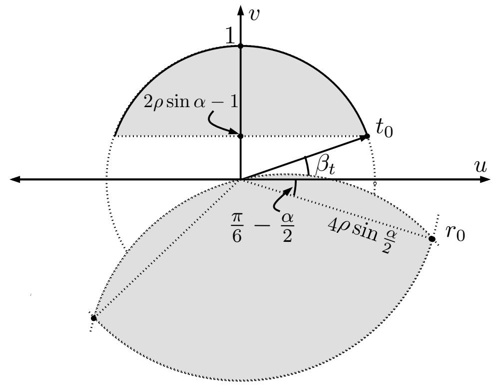


Fig. 10: Translation and rotation space for serial alignment.

(aligned with $\overline{p_1p_2}$). We show them both in Fig. 10 in $(u,v)$ coordinates by rotating the latter counterclockwise by angle $\alpha$.

To derive the overall approximation bound it suffices to find (as a function of $\rho$ and maximizing over all $\alpha$) the two vectors $\vec{t} \in \mathcal{T}_\rho(\alpha)$ and $\vec{r} \in \mathcal{R}_\rho(\alpha)$ whose sum produces the largest total displacement distance $\|\vec{t}+\vec{r}\|$. Because each point of $P$ is initially within distance $h_{opt}$ $(=1)$ of its closest point of $Q$, and so the final approximation bound will be $\|\vec{t}+\vec{r}\|+1$. In order to do this, we need to determine the pair of vectors from these spaces whose sum achieves the greatest distance. A visual inspection of Fig. 10 suggests that the desired pair consists of the rightmost vertex of each region. We establish this in the following lemma.

Recall from Lemma 5.2 the angle $\beta_t$ where $\sin\beta_t = 2\rho\sin\alpha - 1$, which the angle to the rightmost vertex of $\mathcal{T}_\rho(\alpha)$. Recall from the comments after Lemma 5.2 that we may assume that $\beta_t > 0$. Let $\vec{t}_0$ denote the unit length vector to this vertex, whose polar coordinates are $\langle 1, \beta_t\rangle$. Also recall from Lemma 5.3 that the angle (in $(x,y)$ coordinates) to the rightmost vertex of $\mathcal{R}_\rho(\alpha)$ is $\phi_2 = -\left(\frac{\pi}{6}+\frac{\alpha}{2}\right)$. To convert this into $(u,v)$ coordinates we add $\alpha$, and so let us define $\phi_r = \phi_2 + \alpha = -\frac{\pi}{6}+\frac{\alpha}{2}$. Let $\vec{r}_0 \in \mathcal{R}_\rho(\alpha)$ denote the vector to this vertex whose polar coordinates are $\left\langle 4\rho\sin\frac{\alpha}{2}, \phi_r\right\rangle$.

**Lemma 5.4** *Assuming* $\beta_t > 0$,

$$\max_{\substack{\vec{t}\in\mathcal{T}_\rho(\alpha)\\ \vec{r}\in\mathcal{R}_\rho(\alpha)}} \|\vec{t}+\vec{r}\|^2 \;\leq\; 1 + \left(4\rho\sin\frac{\alpha}{2}\right)^2 + 8\rho\sin\frac{\alpha}{2}\sin\left(\frac{\pi}{3}-\beta_t+\frac{\alpha}{2}\right).$$

*The sum of the vectors* $\vec{t}_0$ *and* $\vec{r}_0$ *achieves this maximum length.*

**Proof**: We will show that for all $\vec{t}\in\mathcal{T}_\rho(\alpha)$ and $\vec{r}\in\mathcal{R}_\rho(\alpha)$, $\|\vec{t}+\vec{r}\| \leq \|\vec{t}_0+\vec{r}_0\|$. First we observe that $\phi_r < 0$. To see this recall that $\rho > \rho^*_{ser}$ and $\sin\alpha < 1/\rho$. Thus

$$\sin\alpha \;<\; \frac{1}{\rho^*_{ser}} \;\approx\; \frac{1}{1.72} \;<\; \frac{\sqrt{3}}{2} \;=\; \sin\frac{\pi}{3}.$$

Therefore $\alpha < \pi/3$, implying that $\phi_r < 0$.

Assuming that the angle between the two vectors $\vec{t}$ and $\vec{r}$ is acute, in order to maximize $\|\vec{t}+\vec{r}\|$ we should maximize the lengths of these vectors while minimizing the angle between them. Because the upper boundary of $\mathcal{T}_\rho(\alpha)$ is a circle centered at the origin, it is easy to see that the optimal choice would be at either the leftmost or rightmost vertices of the region. Because of the asymmetry introduced by $\alpha$, the rightmost vertex $t_0$ is easily seen to be the better choice. (This is based on our assumption that $\alpha$ is a clockwise angle. The leftmost vertex would be chosen if the rotation had been counterclockwise.)

