Fluid Dynamics and the Navier-Stokes Equation

CMSC498A: Spring ’12 Semester

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Introduction

I began this project through a desire to simulate smoke and fire through the use of programming and graphics rendering. To do this, I researched the concepts of vector calculus, fluid dynamics, and the Navier-Stokes equation. Upon finding such useful and insightful information, the project evolved into a study of how the Navier-Stokes equation was derived and how it may be applied in the area of computer graphics.

The Navier-Stokes equation is named after Claude-Louis Navier and George Gabriel Stokes. This equation provides a mathematical model of the motion of a fluid. It is an important equation in the study of fluid dynamics, and it uses many core aspects to vector calculus.

Before explaining the Navier-Stokes equation it is important to cover several aspects of computational fluid dynamics. At the core of this is the notion of a vector field. A *vector field* is defined as a mapping from each point in 2- or 3-dimensional real space to a vector. Each such vector can be thought of as being composed of a directional unit vector and a scalar multiplier. In the context of fluid dynamics, the value of a vector field at a point can be used to indicate the velocity at that point. Vector fields are useful in the study of fluid dynamics, since they make it possible to discern the approximated path of a fluid at any given point [12].

Vector Calculus

Vector calculus is the branch of mathematics that is involved with differentiation and integration over vector fields. In this section we present a brief overview of this area. We begin with a very important mathematical operator called *del* (∇). Del is defined as the partial derivatives of a vector. Letting i, k, and j denote the unit vectors for the coordinate axes in real 3-space, the operator is defined [5]:

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}.$$

With del defined, we may now look at four key differential operators that are based on del. Note that we will be using uppercase letters to denote vector fields, and lower case letters to denote scalar fields.

First we have the gradient. *The gradient* is defined as the measurement of the rate and direction of change in a scalar field. The gradient maps a scalar field to a vector field. So, for a scalar field f [6]:

$$grad(f) = \nabla(f) .$$

As an example of gradient, consider the scalar field $f = xy^2 + z$.We take the partial derivatives with respect to *x, y,* and *z*.

$\frac{d}{dx} = y^2$ $\frac{d}{dy} = 2x$ $\frac{d}{dx} = 1$

So, the gradient is:

$$grad(f) = y^2 i + 2xj + k .$$

At any point (*x,y,z*), there is a directional vector that is a part of this vector field. For example, at (1,0,0), the vector would be *2j + k.*

Next we have curl, which is defined as the measurement of the tendency to rotate about a point in a vector field. *The curl* maps a vector field to another vector field. For vector F, we define [6]:

$$curl(F) = \nabla \times F .$$

For example, consider vector field $F = xi - xyj + z^2k$. We can express the $curl(F)$ symbolically as the following determinant:

$$\begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x & -xy & z^2 \end{vmatrix}$$

Letting *F1* = *x, F2* = $-xy,$ and *F3* = z^2this can be expressed using the cross product form as$\left(\frac{dF3}{dy} - \frac{dF2}{dz}\right) i - \left(\frac{dF3}{dx} - \frac{dF1}{dz}\right) j + \left(\frac{dF2}{dx} - \frac{dF1}{dy}\right) k.$

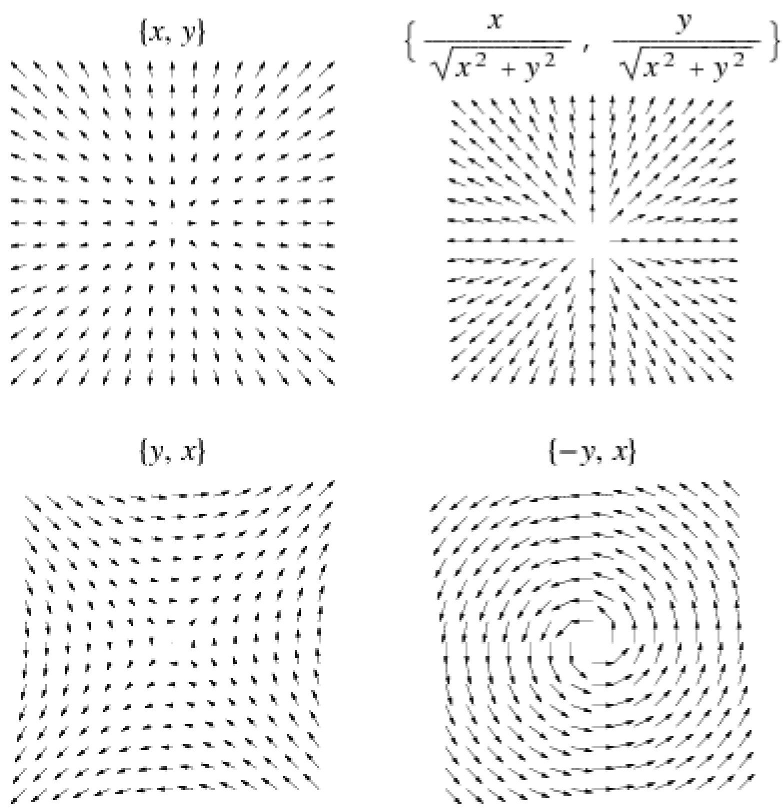
From this we obtain:

