Last update: February 2, 2010

Informed search algorithms

CMSC 421: Chapter 4, Sections 1–2

Motivation

- In Chapter 3 we were talked about trial-and-error search
- In the worst case, most searches take exponential time (unless P=NP)
- Can sometimes do much better on the average, using *heuristic* techniques

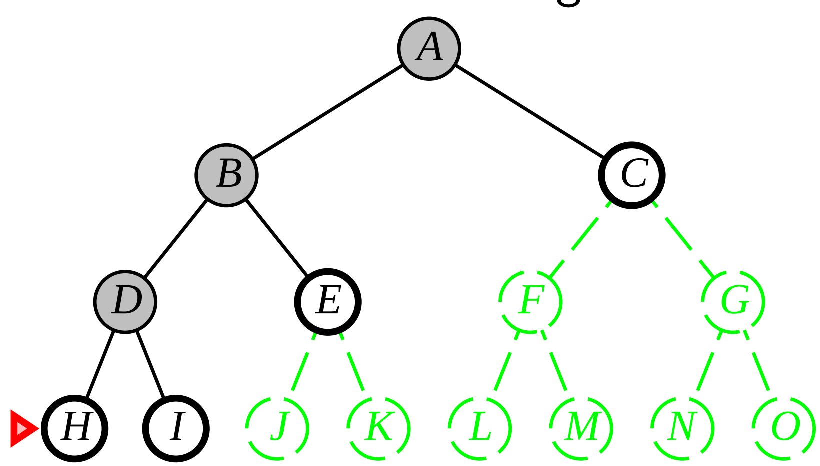
Heuristic:

- Rule of thumb, simplification, or educated guess
- Reduces the search for solutions in domains that are difficult and poorly understood.
- Depending on what heuristic you use, you won't necessarily find an optimal solution, or even a solution at all.

Heuristic tree search

function TREE-SEARCH(*problem*) **returns** a solution, or failure
  fringe ← a list containing MAKE-NODE(INITIAL-STATE[*problem*])
  loop do
    if *fringe* is empty **then return** failure
    node ← REMOVE-FRONT(*fringe*)
    if GOAL-TEST[*problem*] applied to STATE(*node*) succeeds **return** *node*
    fringe ← INSERTALL(EXPAND(*node*, *problem*), *fringe*)

fringe = a list of the nodes that have been generated but not expanded:



Heuristic tree search

```
function TREE-SEARCH( problem) returns a solution, or failure
    fringe ← a list containing MAKE-NODE(INITIAL-STATE[problem])
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST[problem] applied to STATE(node) succeeds return node
        fringe ← INSERTALL(EXPAND(node, problem), fringe)
```

Heuristic choice in search algorithms: **what node to expand next**

Use an *evaluation function* $f(n)$ for each node n

– estimate of "desirability"

$\Rightarrow$ Expand most desirable unexpanded node

INSERTALL keeps *fringe* sorted in decreasing order of desirability,

i.e., if $fringe = \langle s_1, s_2, \ldots, s_k \rangle$

then $f(s_1) \leq f(s_2) \leq \ldots \leq f(s_k)$

Thus REMOVE-FRONT always gets a most-desirable node

Heuristic graph search

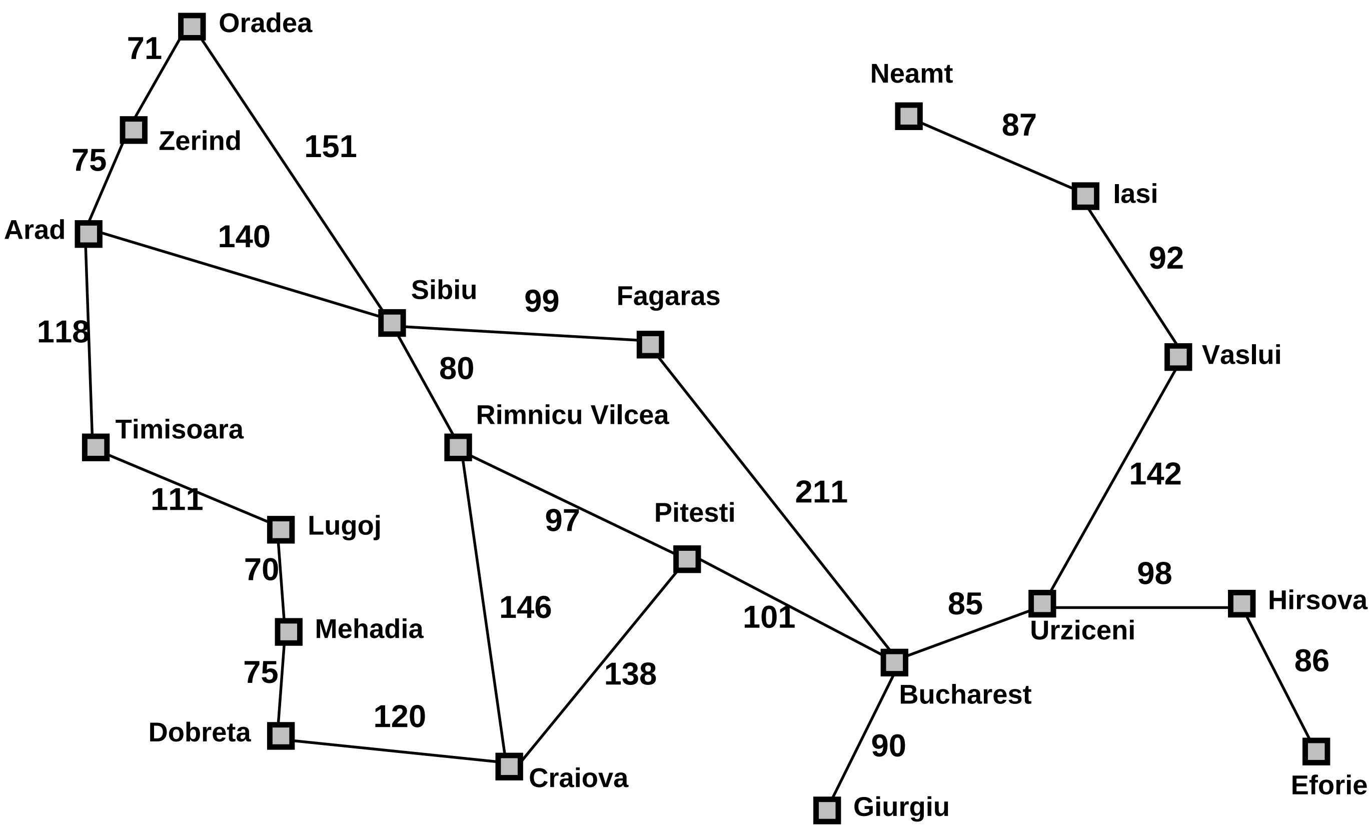
```
function GRAPH-SEARCH( problem, fringe) returns a solution, or failure
    closed ← an empty set
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST(problem, STATE[node]) then return node
        if STATE[node] is not in closed then
            add STATE[node] to closed
            fringe ← INSERTALL(EXPAND(node, problem), fringe)
    end
```

Same as for TREE-SEARCH:

Use INSERTALL to keep *fringe* sorted in decreasing order of desirability

Romania with step costs in km

Recall that we want to get from Arad to Bucharest:



Straight−line distance
to Bucharest

Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
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Zerind	374

Greedy search

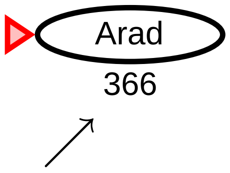
Heuristic function $h(n)$ = estimate of cost from n to the closest goal