The remaining problem is to determine the best choice of $\vec{r}$. The lower boundary of $\mathcal{R}_\rho(\alpha)$ is a circular arc centered at the origin, and so the optimal choice cannot be in the interior of this arc. Thus, the optimal choice must lie along the upper boundary of $\mathcal{R}_\rho(\alpha)$, and in particular, along the arc running from the origin to $\vec{r}_0$. We will show that the optimum is achieved at $\vec{r}_0$. Recall from case (ii) of the proof of Lemma 5.3 that the points of this arc have polar coordinates (relative to $(x,y)$ coordinates)

$$\left\langle 8\rho\cos\theta\sin\frac{\alpha}{2}, -\left(\frac{\pi}{2} - \theta + \frac{\alpha}{2}\right)\right\rangle,$$

where $\pi/3 \le \theta \le \pi/2$. We can convert this to polar coordinates relative to $(u,v)$ coordinates by adding $\alpha$ to the angle. Let

$$\vec{r}(\theta) \;=\; \langle s(\theta), \phi(\theta)\rangle, \qquad \text{where} \quad s(\theta) = 8\rho\cos\theta\sin\tfrac{\alpha}{2}, \qquad \text{and} \quad \phi(\theta) = -\tfrac{\pi}{2} + \theta + \tfrac{\alpha}{2}.$$

Observe that $\vec{r}_0 = \vec{r}(\pi/3)$. Recall from Lemma 5.2 that the polar representation of $\vec{t}_0$ is $\langle 1, \beta_t\rangle$. By the law of cosines, the squared length of their sum can be expressed as a function $f(\theta) = \|\vec{t}_0 + \vec{r}(\theta)\|^2$, where

$$\begin{aligned} f(\theta) &= 1^2 + s(\theta)^2 - 2\cdot 1\cdot s(\theta)\cdot\cos\left(\pi - (\beta_t - \phi(\theta))\right) \\ &= 1 + \left(8\rho\cos\theta\sin\frac{\alpha}{2}\right)^2 + 16\rho\cos\theta\sin\frac{\alpha}{2}\sin\left(\theta - \beta_t + \frac{\alpha}{2}\right). \end{aligned}$$

To determine its maximum value, we consider the partial derivative of $f$ with respect to $\theta$.

$$\begin{aligned} \frac{\partial f}{\partial\theta} &= -128\rho^2\sin^2\frac{\alpha}{2}\cos\theta\sin\theta + \\ &\qquad 16\rho\sin\frac{\alpha}{2}\left(-\sin\theta\sin\left(\theta - \beta_t + \frac{\alpha}{2}\right) + \cos\theta\cos\left(\theta - \beta_t + \frac{\alpha}{2}\right)\right) \\ &= -128\rho^2\sin^2\frac{\alpha}{2}\cos\theta\sin\theta + 16\rho\sin\frac{\alpha}{2}\cos\left(2\theta - \beta_t + \frac{\alpha}{2}\right). \end{aligned}$$

To simplify notation let $w = 8\rho\sin\frac{\alpha}{2}$, and by simple trigonometry we have

$$\begin{aligned} \frac{\partial f}{\partial\theta} &= -2w^2\cos\theta\sin\theta + 2w\cos\left(2\theta - \beta_t + \frac{\alpha}{2}\right) \\ &= -w^2\sin 2\theta + 2w\cos 2\theta\cos\left(\beta_t - \frac{\alpha}{2}\right) + 2w\sin 2\theta\sin\left(\beta_t - \frac{\alpha}{2}\right). \end{aligned}$$

Now we will show that the derivative is negative under our assumption that $\pi/3 \le \theta \le \pi/2$ and $\beta_t > 0$. Observe that $\sin 2\theta \ge 0$ and $\cos 2\theta < 0$. To estimate the other terms we first consider the range of $\beta_t - \frac{\alpha}{2}$. Using the fact that $\sin\beta_t = 2\rho\sin\alpha - 1$ and $\beta_t > 0$ it follows that $0 < \beta_t \le \pi/2$. Since $0 \le \alpha < \pi/3$ for $\rho > \rho^*_{ser}$ we have

$$-\frac{\pi}{6} \;\le\; \beta_t - \frac{\alpha}{2} \;\le\; \frac{\pi}{2}.$$

Thus, $\cos(\beta_t - \alpha/2) \geq 0$. The only remaining term is $\sin(\beta_t - \alpha/2)$. If $\beta_t - \frac{\alpha}{2} \leq 0$ it is obvious that the partial derivative is negative. Otherwise, $\beta_t - \frac{\alpha}{2} > 0$, and so

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= -w^2 \sin 2\theta + 2w \cos 2\theta \cos\left(\beta_t - \frac{\alpha}{2}\right) + 2w \sin 2\theta \sin\left(\beta_t - \frac{\alpha}{2}\right) \\ &< -w^2 \sin 2\theta + 2w \sin 2\theta \;=\; (2-w)w \sin 2\theta.\end{aligned}$$

