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Packing and Covering the Plane with Translates of a Convex Polygon

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A covering of the Euclidean plane by a polygon P is a system of translated copies of P whose union is the plane, and a packing of P in the plane is a system of translated copies of P whose interiors are disjoint. A lattice covering is a covering in which the translates are defined by the points of a lattice, and a lattice packing is defined similarly. We show that, given a convex polygon P with n vertices, the densest lattice packing of P in the plane can be found in $O(n)$ time. We also show that the sparsest lattice covering of the plane by a centrally symmetric convex polygon can be solved in $O(n)$ time. Our approach utilizes results from classical geometry that reduce these packing and covering problems to the problems of finding certain extremal enclosed figures within the polygon.

1. INTRODUCTION

The problems of packing a set of geometric objects into a space and covering a space with a set of objects arise frequently in applications of computational geometry. A family of sets $S_1, S_2, \ldots$ *covers* a space S if $S \subseteq \bigcup S_i$. The family of sets forms a *packing* of S if the sets of the family have pairwise disjoint interiors and $\bigcup S_i \subseteq S$. It is well known that many packing and covering problems are NP-complete. Examples include the knapsack problem, bin-packing problem, and set cover problem [9]. Al-

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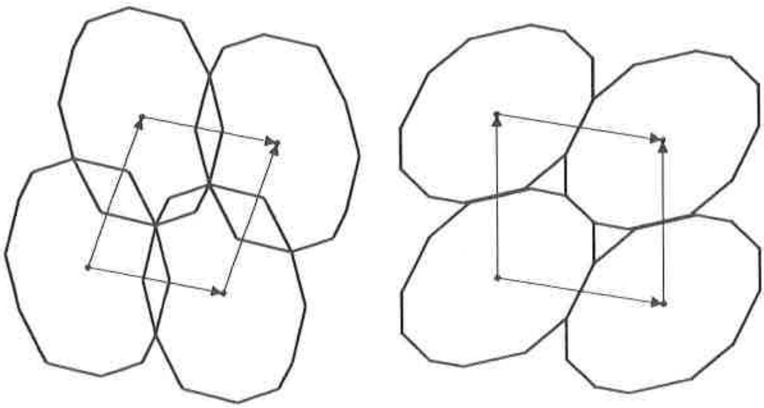
though these problems have natural geometric interpretations, they remain NP-complete even as numeric or discrete problems, ignoring the added intricacies of geometry. Thus, it is of interest to investigate natural geometric packing and covering problems that are solvable in polynomial time.

To motivate the problems considered in this paper, consider an application in stock cutting where the objective is to cut as many identical oriented objects as possible from a flat sheet that has a directional grain, preventing rotation of objects (the oriented cookie cutter's problem). One strategy for solving this problem is to find the densest packing of translates of the object in the plane and then to truncate the optimum plane packing at the sheet's boundary. If the size of the sheet is much larger than the size of the objects being cut out, this strategy will yield a good approximation, since optimizing the utilization of the middle of the sheet will be more important than the wastage caused near the sheet's outer boundary. We will assume that the object to be packed is modeled by a simple polygon. Although it is of interest to solve this problem for arbitrary polygons, in this paper we limit our attention to convex polygons, since it is for these objects that the mathematics of packing is best understood.

A natural dual to the packing problem is the corresponding covering problem. The problem of covering the plane by translated copies of a polygon arises in applications such as vision where a camera with a fixed orientation and a small polygonal field of view must take a series of pictures covering every point of a large planar region. Finding the sparsest covering corresponds to minimizing the overlap between pictures.

Obviously, packing and covering the entire plane implies that the number of objects involved with the solution will be countably infinite; hence the issue of representing the solution is important. Given a convex polygon P and a vector a, the *translate* $P + a$ consists of the set of points $p + a$, where $p \in P$. A *lattice* in the Euclidean plane generated by two linearly independent vectors a_1 and a_2 is the set of vectors of the form $ua_1 + va_2$, where u and v range over the integers. Let $a_1, a_2, a_3, \ldots$ denote an enumeration of the elements of the lattice. The system of translates $\{P + a_i\}$ forms a *lattice covering* if each point of the plane lies in at least one of the members of the system (see Fig. 1a). The system of translates $\{P + a_i\}$ forms a *lattice packing* if no two members of the system share a common interior point (see Fig. 1b). A set P is *centrally symmetric* with respect to a point c if for each point $p \in P$, the reflection of p about c, $2c - p$, is in P. The point c is called the *center* of P. Throughout, we will use the term *symmetric* to mean centrally symmetric.

The *density* of a packing or covering (see Rogers [15] for exact definitions) can be thought of roughly as the ratio of the sum of the areas of the translates to the area of the entire space. This density is at most 1 for a packing and at least 1 for a covering. Densities of 1 are attainable for



(a) (b)

FIGURE 1

parallelograms and symmetric hexagons since translates of these polygons can tessellate the plane.