E.g., $h_{\mathrm{SLD}}(n)$ = straight-line distance from n to Bucharest

Greedy search uses $f(n) = h(n)$,
i.e., keeps $fringe$ ordered in increasing value of h

hence always expands whatever node **appears** to be closest to a goal

Greedy search example



straight-line distance to Bucharest

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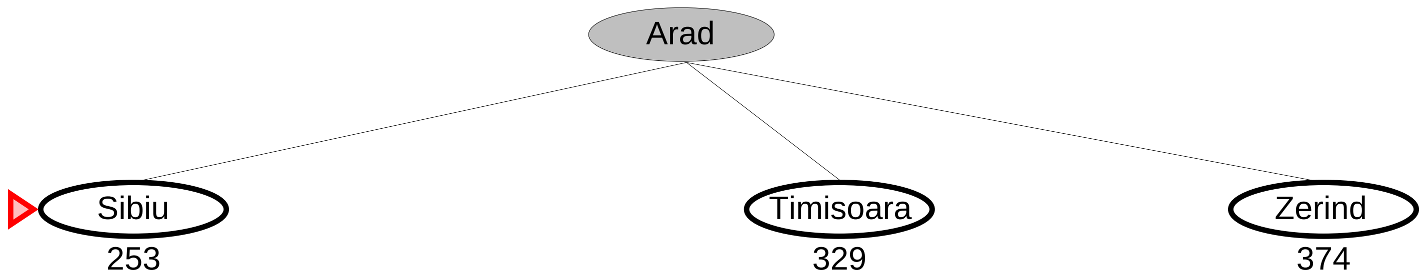

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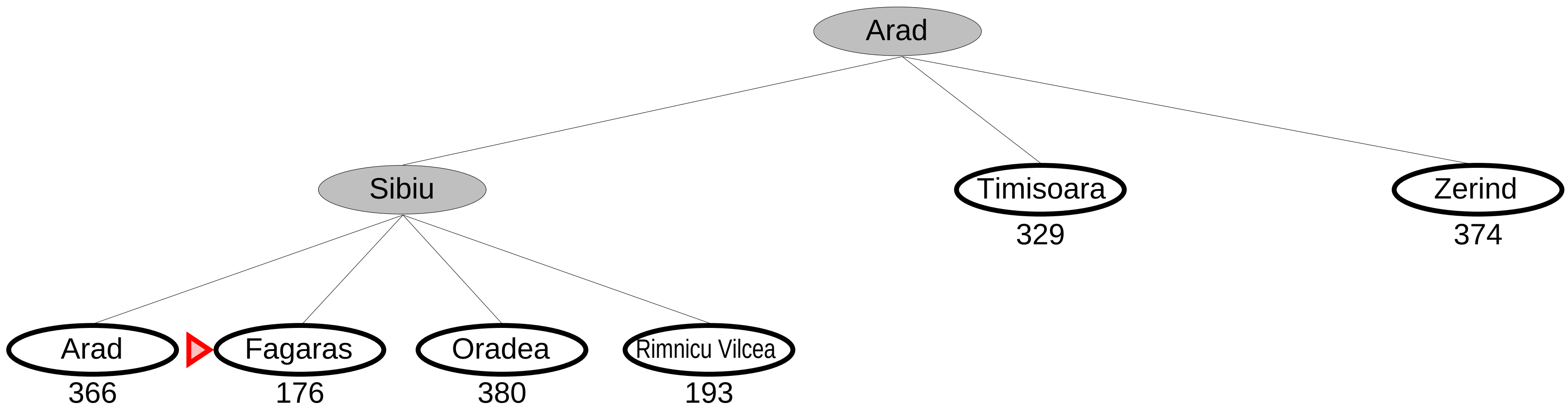

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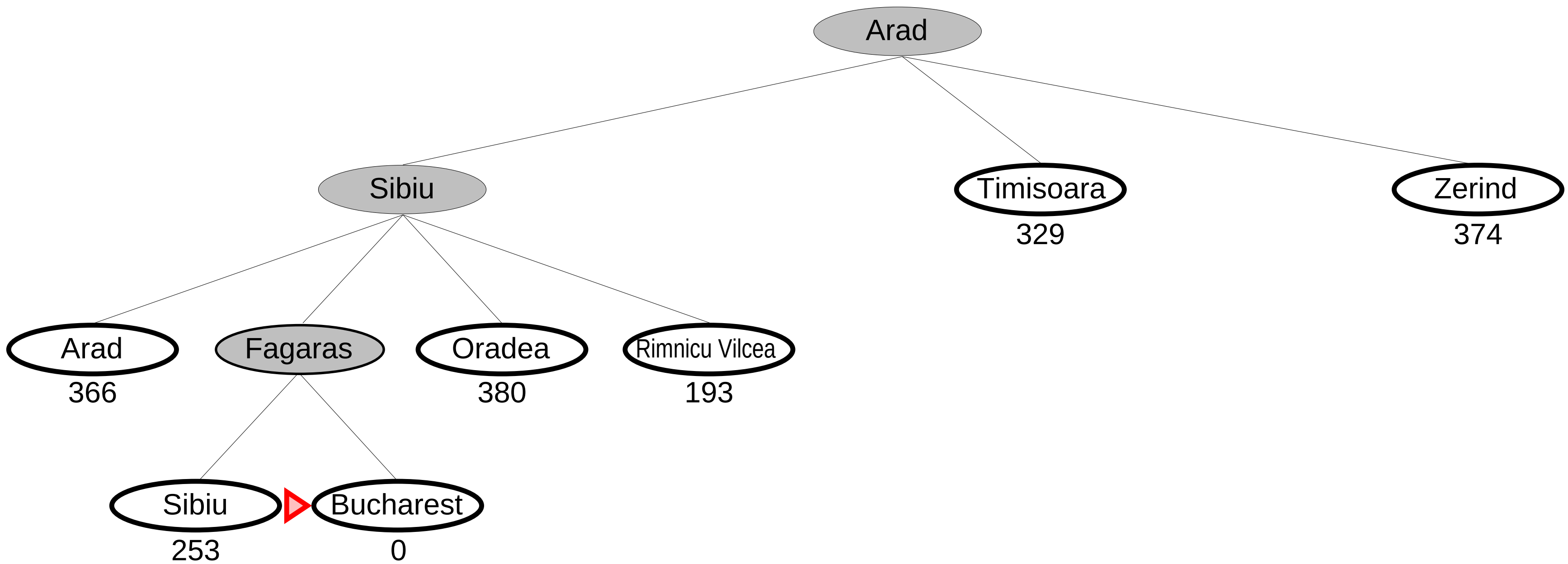

Properties of greedy search

Complete?

Properties of greedy search

Complete? No. Can get stuck in loops:

Iasi $\longrightarrow$ Neamt $\longrightarrow$ Iasi $\longrightarrow$ Neamt $\longrightarrow$

Complete in finite space with repeated-state checking

Time?

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Time? $O(b^m)$, but a good heuristic can give dramatic improvement

Space?

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Space? $O(b^m)$—keeps all nodes in memory

Optimal?

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Optimal? No

Problem with terminology:

Greedy search is not the same as an ordinary **greedy algorithm**.

An ordinary greedy algorithm doesn't remember all of $fringe$.
It remembers only the current path, and never backtracks. Hence:

- ♢ Repeated-state checking cannot make it complete
- ♢ It runs in time $O(l)$ if it finds a solution of length l

A* tree search

Idea: avoid expanding paths that are already expensive

Evaluation function $f(n) = g(n) + h(n)$

$g(n)$ = cost so far to reach n
$h(n)$ = estimated cost to goal from n
$f(n)$ = estimated total cost of path through n to goal

Optimality requirement for A* tree search:

A* needs an *admissible* heuristic, i.e., $0 \leq h(n) \leq h^*(n)$
where $h^*(n)$ is the **true** cost from n.
(Thus $h(G) = 0$ for any goal G.)