Since $\beta_t > 0$ it follows that $2\rho \sin\alpha > 1$ and hence

$$w \;=\; 8\rho \sin\frac{\alpha}{2} \;>\; 4\rho \sin\alpha \;>\; 2.$$

Therefore, the partial derivative is negative for $\pi/3 \leq \theta \leq \pi/2$ and $\beta_t > 0$, and so $f(\theta)$ is a decreasing function of $\theta$. This implies that the maximum rotational displacement is achieved at the minimum value of $\theta = \frac{\pi}{3}$. Thus, the maximum combined displacement is achieved by the sum of $\vec{t}_0$ and $\vec{r}(\pi/3) = \vec{r}_0$, as desired. By plugging $\theta = \pi/3$ into $f(\theta)$ we obtain the maximum squared displacement distance value given in the statement of the lemma. □

Combining this result with our earlier remarks we have

$$\begin{aligned}A_{ser}(\rho) &\leq \max_{\alpha}\left(\max_{\substack{\vec{t}\in\mathcal{T}_\rho(\alpha)\\ \vec{r}\in\mathcal{R}_\rho(\alpha)}} \|\vec{t}+\vec{r}\|\right) + 1 \\ &= \max_{\alpha} \sqrt{1 + \left(4\rho \sin\frac{\alpha}{2}\right)^2 + 8\rho \sin\frac{\alpha}{2} \sin\left(\frac{\pi}{3} - \beta_t + \frac{\alpha}{2}\right)} + 1.\end{aligned}$$

Unfortunately, this function is too complex to reason about easily. However, the resulting approximation ratio can be computed numerically and plotted as a function of $\rho$. (See Fig 11.) By numerical means we determine that the cross-over point, denoted $\rho^*_{ser}$ is approximately 1.72, and $A_{ser}(\rho^*_{ser}) \leq 3.44$. The figure shows that as $\rho$ tends to $\infty$ the approximation bound decreases and converges to a value, which we denote by $c_\infty$. The next lemma provides a characterization of this function.

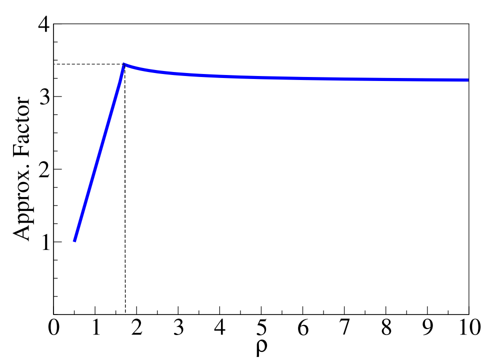


Fig. 11: The approximation ratio for serial alignment as a function of $\rho$.

**Lemma 5.5** *If $\rho > \rho^*_{ser}$ then $A_{ser}(\rho) \le c_\infty + \frac{9}{4\rho}$, where $c_\infty = \lim_{\rho\to\infty} A_{ser}(\rho) \approx 3.196$.*

**Proof**: We will follow the approach we used in Lemma 3.5. We first find an optimal solution for the limiting case when $\rho \to \infty$ and then present the general case. Let $D_\rho(\alpha)$ denote the maximum squared total displacement length as given in Lemma 5.4. When $\rho$ and $\alpha$ are clear from context we simply write $D$.

$$D \;=\; D_\rho(\alpha) \;=\; 1 + \left(4\rho\sin\frac{\alpha}{2}\right)^2 + 8\rho\sin\frac{\alpha}{2}\sin\left(\frac{\pi}{3} - \beta_t + \frac{\alpha}{2}\right).$$

As mentioned above, $A_{ser}(\rho) \le \max_\alpha \sqrt{D} + 1$. Since $0 \le \sin\alpha \le 1/\rho$, in the limit $\alpha$ approaches 0. Using the fact of $\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha} = 1$ and elementary trigonometry we have

$$\begin{aligned}
\lim_{\rho\to\infty} D &= \lim_{\rho\to\infty}\left(1 + \left(4\rho\sin\frac{\alpha}{2}\right)^2 + 8\rho\sin\frac{\alpha}{2}\left(\sin\left(\frac{\pi}{3}+\frac{\alpha}{2}\right)\cos\beta_t - \sin\beta_t\cos\left(\frac{\pi}{3}+\frac{\alpha}{2}\right)\right)\right)\\
&= \lim_{\rho\to\infty} 1 + (2\rho\alpha)^2 + 4\rho\alpha\left(\frac{\sqrt{3}}{2}\sqrt{4\rho\alpha - 4\rho^2\alpha^2} - (2\rho\alpha - 1)\frac{1}{2}\right)\\
&= \lim_{\rho\to\infty} 1 + 2\rho\alpha + 4\sqrt{3}\rho\alpha\sqrt{\rho\alpha - \rho^2\alpha^2}.
\end{aligned}$$