Lattice packings and lattice coverings are special cases of packings and coverings, respectively. The density of the lattice covering or lattice packing of a polygon P generated by $\{a_1, a_2\}$ is easily seen to be the ratio of the area of P to the area of the parallelogram generated by a_1 and a_2, that is $\det(a_1, a_2)$. It has been shown that the densest packing by translates of a convex set in the plane is a lattice packing [14]. It is also known that the densest packing by translates of a nonconvex object need not be generated by a lattice [2]. The sparsest covering of the plane by translates of a *symmetric* convex set is a lattice covering [6, 7].

Our main results are that the densest lattice packing by translates of an n-sided convex polygon in the plane and the sparsest lattice covering of the plane by translates of a symmetric convex polygon are both solvable in $O(n)$ time. Using the same techniques given in [13] it can be shown that this is optimal in the worst case.

To find the sparsest lattice covering by a symmetric convex polygon we employ a result that reduces this problem to finding the largest symmetric hexagon or parallelogram enclosed in the polygon. In Section 2 we give a simple reduction of this problem to the problem of finding the largest triangle enclosed in the polygon, for which an $O(n)$ algorithm exists [4]. Throughout, n denotes the number of vertices of the polygon.

We exploit two results to find the densest lattice packing of a convex polygon. The first result, due to Minkowski, reduces the problem to that of finding the densest packing of a *symmetric* convex polygon [12], and the second result, attributed to Minkowski (see Rogers [15, p. 6]), reduces this problem to that of finding the smallest parallelogram with one vertex at the center of the symmetric polygon and the other three vertices on the boundary of the polygon. We give an $O(n)$ algorithm that finds this parallelogram. The algorithm works by the method of rotating calipers [17, 13] based on an interspersing condition and finiteness criterion, which we prove in Section 3.

2. Covering the Plane with a Symmetric Convex Polygon

In this section we show that the sparsest lattice covering of the plane by a symmetric convex polygon P can be found in $O(n)$ time. The method used is to determine the symmetric hexagon of largest area contained in P. (A parallelogram is considered to be a degenerate case of a hexagon.) It is easy to show that this hexagon will have its vertices on the boundary of P and will share the same center as P [5]. It is a well-known result from the theory of packing and covering that the sparsest lattice covering is uniquely determined by tessellating the plane with this symmetric hexagon [6, 7].

We show that the problem of finding the largest symmetric convex hexagon contained in P can be reduced to the problem of finding the largest triangle contained in P. Let us assume that P is translated so that its center coincides with the origin. The relationship between the largest inscribed triangle and covering density was reported first by Bambah, Rogers, and Zassenhaus [1]. However, their theorem, while quite a bit more general than ours (it applies to asymmetric convex bodies as well), does not directly provide this simple reduction. In Lemma 2.1 we show that the triangle of maximum area contained in P contains the origin in its interior. Clearly the largest triangle contained in P has its vertices on the boundary of P. In Lemma 2.2 we show that given a triangle T inscribed in P (that is, with vertices on the boundary of P) that contains the origin, the convex hull of T and $-T$ defines a symmetric convex hexagon inscribed in P with twice the area of T. Thus, there is a symmetric convex hexagon contained in P with twice the area of the largest triangle contained in P.

Conversely, consider a symmetric convex hexagon H contained in P, where H is centered at the origin and has its vertices on the boundary of P. It is a simple observation that any triangle T formed by selecting alternating vertices of H contains the origin and that the convex hull of T and $-T$ equals H. It follows immediately that the largest inscribed symmetric hexagon in a symmetric convex polygon P can be found by computing the largest inscribed triangle T in P and computing the convex hull of T and $-T$. Dobkin and Snyder [4] show that given a convex polygon P, the

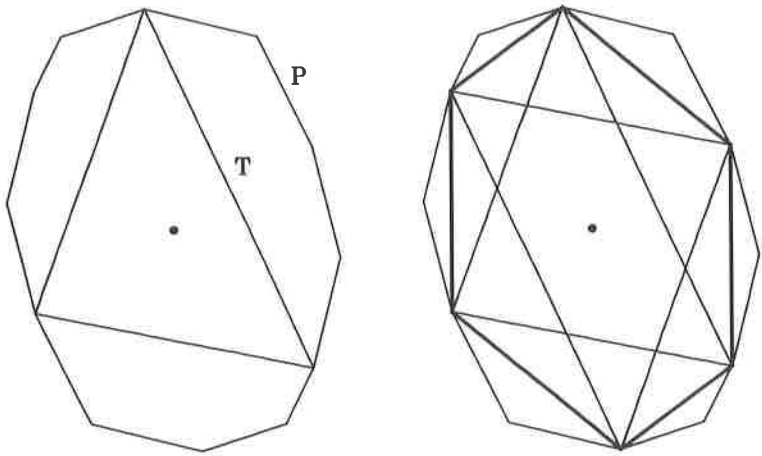


FIGURE 2

inscribed triangle T of maximum area can be found in $O(n)$ time. The convex hull of T and $-T$ can be computed in $O(1)$ additional time. It only remains to prove the two lemmas.