E.g., $h_{\mathrm{SLD}}(n)$ never overestimates the actual road distance

Theorem: If the optimality requirement is satisfied, then A* tree search never returns a non-optimal solution

A* tree search

Completeness requirement for A* tree search:

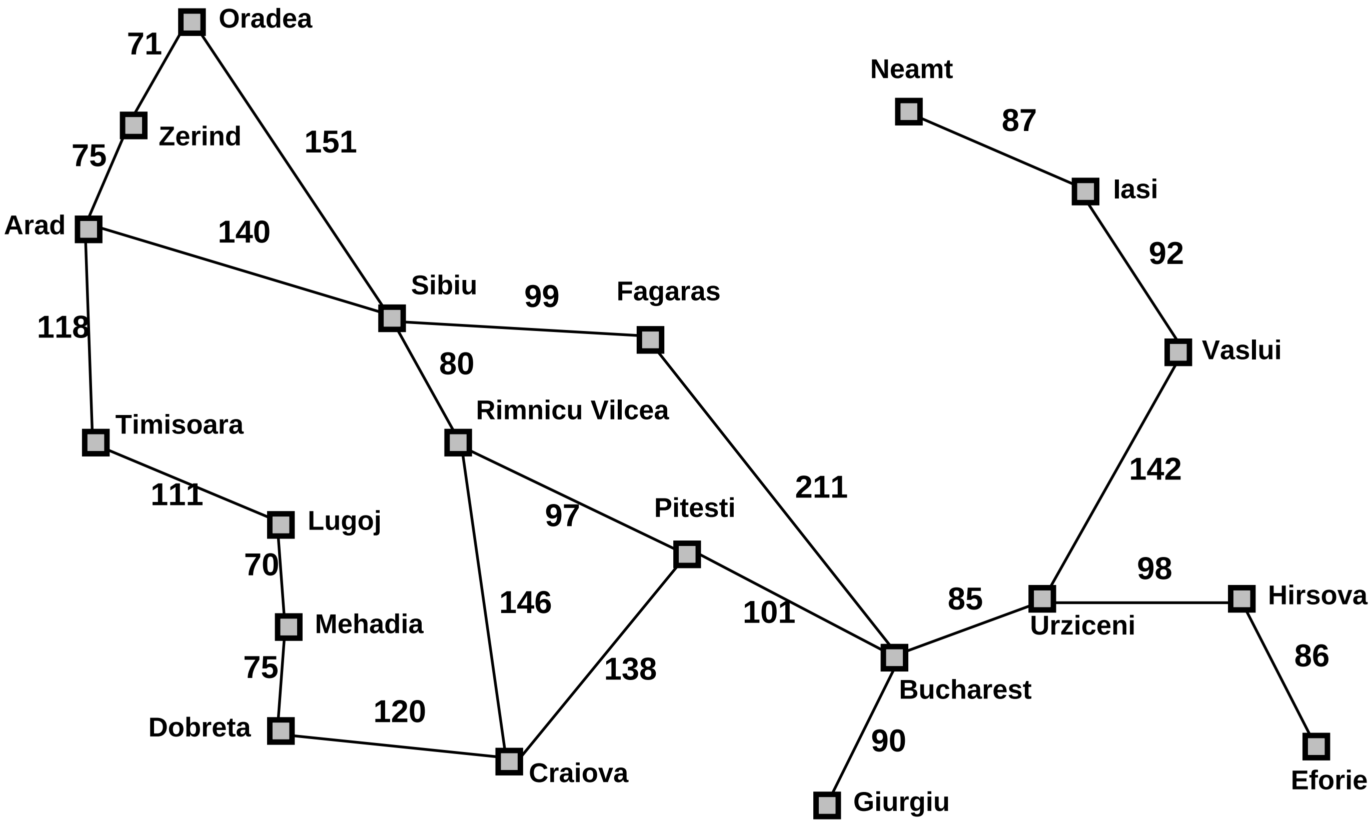
No infinite path has a finite cost

Theorem: On any solvable problem that satisfies the completeness requirement, A* tree search returns a solution.

Corollary: If the optimality requirement also is satisfied, then A* tree search returns an optimal solution.

Romania with step costs in km

Recall that we want to get from Arad to Bucharest:



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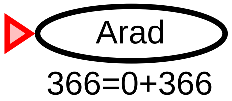
A* tree search



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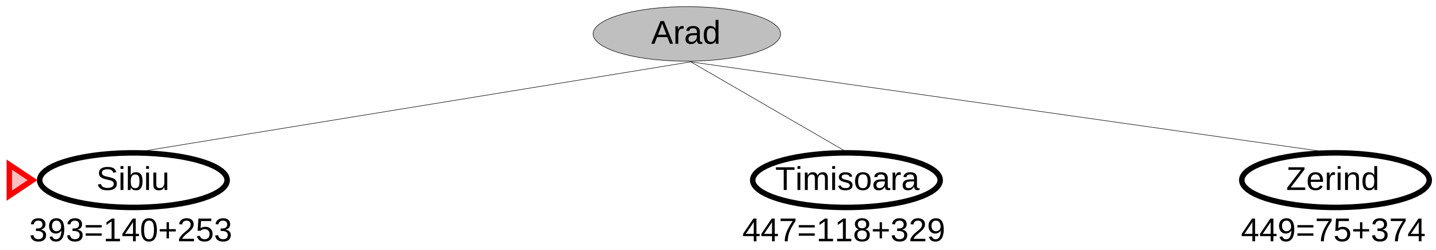

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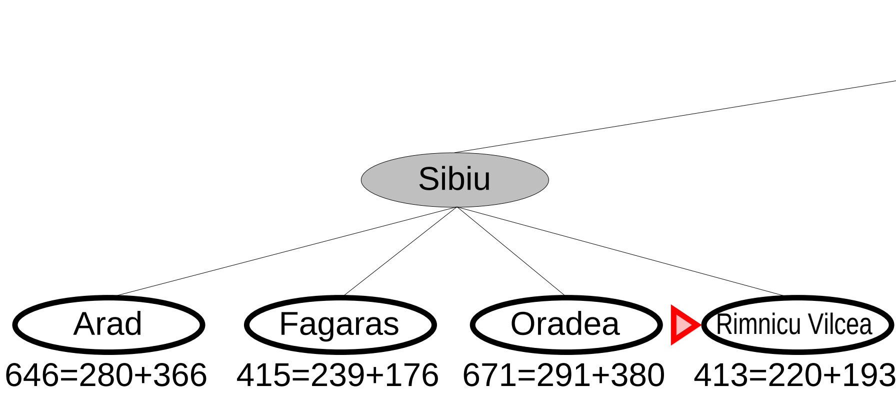

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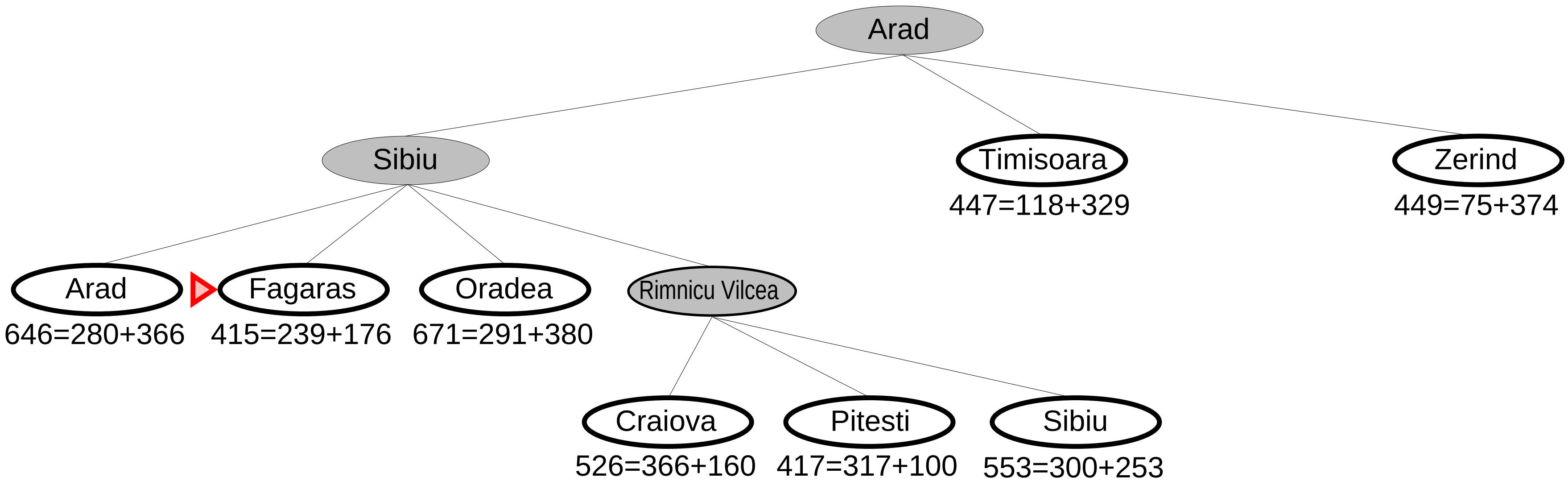