Substituting $x = \rho\alpha$ $(0 \le x \le 1)$, let

$$f(x) = 1 + 2x + 4\sqrt{3}x\sqrt{x - x^2}.$$

To find the value $x = x_0$ at which the maximum is achieved, we set the derivative to 0, from which we obtain

$$x_0 \;=\; \frac{1}{6}\left(3 + \sqrt{2}\cos\phi_0 + \sqrt{6}\sin\phi_0\right), \qquad \text{where } \;\phi_0 = \frac{1}{3}\arctan\frac{\sqrt{47}}{9}. \tag{2}$$

Observe that $x_0$ is slightly greater than $\sqrt{2/3}$. Thus, we have

$$c_\infty \;=\; A_{ser}(\infty) \;=\; \lim_{\rho\to\infty}\max_\alpha \sqrt{D_\rho(\alpha)} + 1 \;=\; \sqrt{f(x_0)} + 1 \;\approx\; 3.196. \tag{3}$$

Next, we consider the general case. Before considering $D$, let us simplify the last term of $D$ first. Let

$$g_\rho(\alpha) \;=\; \sin\left(\frac{\pi}{3} - \beta_t + \frac{\alpha}{2}\right).$$

From basic trigonometry we have

$$\begin{aligned}
g_\rho(\alpha) &= \sin\left(\frac{\pi}{3}+\frac{\alpha}{2}\right)\cos\beta_t - \sin\beta_t\cos\left(\frac{\pi}{3}+\frac{\alpha}{2}\right)\\
&= \sin\frac{\pi}{3}\cos\frac{\alpha}{2}\cos\beta_t + \cos\frac{\pi}{3}\sin\frac{\alpha}{2}\cos\beta_t - \cos\frac{\pi}{3}\cos\frac{\alpha}{2}\sin\beta_t + \sin\frac{\pi}{3}\sin\frac{\alpha}{2}\sin\beta_t.
\end{aligned}$$

To simplify $g_\rho(\alpha)$ we apply a Taylor's series expansion.

$$\begin{aligned}
g_\rho(\alpha) &\le \frac{\sqrt{3}}{2}\cos\beta_t + \frac{\alpha}{4}\cos\beta_t - \frac{1}{2}\left(1 - \frac{\alpha^2}{8}\right)\sin\beta_t + \frac{\sqrt{3}\alpha}{4}\sin\beta_t\\
&\le \left(\sqrt{3} + \frac{\alpha}{2}\right)\sqrt{\rho\alpha - \rho^2\left(\alpha - \frac{\alpha^3}{6}\right)^2} -\\
&\qquad \frac{1}{2}\left(1 - \frac{\alpha^2}{8}\right)\left(2\rho\left(\alpha - \frac{\alpha^3}{6}\right) - 1\right) + \frac{\sqrt{3}\alpha}{4}(2\rho\alpha - 1).
\end{aligned}$$

Now, applying a Taylor's series expansion of $D$ (as a function of $\alpha$) we have

$$\begin{aligned} D &= 1 + \left(4\rho \sin\frac{\alpha}{2}\right)^2 + 8\rho \sin\frac{\alpha}{2} \cdot g_\rho(\alpha) \\ &\leq 1 + (2\rho\alpha)^2 + 4\rho\alpha \cdot g_\rho(\alpha). \end{aligned}$$

Substituting $g_\rho(\alpha)$ for the last term of $D$ and removing all the negative terms we have

$$D \leq 1 + 2\rho\alpha + 4\rho\alpha\left(\left(\sqrt{3} + \frac{\alpha}{2}\right)\sqrt{\rho\alpha - \rho^2\alpha^2 + \frac{\rho^2\alpha^4}{3}} + \frac{7\rho\alpha^3}{24} + \frac{\sqrt{3}\rho\alpha^2}{2}\right). \qquad (4)$$

Since $0 \leq \alpha \leq \arcsin(1/\rho)$, we have $0 \leq \rho\alpha \leq \rho \arcsin(1/\rho)$. For all $\rho > \rho^*_{ser}$, observe that $\rho \arcsin(1/\rho)$ is at most $\rho^*_{ser} \arcsin(1/\rho^*_{ser})$. Let us denote this value by $c_0$, which by numerical evaluation is approximately 1.07. Thus, we have $0 \leq \rho\alpha < c_0$.

We will show that $D \leq (c_\infty - 1)^2 + \frac{9}{\rho}$. We will consider two cases: $0 \leq \rho\alpha \leq 1$ and $1 < \rho\alpha < c_0$. For the first case ($0 \leq \rho\alpha \leq 1$) we substitute $x = \rho\alpha$ ($0 \leq x \leq 1$) and use the fact that $(1 + 2x + 4\sqrt{3}x\sqrt{x - x^2}) \leq f(x_0)$, which we argued above. And so we have

$$\begin{aligned} D &\leq D + \left[f(x_0) - \left(1 + 2x + 4\sqrt{3}x\sqrt{x - x^2}\right)\right] \\ &= f(x_0) + 4\sqrt{3}x\left(\sqrt{x - x^2 + \frac{x^4}{3\rho^2}} - \sqrt{x - x^2}\right) + \\ &\qquad 4x\left(\frac{x}{2\rho}\sqrt{x - x^2 + \frac{x^4}{3\rho^2}} + \frac{7x^3}{24\rho^2} + \frac{\sqrt{3}x^2}{2\rho}\right). \end{aligned}$$