LEMMA 2.1. *Let P be a symmetric convex polygon centered at the origin, and let T be the largest triangle contained in P. Then T contains the origin.*

Proof. Suppose that T does not contain the origin. Then one of the sides of the triangle, say ab, would separate the remaining vertex c from the origin. However, the triangle $(a, b, -c)$ has the same base as (a, b, c), by symmetry, and has a larger altitude, violating maximality. □

LEMMA 2.2. *Let P be a symmetric convex polygon centered at the origin, and let T be a triangle contained in P so that the vertices of T lie on the boundary of P and T contains the origin. Then the convex hull of T and $-T$ is a symmetric convex hexagon (degenerating possibly to a parallelogram) inscribed in P and of twice the area of T (see Fig. 2).*

Proof. Let a, b, c be the points on the boundary of P forming the clockwise triangle T. Since T contains the origin, it follows that a, $-c$, b, $-a$, c, $-b$ is a cyclic order of the points about the origin. By symmetry, all six of these points lie on the boundary of P; thus by convexity these points form the boundary of the convex hull of T and $-T$. Let H denote this convex hull.

The fact that H is symmetric is trivial. Thus, either H is a parallelogram or H is a hexagon. We wish to show that the area of H is twice the area of

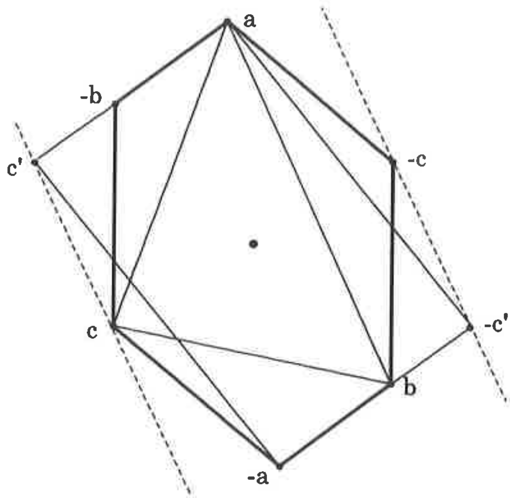


FIGURE 3

T. If H is a hexagon, we map H to a parallelogram H' of equal area and map T to T' of equal area. Otherwise let $H' = H$ and $T' = T$. We then show that the area of H' is twice the area of T'.

Translate the point c along the direction of line ab until reaching a point c' that is collinear with line $a(-b)$ (see Fig. 3). (Note that lines ab and $a(-b)$ are not parallel.) Symmetrically map $-c$ to $-c'$. Clearly, the triangle T' with vertices a, b, and c' has the same area as T. It is easy to verify that the parallelogram $H' = (a, c', -a, -c')$ is equal in area to H. Finally, H' has twice the area of T' because they share the common base ac' and the common altitude from b to ac'. □

3. Packing a Convex Polygon in the Plane

In this section we consider the problem: given a convex polygon P, determine the densest packing of translates of P in the plane. Unlike the previous section, we do not assume that P is symmetric, but, as we will see next, this generalization causes no real problems. The *difference body* of P is the convex sum of polygons P and $-P$; that is, the set

$$P - P = \{x - y | x, y \in P\}.$$

The difference body is clearly symmetric and has at most twice as many vertices as P [18, p. 45]. It can be computed in $O(n)$ time with the procedure given by Schwartz [16] for computing the Minkowski sum of two

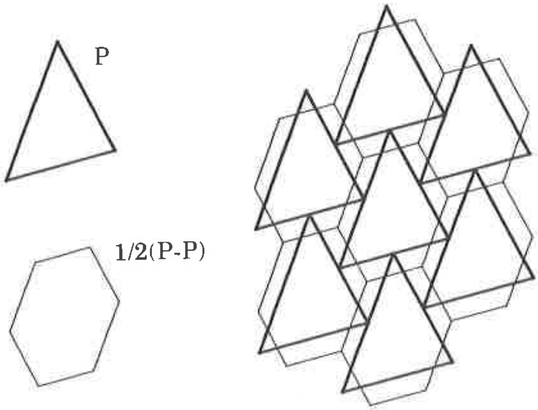


FIGURE 4

convex polygons. Let $\frac{1}{2}(P - P)$ denote the difference body of P scaled by a factor of one half. The following result, due to Minkowski [12], shows that the problem of packing a convex set can be reduced to the problem of packing a symmetric convex set. A proof can be found in Rogers [15, p. 69], and the theorem is illustrated in Fig. 4.