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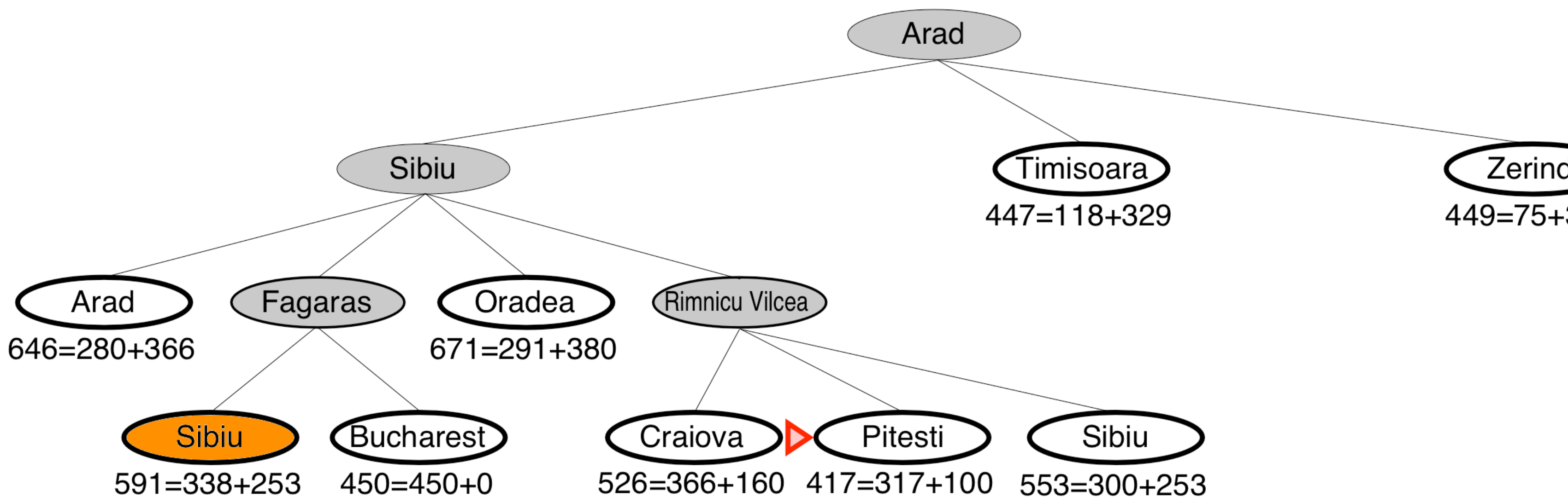

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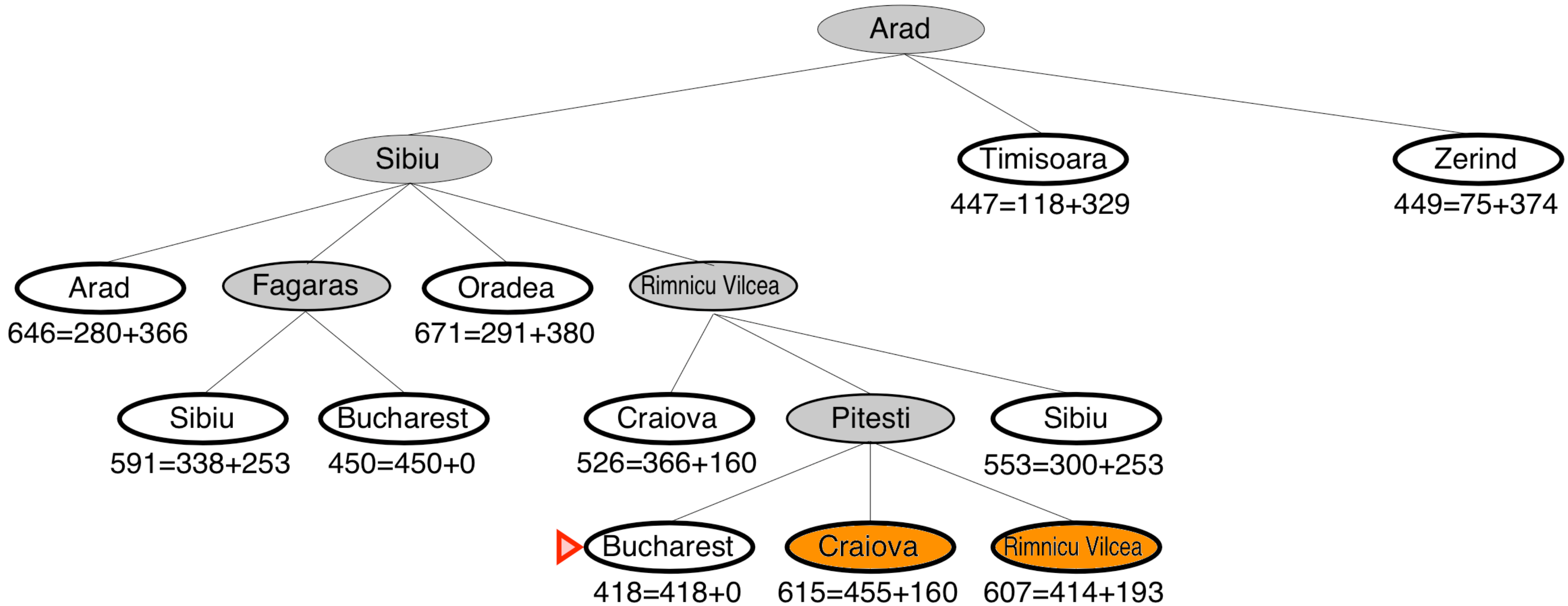
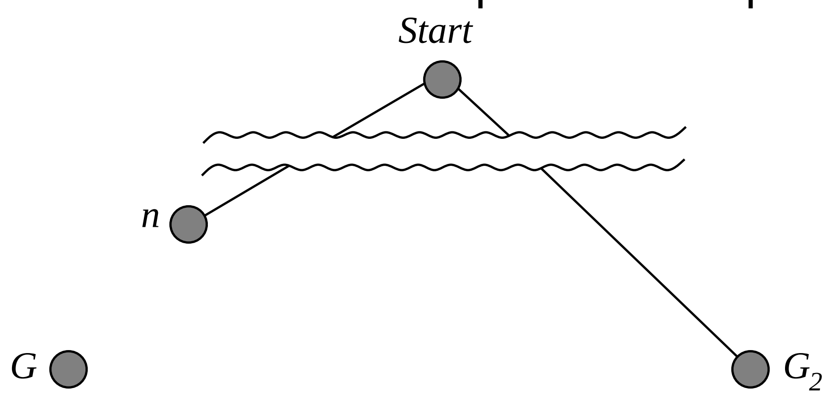


Optimality of A*

Suppose some suboptimal goal G_2 has been generated and is in $fringe$. Let n be an unexpanded node on a shortest path to an optimal goal G_1.



$$
\begin{aligned}
f(G_2) &= g(G_2) && \text{since } h(G_2) = 0 \\
&> g(G_1) && \text{since } G_2 \text{ is suboptimal} \\
&\geq f(n) && \text{since } h \text{ is admissible}
\end{aligned}
$$

Since $f(G_2) > f(n)$, A* will never select G_2 for expansion

A* graph search

A* can also be used with Graph-Search.