It is easily proved that this function is increasing for $0 \leq x \leq 1$. Thus, it achieves its maximum value at $x = 1$. By hypothesis, $\rho > \rho^*_{ser} > 1.72$, from which it follows that

$$D \leq f(x_0) + \left(\frac{4 + 2\sqrt{3}}{\rho} + \frac{7 + 4\sqrt{3}}{6\rho^2}\right) < f(x_0) + \frac{9}{\rho} = (c_\infty - 1)^2 + \frac{9}{\rho},$$

as desired.

For the second case ($1 < \rho\alpha < c_0$), since $\rho\alpha - \rho^2\alpha^2 < 0$, from Eq. (4) we have

$$D < 1 + 2\rho\alpha + 4\rho\alpha\left(\left(\sqrt{3} + \frac{\alpha}{2}\right)\sqrt{\frac{\rho^2\alpha^4}{3}} + \frac{7\rho\alpha^3}{24} + \frac{\sqrt{3}\rho\alpha^2}{2}\right).$$

It is easily observed that the function is an increasing function with respect to $\rho\alpha$ and the maximum value is achieved at $\rho\alpha = 1.07$, and this value is at most $(c_\infty - 1)^2 + \frac{9}{\rho}$, as desired.

Now, using the fact of $c_\infty > 3$ we have

$$D < (c_\infty - 1)^2 + \frac{9}{\rho} < \left(c_\infty - 1 + \frac{9}{4\rho}\right)^2.$$

Therefore,

$$A_{ser}(\rho) \leq \max_\alpha \sqrt{D} + 1 \leq c_\infty + \frac{9}{4\rho},$$

which completes the proof. □

The analysis of Lemma 5.4 was performed under the assumption that $\beta_t > 0$. We can justify this assumption now by showing that any other value would lead to a weaker approximation bound.

**Lemma 5.6** *For $\rho > \rho^*_{ser}$, if $\beta_t \leq 0$ then*

$$d_{ser}(\rho) \;<\; A_{ser}(\rho) h_{opt},$$

*where $d_{ser}(\rho)$ is a Hausdorff distance after running serial alignment.*

**Proof**: To simplify matters, assume that $h_{opt} = 1$ and let $d_{ser}(\rho) \leq \max_\alpha g_\rho(\alpha)$. By the triangle inequality we have

$$g_\rho(\alpha) \;=\; \max_{\substack{\vec{t} \in \mathcal{T}_\rho(\alpha) \\ \vec{r} \in \mathcal{R}_\rho(\alpha)}} (\|\vec{t} + \vec{r}\| + 1) \;\leq\; \max_{\substack{\vec{t} \in \mathcal{T}_\rho(\alpha) \\ \vec{r} \in \mathcal{R}_\rho(\alpha)}} (\|\vec{t}\| + \|\vec{r}\| + 1).$$

By Lemmas 5.2 and 5.3 we have $\|\vec{t}\| \leq 1$ and $\|\vec{r}\| \leq 4\rho \sin\frac{\alpha}{2}$, and so

$$g_\rho(\alpha) \;\leq\; 2 + 4\rho \sin\frac{\alpha}{2} \;=\; 2 + \frac{2\rho \sin\alpha}{\cos\frac{\alpha}{2}}.$$

By the assumption that $\beta_t \leq 0$, it follows $2\rho \sin\alpha - 1 \leq 0$, and so $g_\rho(\alpha) \;\leq\; 2 + \frac{1}{\cos(\alpha/2)}$. For $\rho > \rho^*_{ser}$ and $\sin\alpha \leq 1/(2\rho)$, we have

$$\cos\frac{\alpha}{2} \;=\; \frac{1}{\sqrt{2}}\sqrt{1 + \sqrt{1 - \sin^2\alpha}} \;>\; \frac{1}{\sqrt{2}}\sqrt{1 + \sqrt{1 - 1/(2\rho^*_{ser})^2}} \;\approx\; \frac{1}{1.01}.$$

Thus, we see that the bound on $d_{ser}(\rho)$ generated under the hypothesis $\beta_t \leq 0$ is significantly smaller than $c_\infty$. In the proof of Lemma 5.5 it was shown that for all $\rho > \rho^*_{ser}$, the approximation bound is never smaller than $c_\infty$. □

Combining Lemmas 5.1, 5.5, and 5.6 completes the proof of Theorem 3.

# 6 Serial Alignment: Lower Bound

In this section we present a result analogous to Theorem 2, by proving that the upper bound on the approximation ratio for serial alignment given in the previous section is tight in the limit as $\rho$ approaches $\infty$. Recall that $c_\infty = \lim_{\rho \to \infty} A_{ser}(\rho)$. (See Eq. (3) in Lemma 5.5.)