THEOREM 3.1 (Minkowski). *Let P be a bounded convex set in the plane with positive measure. The system $\{P + a_i\}$ is a packing of P if and only if the system $\{\frac{1}{2}(P - P) + a_i\}$ is a packing of $\frac{1}{2}(P - P)$.*

Let us assume that the polygon P to be packed is symmetric and centered at the origin with n vertices. It is well known that the densest packing of P results by tessellating the plane with the smallest symmetric hexagon (or parallelogram) that encloses P [8, p. 86]. This suggests that the problem can be solved by a method analogous to that of the previous section by reducing the problem to that of finding the smallest triangle enclosing the object. We know of no such transformation (although the interested reader is referred to DePano's thesis in which such a transformation is used for generating efficient packings of polygons [3]). Alternatively, since a symmetric hexagon is determined by only three vertices, one would expect that the known algorithms for computing smallest enclosing triangles [10, 13] could easily be adapted to find smallest enclosing symmetric hexagons. Unfortunately, the critical monotonicity properties of anchored triangles that are exploited by these algorithms do not seem to hold for their generalization to anchored symmetric hexagons.

We solve this problem by reducing the packing problem to an alternative characterization, which was first given by Minkowski [15, p. 6]. Define a *central parallelogram* for the symmetric convex set P to be a parallelogram with one vertex at the center of P and the other three vertices on the boundary of P. Minkowski's theorem shows that the problem of finding the densest packing of a convex symmetric set in the plane can be reduced to the problem of finding the smallest central parallelogram in P.

THEOREM 3.2 (Minkowski). *Let P be a symmetric convex set centered at the origin, and consider a lattice packing of P generated by some pair of vectors a_1 and a_2. This packing is the densest lattice packing for P if and only if the origin together with the points $\frac{1}{2}a$, $\frac{1}{2}a_2$, and $\frac{1}{2}(a_1 + a_2)$ define a central parallelogram for P of minimum area.*

Let O denote the origin. Let us assume that P is a symmetric convex polygon centered at O. We use the notation $\langle O, a, b, c\rangle$ to denote the central parallelogram whose clockwise vertices are O, a, b, and c. If $\langle O, a, b, c\rangle$ is a central parallelogram, then $b = a + c$.

Thus our objective is to find the smallest central parallelogram for P. We begin with some observations about the structures of these parallelograms. Our approach is quite similar in structure to the algorithms for computing enclosed and enclosing triangles [4, 13] based on the ideas of rotating calipers [17]. We first define the notion of a central parallelogram *anchored* at a given point and then prove the fundamental interspersing property needed for the rotating caliper. This property states that as the anchored point moves clockwise around the boundary of the polygon, the other points of the anchored central parallelogram also move clockwise. We show that there are finitely many anchor points (in fact, only linearly many) that might lead to the smallest central parallelogram.

Our first lemma formalizes the notion of an anchored central parallelogram by stating that once the first boundary point of a central parallelogram is fixed, then either the other two boundary points are uniquely determined or there are infinitely many such pairs lying on a common edge, all generating parallelograms of equal area. Let $|a|$ denote the length of vector a.

LEMMA 3.1. *Let P be a symmetric convex polygon, and let $C = \langle O, a, b, c\rangle$ and $C' = \langle O, a', b', c'\rangle$ be two distinct central parallelograms for P. Then $a = a'$ if and only if b, c, b' and c' all lie on a common edge of P. If $a = a'$ then C and C' have equal area.*

Proof. Suppose that $a = a'$. Let us assume that P is oriented to that a is directed upwards along the y-axis. Because the orientation of points is clockwise, b, b', c, and c' all lie on the positive side of the y-axis. Let $L(x_0)$ denote the intersection of the vertical line $x = x_0$ with P. Let x_r denote the maximum x for which $L(x)$ is nonempty. By convexity, as x varies from $-x_r$ to x_r the length of $L(x)$ as a function of x is unimodal.

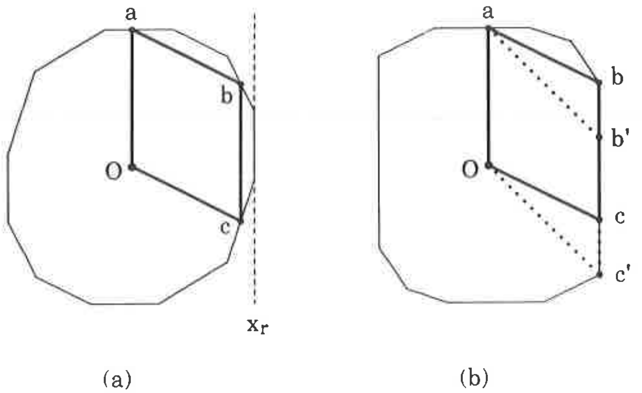


FIGURE 5

By symmetry, this function achieves its maximum at $x = 0$. Thus, as x varies from 0 to x_r, the length of $L(x)$ may remain constant at $2|a|$ for some period, and then decreases strictly monotonically to some minimum value at x_r. Let d be this minimum value.