Optimality requirement is same as before:

- ♢ The heuristic must be admissible

There are two completeness requirements:

- ♢ One is the same as before: no infinite path has a finite cost
- ♢ The other is something that Russell & Norvig don't mention:

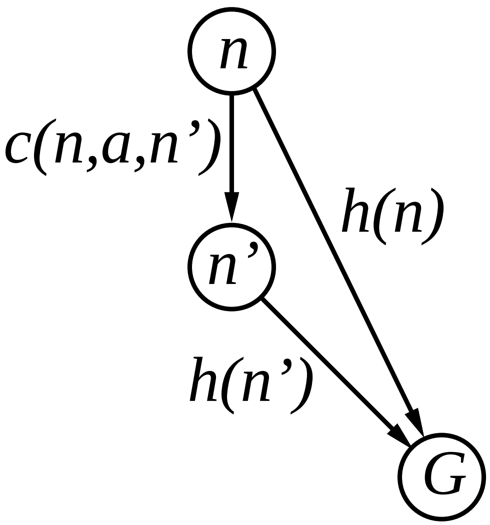
  Either the heuristic needs to be *consistent*,
  or else we need to modify the Graph-Search algorithm

Consistent heuristics

Consistency is analogous to the *triangle inequality* from Euclidian geometry

A heuristic is *consistent* if

$$h(n) \leq c(n, a, n') + h(n')$$



If h is consistent, then for every child n' of n,

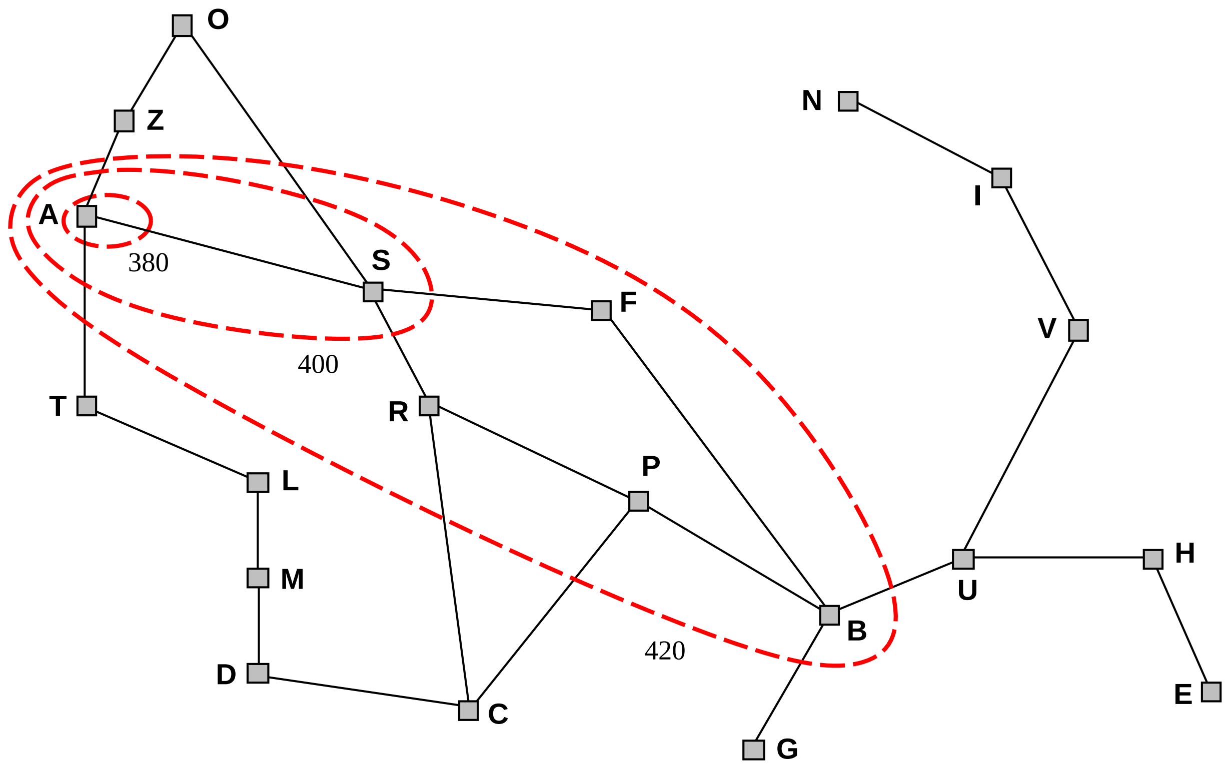
$$\begin{aligned} f(n') &= g(n') + h(n') \\ &= g(n) + c(n, a, n') + h(n') \\ &\geq g(n) + h(n) \\ &= f(n) \end{aligned}$$

I.e., $f(n)$ is nondecreasing along any path.

Behavior of A* with consistent heuristic

If h is consistent, then A* expands nodes in order of increasing f value

Gradually adds "f-contours" of nodes (cf. breadth-first adds layers)
Contour i has all nodes with $f = f_i$, where $f_i < f_{i+1}$



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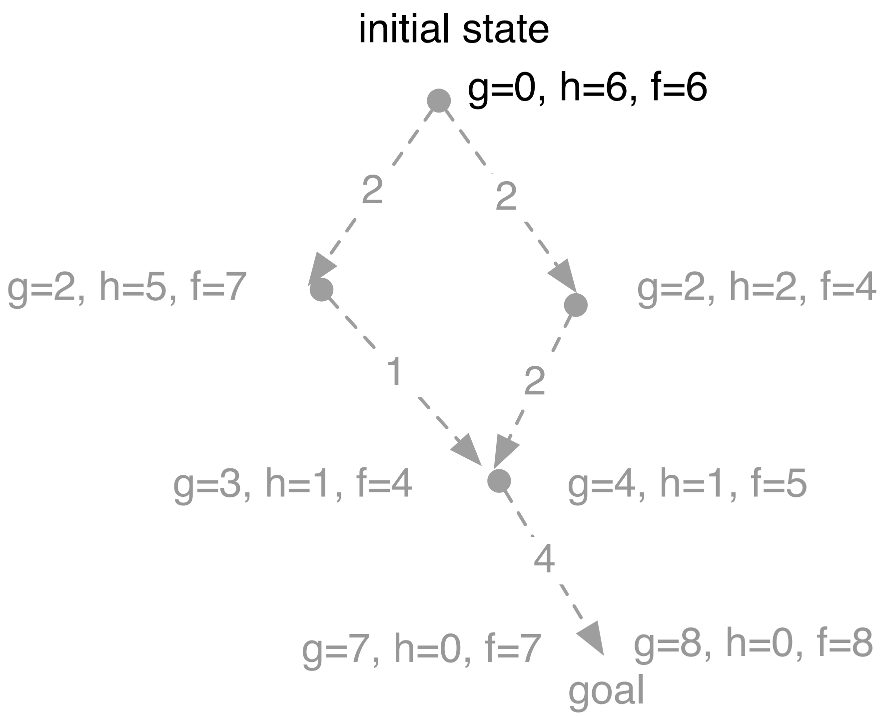
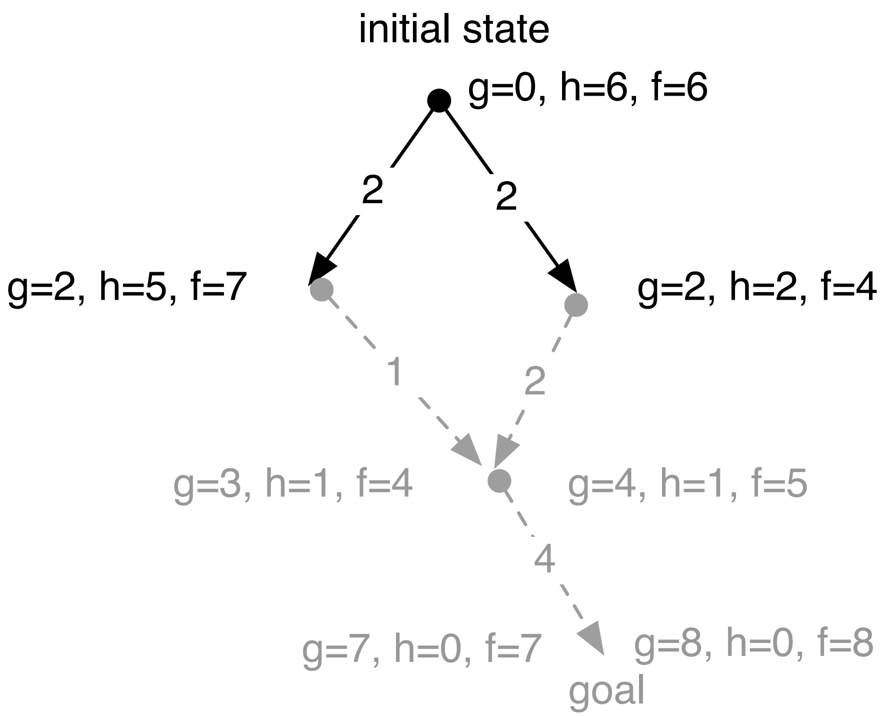