**Theorem 4** *For all sufficiently large $\rho$ there exists an input on which serial alignment achieves an approximation factor of at least $c_\infty + \frac{1}{27\rho^2}$*

The input configuration is structurally similar to the one used in the proof of Theorem 2, except for the choice of the angle $\alpha$. Here we select $\alpha$ such that $\sin\frac{\alpha}{2} = x_0/(2\rho)$, where $x_0$ is as defined in Eq. (2) in the proof of Lemma 5.5. Consider a fixed value $\rho > 2A_{ser}$. As in Section 4 we define the pattern point set $P = \{p_1, p_2, p_3, p_4\}$, where the first three points form an equilateral triangle

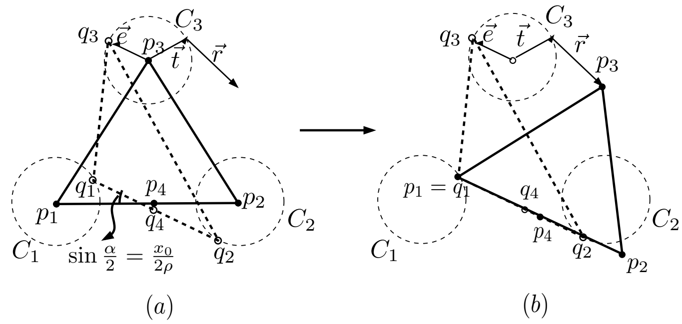


Fig. 12: The lower bound on $A_{ser}$.

of side length $2\rho$, and $p_4$ is placed at the midpoint of $\overline{p_1p_2}$. (See Fig. 12(a).) By an infinitesimal perturbation of the points, we may assume that the pair $(p_1, p_2)$ is the unique diametrical pair for $P$.

We define the background set $Q = \{q_1, q_2, q_3, q_4\}$ so that the optimum Hausdorff distance will be at most 1. For $1 \leq i \leq 4$, let $C_i$ be a circle of unit radius centered $p_i$. Consider a lower tangent line at $C_2$ forming an angle $\alpha$ with line $\overline{p_1p_2}$. It is easy to verify that this line intersects $C_1$. Place $q_1$ at the rightmost intersection point of the tangent line and $C_1$, and place $q_2$ at the point of tangency with $C_2$. (See Fig. 12(a).) Running serial alignment so that $(p_1, p_2)$ is aligned with $(q_1, q_2)$ results in a translation and rotation. Let $\vec{t}$ and $\vec{r}$ denote the translation and rotational displacement vectors for $p_3$, respectively. Let $q_3 = p_3 + \vec{e}$, where $\vec{e}$ is a vector of unit length whose direction is chosen to be directly opposite that of $\vec{t} + \vec{r}$. Intuitively, $q_3$ has been chosen to be as far away as possible from $p_3$ after alignment. Finally, place $q_4$ at the midpoint of $\overline{q_1q_2}$. Observe that prior to alignment, each $p_i$ is within distance 1 from $q_i$, and therefore $h_{opt} \leq 1$.

Consider the placement of $P$ and $Q$ after running serial alignment. (See Fig. 12(b).) We will analyze the displacement distance of $p_3$ relative to $q_3$. As in the construction of Section 4, $p_4$ and $q_4$ were introduced to induce an asymmetry that forces the alignment of $(p_1, p_2)$ with $(q_1, q_2)$. We establish this formally in the next lemma.

**Lemma 6.1** *For all sufficiently large $\rho$, serial alignment will align $(p_1, p_2)$ with $(q_1, q_2)$.*

**Proof**: We reduce the proof to the analogous result for symmetric alignment, namely Lemma 4.1. In the proof of that lemma the only facts that we used were that $\rho$ is sufficiently large, and prior to alignment, each point of $P$ lies within distance $h_{opt} \leq 1$ of the corresponding point of $Q$. These are both true in the present scenario.

Suppose that when we run serial alignment on the above input instance $(p_1, p_2)$ was not aligned with $(q_1, q_2)$. In Lemma 2.3 we showed that on all input instances, if serial alignment produces a Hausdorff distance of $d_{ser}$ then (without altering the individual point assignments) symmetric alignment will produce a Hausdorff distance of at most $d_{sym} \leq d_{ser} + h_{opt}$. In the proof of Lemma 4.1 it was argued that if $(p_1, p_2)$ was not aligned with $(q_1, q_2)$ in symmetric alignment, the resulting Hausdorff distance would be at least $\rho - h_{opt}$. Thus we have

$$\rho - h_{opt} \;\leq\; d_{sym} \;\leq\; d_{ser} + h_{opt},$$

which implies that $d_{ser} \geq \rho - 2h_{opt} \geq \rho - 2$. This distance will exceed $A_{ser}h_{opt}$ for all sufficiently large $\rho$, which yields the desired contradiction. $\square$