If $d \leq |a|$, there is exactly one value of x for which the length of $L(x)$ equals $|a|$, and for this one value of x the upper and lower endpoints of $L(x)$ complete a central parallelogram (see Fig. 5a). If $d > |a|$ then d is the length of a vertical edge, and the central parallelogram can be completed if and only if its other two vertices lie along this edge at distance $|a|$ from each other. Clearly all parallelograms generated in this way have the same area (see Fig. 5b).

Conversely, if b, c, b', and c' all lie on a common edge, then a and a' both lie on the intersection of the boundary of P and a ray directed as $b - c$ (which equals the direction of $b' - c'$). Since P is convex, this point is unique. □

With the above result in mind, for an arbitrary point a on the boundary of P, we define a central parallelogram *anchored* at a to be any central parallelogram $\langle O, a, b, c \rangle$ with a as its first boundary point. If the central parallelogram anchored at a is not unique, Lemma 3.1 shows that, with respect to area, it does not matter which one we use. In this case a *canonical* parallelogram can be chosen by shifting the edge bc of the parallelogram as far clockwise as possible, so that c is a vertex of P.

Next we show that as the point a moves clockwise along the boundary of P the points b and c in the central parallelogram anchored at a also move clockwise (or remain fixed). This is the interspersing condition will be exploited in our algorithm. For two distinct points a and b on the

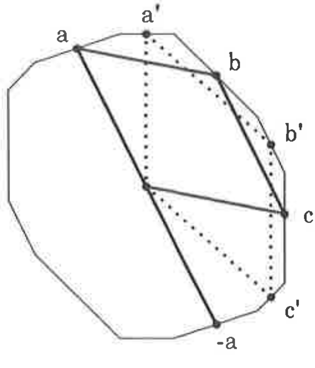


FIGURE 6

boundary of P, let (a, b) denote the open clockwise arc from a to b along the boundary of P, and let $[a, b]$ denote its closure.

LEMMA 3.2. *For any point a on the boundary of a symmetric convex polygon P, let $\langle O, a, b, c\rangle$ be the canonical central parallelogram anchored at a, and let a' be any point in (a, b). If $\langle O, a', b', c'\rangle$ is the canonical central parallelogram anchored at a', then $b' \in [b, c]$ and $c' \in [c, -a]$ (see Fig. 6).*

Proof. For a point q, let q_x and q_y denote the x and y coordinates of q. By rotating P we may assume that the vector a' is directed upwards along the y-axis, implying that the segment $b'c'$ is vertical. This also implies that $a_x < 0$ and $b_x > 0$. Since $c = b - a$, we have $c_x > \max(-a_x, b_x)$.

It follows by convexity and symmetry that any point on the contour $[a, b]$ is contained in the triangle $T = (a, a + b, b)$. Thus, a' is contained in the triangle T. We divide the proof into two cases, depending on the location of c. If c is a point on a vertical edge e of P, and $|e| \geq |a'|$, then b' and c' lie on this edge as shown in Lemma 3.1. Neither b nor $-a$ can lie on this edge, since $c_x > \max(-a_x, b_x)$. Thus, $b \in [a', b']$ and $-a \in [c', -a']$. By the definition of a canonical anchored central parallelogram, c' must be the lower endpoint of the edge e. If $|c - c'| \leq |a'|$, then $c \in [b', c']$ and the conclusion follows (see Fig. 7a). If $|c - c'| > |a'|$ then, since $a' \in T$, the point c' lies within or below the triangle $c - T = (c - a, -a, -2a)$, whose intersection with P is empty, contradicting the fact that c' lies on the boundary of P.

On the other hand, if c does not lie on a vertical edge e of P where $|e| \geq |a'|$, then by Lemma 3.1 there is exactly one central parallelogram anchored at a'. We demonstrate the existence of such a parallelogram satisfying the required conditions. Let B denote the contour of the boundary of P in the positive-x halfplane, that is, $B = [a', -a']$. For each point