If h is inconsistent, then

- ◇ As we go along a path, f may sometimes decrease
- ◇ A* doesn't always expand nodes in order of increasing f value, because A* may find lower-cost paths to nodes it has already expanded
- ◇ A* will need to re-expand these nodes
- ◇ Problem: Graph-Search won't re-expand them

Behavior of A* with inconsistent heuristic



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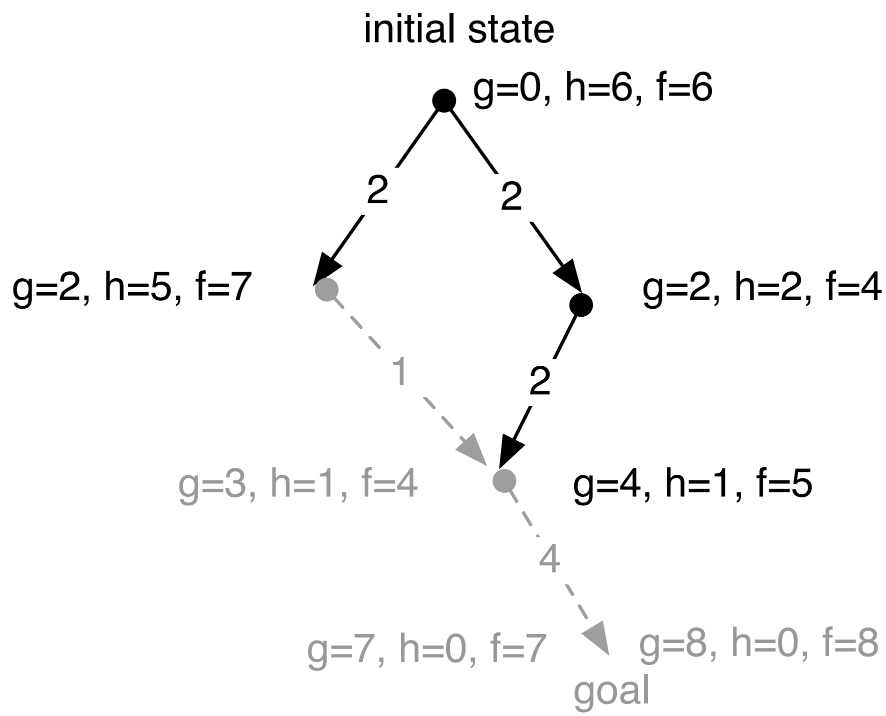

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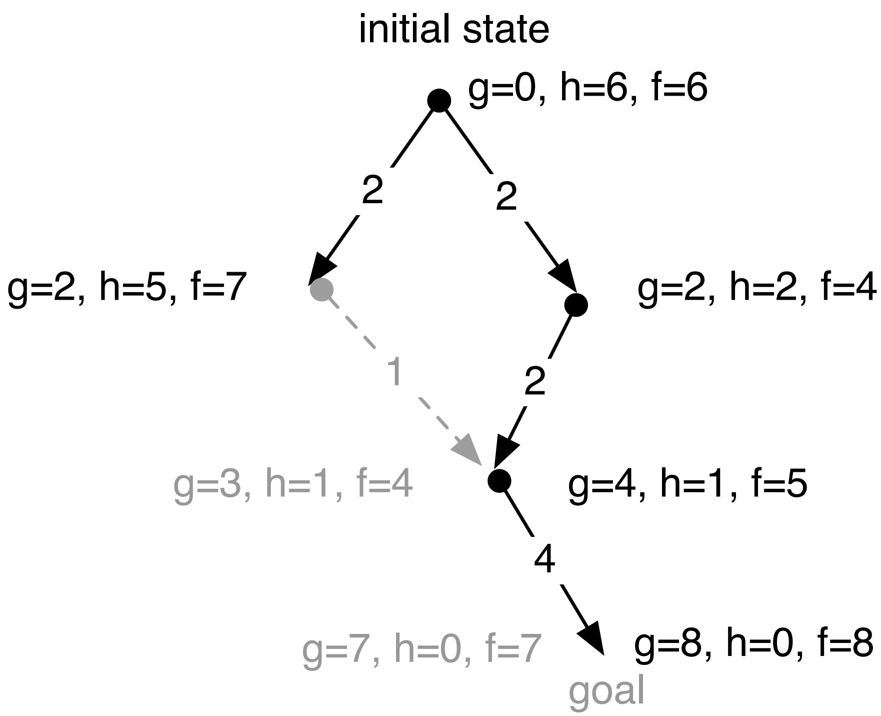

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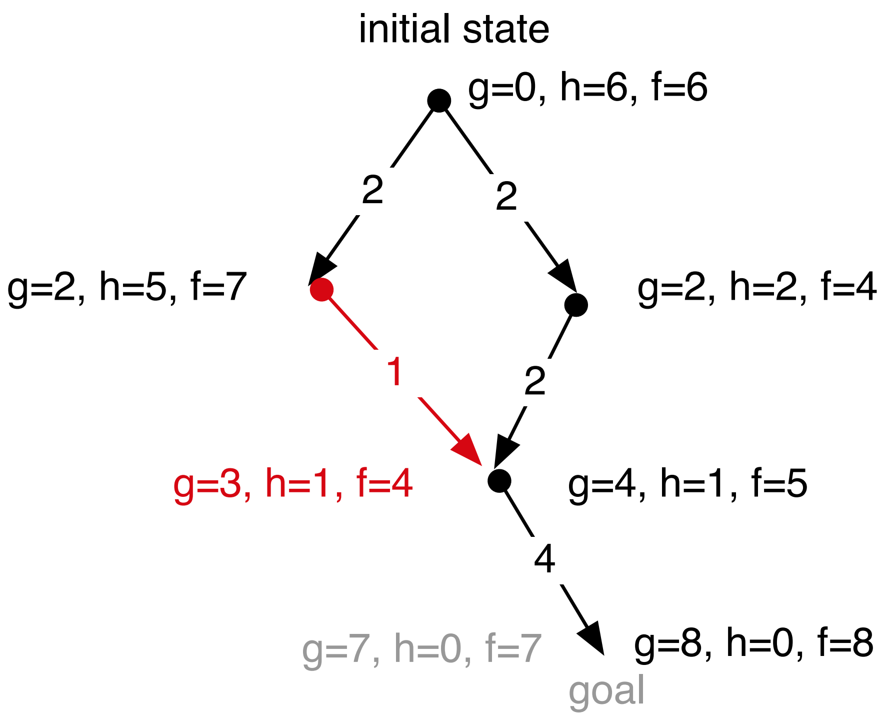

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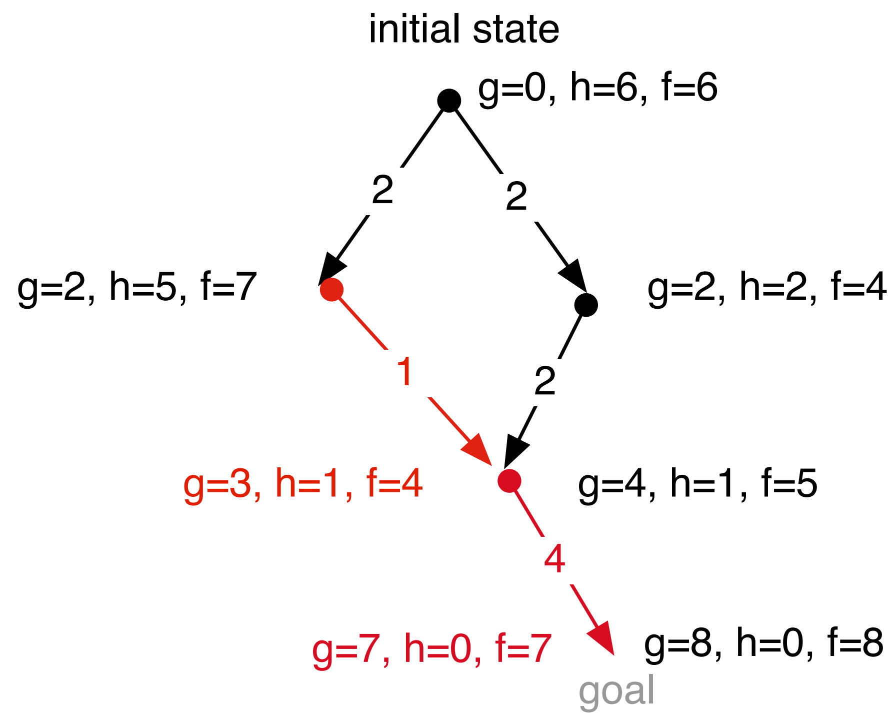