To complete the analysis let us consider the translational and rotational displacement distances of $p_3$ that result from symmetric alignment. It is easy to see that $\vec{t} = \overrightarrow{p_1 q_1}$, and so $\|\vec{t}\| = 1$. To determine the direction of $\vec{t}$, let us consider the points of $P$ prior to alignment. Let $\ell$ denote the line passing through $p_1$ that is parallel to $\overline{q_1 q_2}$. Let $h_1$ and $h_2$ denote the orthogonal projections of $q_1$ and $q_2$ onto $\ell$, respectively. Since line $\overline{q_1 q_2}$ is the lower tangent line of the circle $C_2$ at $q_2$, it follows that $p_2$, $q_2$, and $h_2$ are collinear. Combining the facts that $\|p_1 p_2\| = 2\rho$, $\|p_2 q_2\| = 1$, and $\ell$ is perpendicular to $\overline{p_2 q_2}$, we have

$$\|q_2 h_2\| \;=\; \|p_1 p_2\| \sin\alpha - \|p_2 q_2\| \;=\; 2\rho\sin\alpha - 1.$$

Because $\|q_1 h_1\|$ is equal to $\|q_2 h_2\|$, the angle between the line $\overline{p_1 q_1}$ and $\ell$, denoted $\beta_t$, satisfies

$$\sin\beta_t \;=\; \frac{\|q_1 h_1\|}{\|p_1 q_1\|} \;=\; \frac{\|q_2 h_2\|}{1} \;=\; 2\rho\sin\alpha - 1.$$

Since $\ell$ forms an angle of $\alpha$ with $\overline{p_1 p_2}$ it follows that $\vec{t}$ forms an angle of $\phi_t = \beta_t - \alpha$, and hence expressing $\vec{t}$ in polar coordinates we have $\vec{t} = \langle 1, \phi_t \rangle$.

Next, let us consider the effect of rotation at $p_3$. Since $\triangle p_1 p_2 p_3$ is an equilateral triangle we have $\|p_1 p_3\| = 2\rho$. By simple trigonometry (or see the comments at the start of the proof of Lemma 5.3) it follows that $p_3$'s displacement distance due to the rotation about $p_1$ by angle $\alpha$ is $2\rho \cdot 2\sin\frac{\alpha}{2} = 4\rho\sin\frac{\alpha}{2}$ and the displacement direction is

$$\left(\frac{\pi}{3} - \frac{\alpha}{2}\right) - \frac{\pi}{2} \;=\; -\frac{\pi}{6} - \frac{\alpha}{2}.$$

Let $\phi_r$ denote this angle. Thus, expressing $\vec{r}$ in polar coordinates we have

$$\vec{r} \;=\; \left\langle 4\rho\sin\frac{\alpha}{2}, \phi_r \right\rangle.$$

Let $D = \|\vec{t} + \vec{r}\|^2$ denote the length of the combined displacements. By the law of cosines we have

$$\begin{aligned} D &= \|\vec{t}\|^2 + \|\vec{r}\|^2 - 2\|\vec{t}\|\,\|\vec{r}\| \cos\left(\pi - (\phi_t - \phi_r)\right) \\ &= 1 + \left(4\rho\sin\frac{\alpha}{2}\right)^2 - 8\rho\sin\frac{\alpha}{2}\cos\left(\pi - (\beta_t - \alpha) + \left(-\frac{\pi}{6} - \frac{\alpha}{2}\right)\right) \\ &= 1 + \left(4\rho\sin\frac{\alpha}{2}\right)^2 + 8\rho\sin\frac{\alpha}{2}\sin\left(\frac{\pi}{3} + \frac{\alpha}{2} - \beta_t\right). \end{aligned}$$

Using the facts that $\sin\frac{\alpha}{2} = \frac{x_0}{2\rho}$ and $0 \leq \alpha \leq \pi/3$ and applying basic trigonometry we have

$$\begin{aligned} D &= 1 + (2x_0)^2 + 4x_0\left(\sin\left(\frac{\pi}{3} + \frac{\alpha}{2}\right)\cos\beta_t - \cos\left(\frac{\pi}{3} + \frac{\alpha}{2}\right)\sin\beta_t\right) \\ &\geq 1 + 4x_0^2 + 4x_0\left(\frac{\sqrt{3}}{2}\cos\beta_t - \frac{1}{2}\sin\beta_t\right). \end{aligned}$$

Note that $x_0$ is a constant slightly greater than $\sqrt{2/3}$. (See Eq. (2) in Lemma. 5.5.) To relate this to $c_\infty$ we begin with a variable substitution. Let

$$s \;=\; \cos\frac{\alpha}{2} \;=\; \sqrt{1-\left(\frac{x_0}{2\rho}\right)^2}.$$

Because $0 \le x_0 < 1$ and $\rho > \rho^*_{ser} > 1.5$ it follows directly that $3/4 < s \le 1$. We can restate a number of quantities in terms of $x_0$ and $s$.