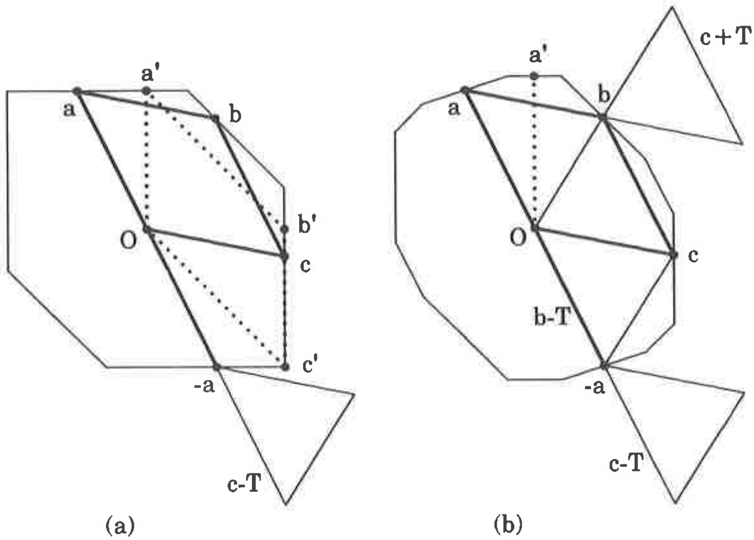


FIGURE 7

q on B, let $L(q)$ denote the intersection of P with the vertical line passing through q. By convexity and symmetry, as q travels along B from a' to $-a'$, the length $|L(q)|$ is a continuous unimodal function, decreasing from $|2a'|$ at a' to a minimum at those points of vertical tangency and then increasing back to $|2a'|$ at $-a'$. Since $c_x > \max(-a_x, b_x)$, it follows that b is the upper endpoint of $L(b)$ and $-a$ is the lower endpoint of $L(-a)$.

To show that there is a point $b' \in [b, c]$ and $c' \in [c, -a]$ such that $b' - c' = a'$, it suffices by unimodality to show that $|L(b)| \geq |a'|, |L(c)| \leq |a'|, |L(-a)| \geq |a'|$. The first assertion follows noting that b is the upper endpoint of $L(b)$, and since $a' \in T = (a, a+b, b)$ we have that $b - a'$ lies in the triangle $b - T = (c, -a, O) \subseteq P$ (see Fig. 7b). The third assertion follows, since $-a$ is the lower endpoint of $L(-a)$, and so $-a + a'$ lies in the triangle $-a + T = (O, b, c) \subseteq P$. If c lies on a vertical edge e (implying $|e| < |a'|$) then $|L(c)| < |a'|$. Otherwise, c is the upper or lower endpoint of $L(c)$. However, $c - a'$ lies in the triangle $c - T = (c - a, -2a, -a)$ and $c + a'$ lies in the triangle $c + T = (b, 2b, c + b)$, and neither of these triangles intersects the interior of P. □

Lemma 3.2 suggests that we can imagine a point a rotating clockwise around the boundary of the polygon P and for each such point computing a central parallelogram anchored at a. However, we need to limit the search to a finite set of points a at which the minimum area central parallelogram

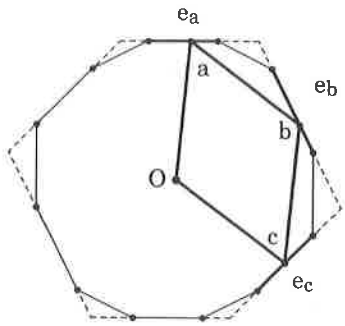


FIGURE 8

may occur. Consider an arbitrary point a on the boundary of P generating a canonical central parallelogram $\langle O, a, b, c\rangle$, where a, b, and c lie on the interior of the edges e_a, e_b, and e_c, respectively. By extending these edges and their negations until they meet, we form a symmetric convex hexagon circumscribing P, called the *extension* of a, b, and c (see Fig. 8). We say that the point a is *viable* if either

(i) a, b, or c is a vertex of P, or

(ii) a, b, and c are the midpoints of the sides of their extension.

(Notice that although the anchoring condition applies to the parallelogram in general, only the anchoring vertex a is defined as being viable.) Viable points of the second type are not as rare as one might think, because it is easy to show that the midpoints of any three consecutive sides of a symmetric hexagon form a central parallelogram. The viable points provide us with the finite and, in fact, linear sized set of candidates for the smallest central parallelogram. These facts are presented in our next lemma.

LEMMA 3.3. (i) *Every minimum area central parallelogram for P is anchored at a viable point.*

(ii) *Given a convex polygon P with n vertices, the number of viable points on P is at most $6n$.*

Proof. To prove (i), suppose to the contrary that the minimum central parallelogram $\langle O, a, b, c\rangle$ is not anchored at a viable point. This implies that the points a, b, and c are not vertices of P. Form the hexagonal extension H of these points. Clearly $P \subseteq H$. Since $\langle O, a, b, c\rangle$ is minimum over all central parallelograms in P, it must be a minimum among the central parallelograms for H. Since H is a symmetric hexagon, its unique densest packing results by tessellating the plane with H, which in turn is generated by the central parallelogram whose points are the midpoints of H. Thus, the points a, b, and c are the midpoints of their extension, implying that a is viable. (In fact, it can be shown that the only viable points that can really lead to the minimum central parallelogram are either of this second type, or else they are the viable points of the first type for which H degenerates to a parallelogram. This result is somewhat analogous to Theorem 1.1 in [10] that states that the minimum enclosing triangle of a polygon touches the polygon at the midpoints of its sides. We introduce viable points of the former type to facilitate description of the algorithm.)