$(0 - 0)i - (0 - 0)j + (-y - 0)k,$ that is, $curl(F) = -yk$**.**

Third, we have divergence. *Divergence* is models the magnitude of a source or sink at a given point in a vector field. Divergence maps a vector field to a scalar field. For a vector filed *F* [6]:

$$div(F) = \nabla \cdot F$$

At any point in a vector field, divergence is positive if there is an outflow, negative if there is an inflow, and zero if there is no convergence or divergence [12]. For example, the upper left vector field, *F = xi +yj*, where $div(F) = 1 + 1 = 2$, there is an outflow, which makes sense as the divergence is positive. If we now look at the bottom left vector field, *F = yi +xj*, where $div(F) = 0 + 0$, there is neither outflow or inflow, which again makes sense due to the divergence being 0.



[13]

Figure 1.

As an example, consider once again $F = xi - xyj + z^2k$.

$\nabla \cdot F = \frac{df1}{dx} + \frac{dF2}{dy} + \frac{dF3}{dz} = i - xj + 2zk$.

And finally, we have the Laplacian, represented as Δ. *The Laplacian* is defined as the composition of the divergence and gradient operations. This maps a scalar field onto another scalar field. The Laplacian of *f* is defined as [6]:

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f$$

For example, consider field $f = xy^2 + z^3$

$\nabla^2 f = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} + \frac{d^2 f}{dz^2} = \frac{d}{dx}(y^2 + 0) + \frac{d}{dy}(2xy + 0) + \frac{d}{dz}(0 + 3z^2) = 2x + 6z$

The Navier-Stokes Equation

Let us now consider the Navier-Stokes equation, what it means, and how it can be used to simulate something physical phenomena like smoke and fire. As observed in [1], the Navier-Stokes equation can be viewed as an application of Newton's second law, $F = ma$, which states that force is the product of the mass of an object times its acceleration. (Note, we will now be using f to represent forces, not scalar or vector fields.) In this equivalent equation, we see the use of density and shear stress. *Density* is a measurement of an objects mass per unit volume, while *shear stress* is defined as the component of stress coplanar with a material cross section, where the force vector component runs parallel to the cross section. Consider [1]:

$$\rho\left[\frac{du}{dt} + u \cdot \nabla u\right] = \nabla \cdot \sigma + f$$

where ρ denotes the density of the fluid and is equivalent to mass, $\frac{du}{dt} + u \cdot \nabla u$ is the acceleration and u is velocity, and $\nabla \cdot \sigma + f$ is the total force, with $\nabla \cdot \sigma$ being the shear stress and f being all other forces. We may also write this as

$$\rho\left[\frac{du}{dt} + u . \nabla u\right] = -\nabla p + \mu\nabla^2 u + f \quad [1]$$

Where ρ is pressure and μ is dynamic viscosity. *Viscosity* is defined as the measure of the resistance of a fluid which is being deformed by the shear stress. Finally, by dividing out ρ and subtracting $u \cdot \nabla u$, we obtain the traditional form of the Navier-Stokes equation [10]:

$$\frac{du}{dt} = -(u \cdot \nabla) \cdot u - \frac{1}{\rho}\nabla p + \gamma\nabla^2 u + f$$

Notice that Navier-Stokes explicitly models changes in the directional velocity using four components.

- The first of these is $-(u \cdot \nabla) \cdot u$, which is the divergence on a velocity, or in simpler terms, it is how the divergence affects the velocity. One image that may help explain this is that of a river. When the river converges, the narrowing acts like a funnel, and the overall velocity of the flow increases (see Figure 2). Conversely, if the river diverges, the particles spread out, and the overall speed of the flow decreases (see Figure 3).

[3]

Figure 2.

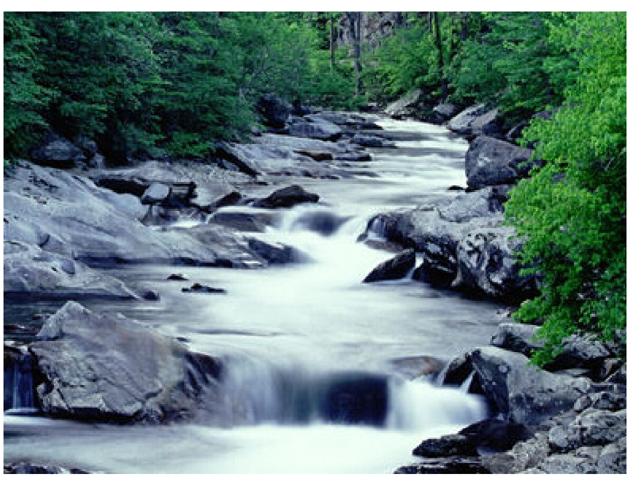
[7]

Figure 3.