$$\begin{aligned}\rho\sin\alpha &= 2\rho\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \;=\; x_0 s,\\ \sin\beta_t &= 2\rho\sin\alpha - 1 \;=\; 2x_0 s - 1, \quad \text{and}\\ \cos\beta_t &= 2\sqrt{x_0 s - x_0^2 s^2}.\end{aligned}$$

Substituting these values yields

$$D \;\ge\; 1 + 4x_0^2 + 4x_0\left(\sqrt{3}\sqrt{x_0 s - x_0^2 s^2} - x_0 s + \frac{1}{2}\right).$$

By definition of $c_\infty$ we have $(c_\infty - 1)^2 = 1 + 2x_0 + 4\sqrt{3}x_0\sqrt{x_0 - x_0^2}$, and so it follows that

$$\begin{aligned}D \;&\ge\; 1 + 4x_0^2 + 4x_0\left(\sqrt{3}\sqrt{x_0 s - x_0^2 s^2} - x_0 s + \frac{1}{2}\right) + \\ &\quad \left[(c_\infty - 1)^2 - \left(1 + 2x_0 + 4\sqrt{3}x_0\sqrt{x_0 - x_0^2}\right)\right]\\ &= (c_\infty - 1)^2 + 4x_0^2(1-s) + 4\sqrt{3}x_0\left(\sqrt{x_0 s - x_0^2 s^2} - \sqrt{x_0 - x_0^2}\right).\end{aligned}$$

It is straightforward to verify that $1/(2x_0) < s \le 1$ for $\rho > \rho^*_{ser}$, and so it follows that $\sqrt{x_0 s - x_0^2 s^2} - \sqrt{x_0 - x_0^2} \ge 0$. We have

$$D \;\ge\; (c_\infty - 1)^2 + 4x_0^2(1-s).$$

Using the fact that $\sqrt{1-x} \le 1 - x/2$ and $s = \sqrt{1 - \left(\frac{x_0}{2\rho}\right)^2}$ we have

$$D \;\ge\; (c_\infty - 1)^2 + 4x_0^2\left[1 - \left(1 - \frac{1}{2}\left(\frac{x_0}{2\rho}\right)^2\right)\right] \;=\; (c_\infty - 1)^2 + \frac{x_0^4}{2\rho^2}.$$

Since $c_\infty < 3.2$ and $\sqrt{2/3} < x_0 < 1$ it is easy to verify that

$$D \;\ge\; (c_\infty - 1)^2 + \frac{x_0^4}{2\rho^2} \;\ge\; \left(c_\infty - 1 + \frac{x_0^4}{12\rho^2}\right)^2 \;\ge\; \left(c_\infty - 1 + \frac{1}{27\rho^2}\right)^2.$$

Therefore,

$$A_{ser}(\rho) \;\ge\; \frac{1+\sqrt{D}}{h_{opt}} \;\ge\; c_\infty + \frac{1}{27\rho^2},$$

which completes the proof of Theorem 4.

## 7 Summary and Concluding Remarks

We have presented a simple modification to the alignment-based algorithm of Goodrich, Mitchell, and Orletsky [13]. Our modification has the same running time and retains the simplicity of the original algorithm. We have analyzed the approximation ratios for these algorithms as a function of the distance ratio $\rho$. We summarize and compare these approximation ratios in Fig. 13.

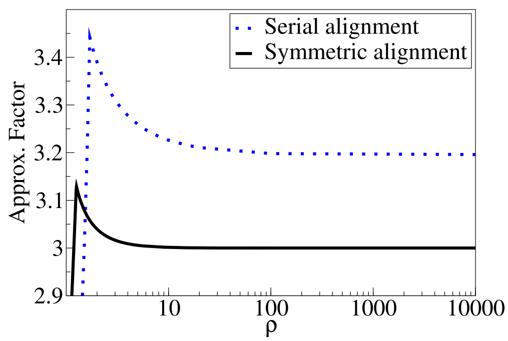


Fig. 13: The approximation ratios for serial alignment and symmetric alignment. (Note that the $y$-axis does not start at 0.)

Note that the approximation ratio for symmetric alignment is smaller for all sufficiently large values of $\rho$. The cross-over point of two bounds is at $(\rho, A) \approx (1.54, 3.08)$. As a function of $\rho$, the highest upper bound for symmetric alignment is at $(\rho, A_{sym}) \approx (1.26, 3.14)$. The highest upper bound for serial alignment is at $(\rho, A_{ser}) \approx (1.72, 3.44)$. As $\rho$ approaches $\infty$, $A_{sym}(\rho)$ converges to 3 and $A_{ser}(\rho)$ converges to a value that is approximately 3.19. As mentioned earlier, in many applications of point pattern matching, large values of $\rho$ (exceeding 10, say) are to be expected. We have also shown that both bounds are nearly tight for large $\rho$, and they are tight in the limit.

This paper opens the question of what are the performance limits of approximations based on point alignments. It is natural to consider generalizations of this approach to higher dimensions, to other geometric groups (including, for example uniform scaling), to matchings based on more than two points, and to other sorts of point-set similarity measures.

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