To prove (ii), note that the boundary of P consists of n vertices and n edges. Each central parallelogram $\langle O, a, b, c\rangle$ can be identified with a triple consisting of the names of the edges or vertices that are incident with a, b, and c, respectively. Let S_1 denote the clockwise sequence of central parallelograms (in canonical form) where either a, b, or c is a vertex and let S_2 denote the sequence of central parallelograms in which all the vertices lie in the interiors of edges of P. By Lemma 3.1, once one point of a canonical central parallelogram is fixed, the other two points of the parallelogram are determined. Thus there are at most $3n$ parallelograms in set S_1; n for each of the points a, b, and c. On the other hand, if all three points lie in the interiors of edges, then by the definition of viability, this is the only central parallelogram incident on this triple of edges. Between every two consecutive parallelograms in S_2 there is at least one vertex, hence at least one member of S_1. Therefore the number of parallelograms in S_2 is no greater than S_1, implying that the total number of parallelograms is at most $6n$. □

We can now describe the algorithm for finding the smallest central parallelogram. As mentioned earlier, for some vertex a_0, let $B = [a_0, -a_0]$ be a half-boundary of P. The algorithm operates by visiting each viable point along B in clockwise order and generating the canonical central parallelogram anchored at that point. (By symmetry the viable points on the remaining half-boundary can be ignored.) We compute the area of this parallelogram and advance the scan to the next viable point in clockwise order. We will show first that the central parallelogram anchored at a_0 can be generated in $O(n)$ time, and second that all subsequent viable points as well as their corresponding central parallelograms can be generated in $O(1)$ time each. Thus the overall running time will be $O(n)$, by Lemma 3.3(ii).

The proof of Lemma 3.1 provides an algorithm for generating the canonical central parallelogram $\langle O, a, b, c\rangle$ anchored at vertex a_0. The segment b_0c_0 is the unique (furthest clockwise) segment whose endpoints are on the boundary of P and which is parallel to and of equal length with the vector a_0. Assume as we did in the proof of Lemma 3.1 that the vector a_0 is directed upwards. The vertical length function $L(x)$, introduced in the proof of Lemma 3.1, is monotonically decreasing from $2|a_0|$ and piecewise linear. This function can be constructed in linear time by merging the vertices in the half-boundary $[a_0, -a_0]$ according to the x-coordinates. After merging, between any two consecutive vertices, the function $L(x)$ is linear and computable in $O(1)$ time. Once the function has been constructed, it is a simple matter to find the largest point x for which $L(x) \geq |a_0|$. The vertex c_0 is the lowest point on the boundary of P at x, and the b_0 is the point $c_0 + a_0$. (In fact, a single central parallelogram can be found in only $O(\log^2 n)$ time by binary search, but this simple linear time algorithm suffices for our purposes.)

Given an arbitrary central parallelogram $\langle O, a, b, c\rangle$ we wish to find the canonical central parallelogram $\langle O, a', b', c'\rangle$ anchored at the next viable point a'. Let us assume that $\langle O, a, b, c\rangle$ is in canonical form. If not, convert it to canonical form in $O(1)$ time. By Lemma 3.2, the points $a,'$, b', and c' are all clockwise from or equal to their corresponding points a, b, and c. Let e_a, e_b, and e_c be the edges lying just clockwise from a, b, and c, respectively, and let a'', b'', and c'' denote the clockwise endpoints of these edges. By the choice of edges, each point like a'' is distinct from the corresponding point a. The edges e_b and e_c cannot be equal because $\langle O, a, b, c\rangle$ is in canonical form. Let H denote the hexagonal extension of these edges, which can be constructed in $O(1)$ time.

If H degenerates to a parallelogram then either e_a and e_b are equal, or e_a and $-e_c$ are equal. In the first case the next central parallelograms in clockwise order result by translating the side ab clockwise along this common edge until b coincides with b''. This operation is analogous to the sliding operation used to transform a parallelogram into canonical form. All parallelograms formed in this way have equal area. In the second case H is a parallelogram centered at the origin and a, b, and c lie on three consecutive edges of H. There is a similar sliding movement here. It follows from elementary geometry that b is fixed as the midpoint of its side of H, and as a moves clockwise by a given distance on its side, c must move clockwise by an equal distance to preserve the parallelogram. All the parallelograms that arise from this sliding operation have equal area, so we slide a and c until one first encounters a vertex, a'' or c'', respectively.