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- ◇ A* will need to re-expand these nodes
- ◇ Problem: GRAPH-SEARCH won't re-expand them

A* for graphs

- Re-expands a node if it find a better path to the node
- Finds optimal solutions even if the heuristic is inconsistent

function A*(*problem*) **returns** a solution, or failure

    closed ← an empty set
    fringe ← a list containing MAKE-NODE(INITIAL-STATE[*problem*])
    loop do
        if *fringe* is empty **then return** failure
        node ← REMOVE-FRONT(*fringe*)
        if GOAL-TEST[*problem*] applied to STATE(*node*) succeeds **return** *node*
        insert *node* into *closed*
        for each node $n \in$ EXPAND(*node*, *problem*) **do**
            if there is a node $m \in$ *closed* $\cup$ *fringe* such that
                STATE(m) = STATE(n) and $f(m) \leq f(n)$
            then do nothing
            else
                insert n into *fringe* after the last node m such that $f(m) \leq f(n)$
    end

Properties of A*

Complete?

Properties of A*

Complete? Yes, unless there are infinitely many nodes with $f \leq f(G)$

Time?

Properties of A*

Complete? Yes, unless there are infinitely many nodes with $f \leq f(G)$

Time? $O(\text{entire state space})$ in worst case, $O(d)$ in best case

Space?

Properties of A*

Complete? Yes, unless there are infinitely many nodes with $f \leq f(G)$

Time? $O($entire state space$)$ in worst case, $O(d)$ in best case

Space? Keeps all nodes in memory

Finds optimal solutions?

Properties of A*

Complete? Yes, unless there are infinitely many nodes with $f \leq f(G)$

Time? $O($entire state space$)$ in worst case, $O(d)$ in best case

Space? Keeps all nodes in memory

Finds optimal solutions? Yes

Additional properties:

A* expands all nodes in $fringe$ that have $f(n) < C^*$
A* expands some nodes with $f(n) = C^*$

If f is *consistent*, A* expands no nodes with $f(n) > C^*$

How to create admissible heuristics

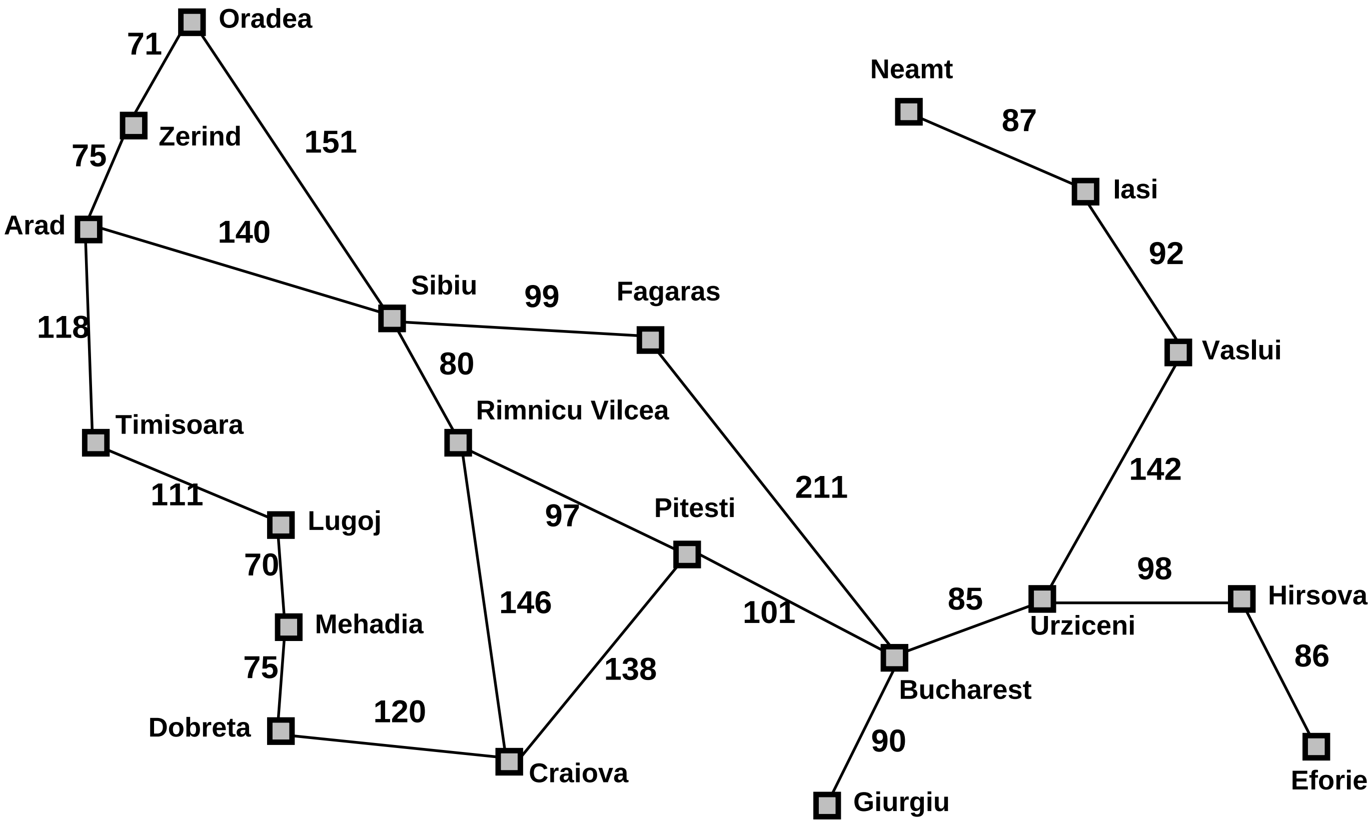
- Suppose P is a problem we're trying to solve
  Let $h^*(s)$ = minimum cost of solution path

- Let P' be a **relaxation** of P
  Remove some constraints on what constitutes a solution

- Every solution path in P is also a solution path in P'
  P' may have additional solution paths that aren't solution paths in P

- Suppose we can find optimal solutions to P' quickly
  Let $h(s)$ = minimum cost of solution path **in** P'
  Then $h(s) \leq h^*(s)$, i.e., h is an admissible heuristic for P

Example: Romania with step costs in km



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$h(\text{at } c) =$ cost of a straight line from city c to Bucharest

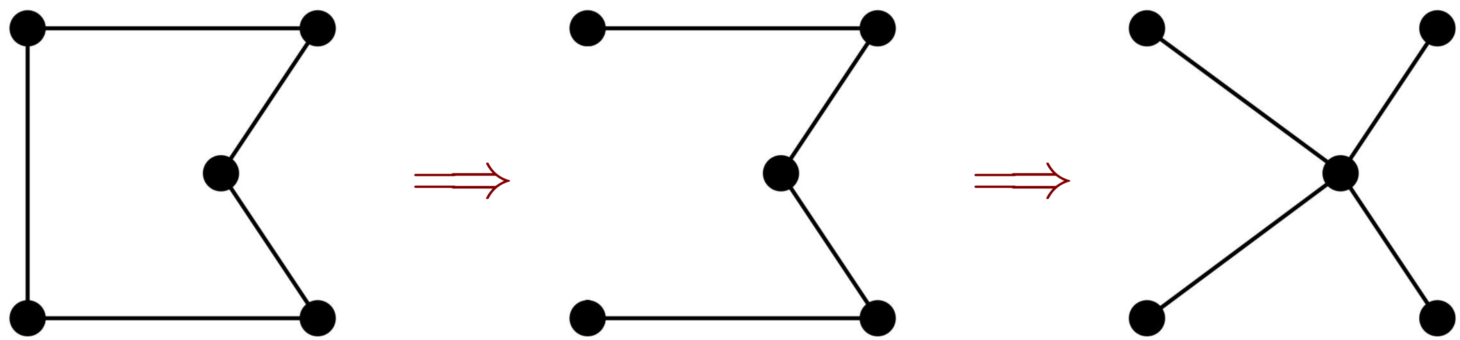
We relaxed the problem to allow paths that are straight lines