- Second, there is $-\frac{1}{\rho}\nabla p$. This may be thought of as how the particles move as pressure changes, specifically, the tendency to move away from areas of higher pressure. Consider a flock of birds flying together, and imagine a hawk attacking them. Think of the hawk as a source of "pressure" being applied to the "fluid" of birds. Those birds would want to move away from an area of high pressure (see Figure 4). If there is a dense group, it will be harder for all of them to get away. So, the less dense a group, the more of the units that can move rapidly when pressure is exerted.

 [11]

Figure 4.

We may also look at this from the perspective of placing pressure on a polymer versus a solid. On a polymer, which is less dense than a solid, the pressure forces the material to spread out (see Figure 5).

[8]

Figure 5.

Now, for the solid, the material stays together (see Figure 6).

[2]

Figure 6.

- Next we consider the term $\gamma\nabla^2 u$. The two key parts are viscosity (γ) and Laplacian (∇^2). It may be a little hard to make sense of this part, but think of it as the difference between what a particle does and what its neighbors do. Think of a high viscous substance, such as syrup. The motion of a particle in a pool of thick syrup will tend to induce nearby particles to move (see Figure 7). In contrast, in a less viscous fluid, such as water, the motion of a particle induces a lower effect on its neighbors (see Figure 8).

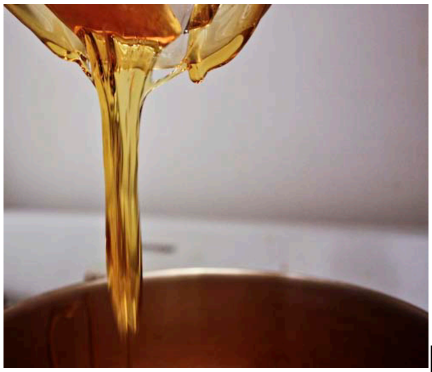
[9]

Figure 7.

[4]

Figure 8.

- And last we have *f*, which again, is any other forces acting on the substance.

Application

An application of the Navier-Stokes equation may be found in Joe Stam's paper, Stable Fluids, which proposes a model that can produce complex fluid like flows [10]. It begins by defining a two-dimensional or three-dimensional grid using the dimensions origin `O[NDIM]` and length `L[NDIM]` of each side of the grid, and the number of cells in each coordinate `N[NDIM]`, with the size of each voxel being `D[i] = L[i]/N[i]`. Next, two cell-centered grids `U0[NDIM]` and `U1[NDIM]` are created, where at each step of the algorithm, one grid represents the result of the previous grid, with the new solution stored in the second grid, after which the grids are swapped. Grids `S0` and `S1` are created to hold scalar fields corresponding to substances transported by the flow. The variable `dt` represents the speed of interactivity. Lastly, the variable `visc` is the viscosity, `kS` is the diffusion rate, and `aS` is the dissipation rate. The forces that set the fluid into motion are given in the array `F[NDIM]`, along with an array `Ssource` for the scalar field.

In the simulator, there are two key steps, `Vstep` and `Sstep`. `Vstep`, a velocity solver, takes grids `U0` and `U1`, `visc`, `F` (the forces) and `dt`. As described in the paper, forces are added to the field, the field is advected by itself, the field diffuses due to viscous friction within the fluid, and in the final step the velocity is forced to conserve mass [10].

`Sstep`, a scalar solver, takes the scalar grids `S0` and `S1`, diffusion constant `kS`, dissipation constant `aS`, a grid `U1`, a source variable `Ssource`, and `dt`. It has four steps: adding `Ssource`, transporting the particles in field `U`, diffusing the field, and dissipating the field. The aforementioned transport is used to resolve the non-linearity of the Navier-Stokes equations, by tracing a path back starting at X (which is, given Origin `O`, cell (*i, j, k*), and size `D`, $X = O + (i + 0.5, j + \ 0.5, k + 0.5) * D$) through the field `U` over time `-dt`. The function `LinInterp` is then called to linearly interpolate the value of the scalar field `S` at location `X0`.

In the end, this allows for the movement of the fluid particles in the grid, allowing for complex animations, such as fire and smoke, to occur. Hopefully, this now gives a better understanding to how the Navier-Stokes equation works.

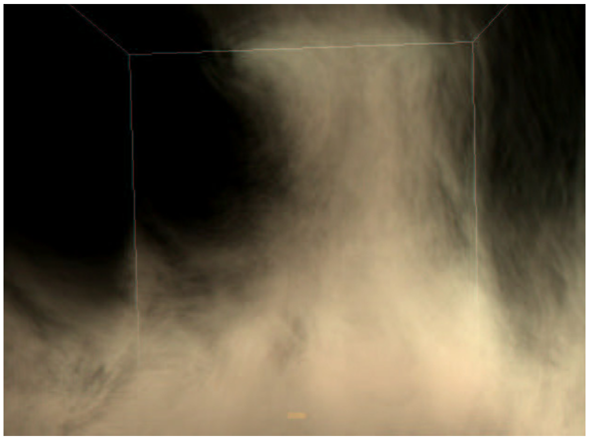
[10]

Figure 9 – Smoke simulated using the Navier-Stokes application.

[10]

Figure 10 – Fire simulated using the Navier-Stokes application.

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