On the other hand, if H is a hexagon, then there are four candidates for the next anchoring point. Let a_1 be the midpoint of the side of H extending e_a. Clearly a_1 is viable by the midpoint criterion, and hence a candidate for the next anchoring point provided that a_1 lies on edge e_a strictly clockwise from a about P. (Note that we do not need to consider corresponding points b_1 and c_1, which lie at the midpoints of their respective edges of the hexagon, because once a_1 is fixed to be the

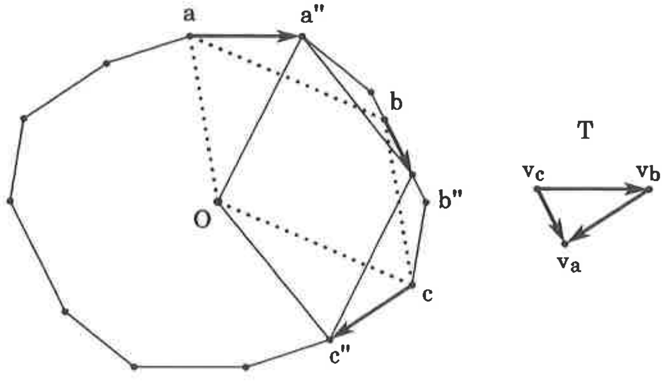


FIGURE 9

midpoint of its side we force the other two vertices of the central parallelogram to lie on the midpoints of their respective sides.) To determine the other three anchor candidates we use the fact that as the anchor point moves continuously clockwise along the segment aa'', the corresponding points of the resulting central parallelograms move continuously along the segments bb'' and cc''. The next significant event occurs when any one of the vertices of the central parallelogram first encounters one of the vertices a'', b'', or c''. Our strategy is to construct three central parallelograms, one having its first vertex at a'', one having its second vertex at b'', and one having its third vertex at c'', and then select that parallelogram whose anchor is closest to a.

To determine these parallelograms, let $T = (v_a, v_b, v_c)$ denote a triangle (of unspecified size and location) whose sides are parallel to e_a, e_b, and e_c. Since H is a hexagon no two of these sides are parallel. The vertices of T are named, for example, so that vertex v_a is opposite the side parallel to e_a (see Fig. 9). Thinking of the vertices of T as vectors, we claim that $\langle O, (a + v_b - v_c), (b + v_a - v_c), (c + v_a - v_b)\rangle$ is a central parallelogram whose vertices lie on the lines extending e_a, e_b, and e_c, respectively. It is easy to see that the points have all been translated along their respective edges. These four points form a parallelogram because $(a + v_b - v_c) + (c + v_a - v_b) = b + v_a - v_c$.

By increasing the scale factor of T upwards from zero, we can generate the local clockwise transformations of the central parallelogram $\langle O, a, b, c\rangle$. We wish to determine the smallest scale factor for which the transformed parallelogram encounters one of the vertices a'', b'', or c''. Let T_a be triangle T scaled so that the side opposite v_a has length $|aa''|$, let T_b be T scaled so that the side opposite v_b has length $|bb''|$, and let T_c be T scaled so that the side opposite v_c has length $|cc''|$. All of these triangles result by a nonzero scaling factor. It follows easily that the parallelograms generated by T_a, T_b, and T_c have vertices coinciding with a'', b'', and c'', respectively. Among these triangles, select the side opposite vertex v_a whose length is minimum. The central parallelogram resulting by translating a through this length is the next anchored central parallelogram having a vertex that coincides with a vertex of P, and hence is viable. By selecting the minimum of this translation (which is necessarily strictly clockwise) and the translation $a_1 - a$ (provided it is strictly clockwise) we have the next viable point a'. The corresponding points b' and c' are determined by the appropriate scale factor applied to T. The time required to determine this next viable point and the corresponding central parallelogram is $O(1)$. Since there are $O(n)$ viable central parallelograms, the algorithm's overall running time is $O(n)$.

4. REMARKS

We have shown that the problems of finding the sparsest covering of a symmetric convex polygon and the densest packing of a (general) convex polygon in the plane are both solvable in $O(n)$ time. The covering problem was solved by a simple reduction to the problem of finding the maximum enclosed triangle, and the packing problem was solved by an algorithm that finds the smallest central parallelogram by the method of rotating calipers.

We know of no general bounds relating the densities of lattice packings and coverings to more general packings and coverings where, for example, rotating the object is permitted. Without rotation a triangle can be packed with density no greater than $\frac{1}{2}$, but allowing for a rotation of 180° the triangle can tile the plane. G. Kuperberg and W. Kuperberg have shown that by using a *double-lattice packing*, in which at even points in the lattice the object is placed, and at odd points of the lattice a 180° rotation of the object is placed, any convex set can be packed in the plane with density at least $\sqrt{3}/2$ [11]. An interesting open problem is to determine an efficient algorithm for computing the densest double lattice packing for a convex polygon.

There are a number of other interesting open problems raised by this work. The first question is, can the results of this paper be extended to more general polygons, for example, to covering the plane by convex asymmetric polygons or packing star-shaped polygons? Unfortunately the techniques of this paper rely heavily on both convexity and symmetry to reduce the number of degrees of freedom in selecting the minimum and maximum enclosing symmetric hexagons. Finally, a very interesting practical question is whether these techniques can be extended to packing systems of multiple objects.

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