Example: TSP

Well-known example: *traveling salesperson problem* (TSP)

♢ Given a *complete* graph (edges between all pairs of nodes)

♢ Find a least-cost *tour* (simple cycle that visits each city exactly once)



Relax the problem twice:

(1) Let {solutions} include paths that visit all cities

(2) Let {solutions} include trees

Minimum spanning tree can be computed in $O(n^2)$

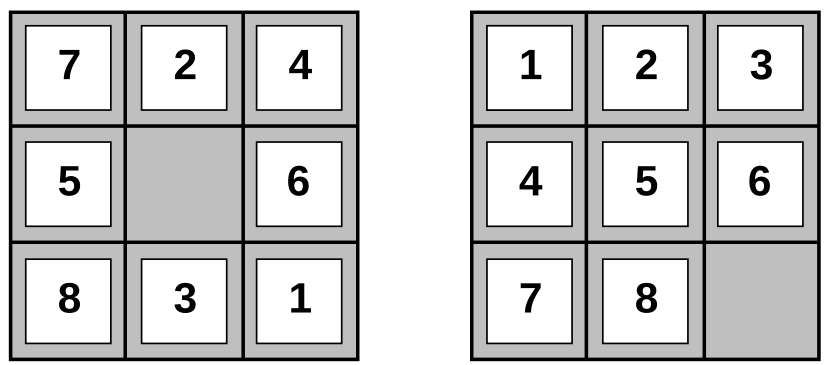
$\Rightarrow$ lower bound on the least-cost path that visits all cities

$\Rightarrow$ lower bound on the least-cost tour

Example: the 8-puzzle

Relaxation 1: allow a tile to move to any other square
regardless of whether the square is adjacent
regardless of whether there's another tile there already

This gives us $h_1(n)$ = number of misplaced tiles



Start State   Goal State

$h_1(S)$ =?

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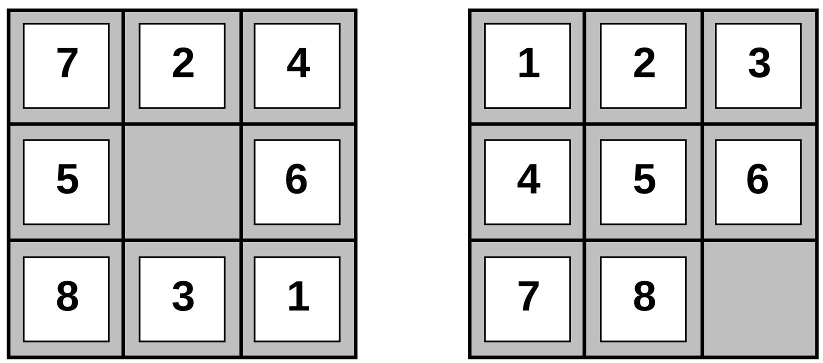


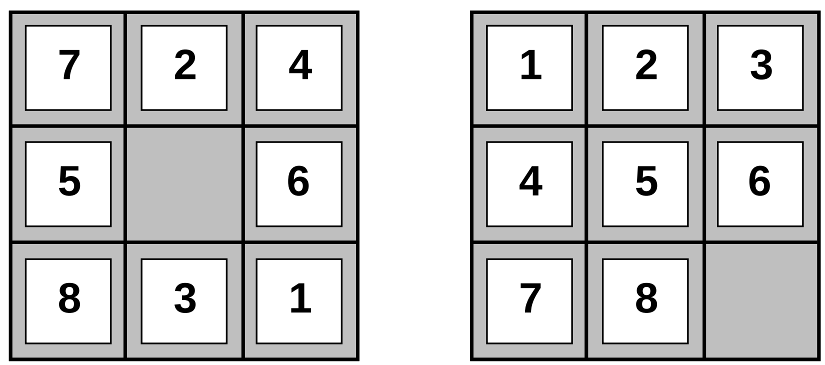
Start State Goal State

$h_1(S)$ =? 6

Example: the 8-puzzle

Relaxation 2: allow a tile to move to any adjacent square, regardless of whether there's another tile there already

This gives us $h_2(n)$ = total *Manhattan* distance



Start State **Goal State**

$h_2(S)$ =?

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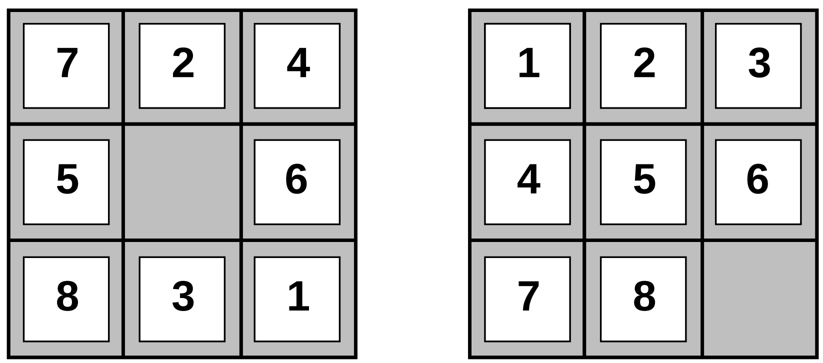

Start State Goal State

$h_2(S)$ =? 4+0+3+3+1+0+2+1 = 14

Dominance

$h_1(n)$ = number of misplaced tiles
$h_2(n)$ = total *Manhattan* distance

Notice that $h_1(n) \leq h_2(n) \leq h^*(n)$ for all n,
i.e., h_2 *dominates* h_1,

h_2's estimate of h^* is never worse than h_1's, and is often better than h_1's

Hence h_2 is better for search. Typical search costs:

$d = 14$ IDS = 3,473,941 nodes
A$^*(h_1)$ = 539 nodes
A$^*(h_2)$ = 113 nodes
$d = 24$ IDS $\approx$ 54,000,000,000 nodes
A$^*(h_1)$ = 39,135 nodes
A$^*(h_2)$ = 1,641 nodes

One way to get dominance

If h_a are h_b admissible heuristic functions, then
$h(n) = \max(h_a(n), h_b(n))$ is admissible and dominates h_a, h_b

Iterative-Deepening A*

```
function IDA*(problem) returns a solution
    inputs: problem, a problem

    f_0 ← h(initial state)
    for i ← 0 to ∞ do
        result ← COST-LIMITED-SEARCH(problem, f_i)
        if result is a solution then return result
        else f_{i+1} ← result
    end

function COST-LIMITED-SEARCH(problem, fmax) returns solution or number

    depth-first search, backtracking at every node n such that f(n) > fmax
    if the search finds a solution then
        return the solution
    else
        return min{f(n) | the search backtracked at n}
```

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Time? Like A* *if* $f(n)$ is an integer and the number of nodes with $f(n) \leq k$ grows exponentially with k

Space?

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Space? $O(bd)$

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Space? $O(bd)$

Optimal? Yes

With consistent heuristic:
IDA* cannot expand f_{i+1} until f_i is finished
IDA* expands all nodes with $f(n) < C^*$
IDA* expands no nodes with $f(n) \geq C^*$

Summary

Heuristic functions estimate costs of shortest paths

Good heuristics can dramatically reduce search cost

Greedy search expands lowest h
– incomplete and not always optimal

A* search expands lowest $g + h$
– complete, returns optimal solutions

IDA* is like a combination of A* and IDS
– complete, returns optimal solutions
– much lower space requirement than A*
– same big-O time *if* number of nodes grows exponentially with cost

Admissible heuristics can be derived from exact solution of relaxed problems