Tug of War Game: An Exposition

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Abstract

This paper proves that there is a winning strategy for Player L in the tug of war game.

1 Introduction

We describe an analyze the continous case of the *Tug of War Game*. This game is a special case of Richman Games[2, 1]. Our version of the game has some difference from Richman Games which we won't go into here.

We would like to thank Dr. William Gasarch for his guidance through the material.

1.1 The Game

1. The board is a line with $2n + 1$ points marked off on it, and the ends are labelled 0 and $2n$. The middle node is labelled n.

   ○−−−−−○−−−−−−○−−−−−−○−−−−−−○−−−−−−○−−−−−−○

   0 n $2n$

   The players are L and R. The game starts with a token on the node n. Node i is the node that is i nodes away from the 0 node. It is Player L's goal to get the token to the 0 node. It is Player R's goal to get the token to the $2n$ node. L starts with d dollars. R starts with d dollars.

2. A turn proceeds as follows:

   L says how much he is willing to pay for a move, say b. b is a rational that is $\leq$ how much L has.

   (a) If R agrees then token is moved towards L node (to the left) BUT R **gets** b **dollars from** L

(b) If R disagrees then he must pay L an amount $> b$ but then gets to move the token one towards R node (to the right).

3. The game ends when the token is either on the L vertex (so L wins) or the R vertex (so R wins). The players play turns until one of them wins. (It could go on forever.)

1.2 The Problem

We will prove that there is a winning strategy for L.

2 Strategy and Analysis

2.1 Phase One: Gain a Small Advantage

L will begin by bidding $\frac{d}{n}$ dollars at every node. An *advantage* is defined as having *more than* $\frac{id}{n}$ dollars at node i.

Lemma 2.1 *If L bids $\frac{d}{n}$ at every node, then either L will win OR L will gain an advantage.*

Proof There are two possible cases:

1. R never chooses to outbid L. L will run out of money just as he reaches the winning node, so L will win OR

2. R outbids L at node $i-1$ by an amount Δ. After this move, L will have *at least*
$$\frac{(i-1)d}{n} + \frac{d}{n} + \Delta = \frac{id}{n} + \Delta$$
dollars and will have gained an advantage Δ. (We think of Δ as being small.)

2.2 Phase Two: Gain a Huge Advantage

L has a *run* when L wins zero or more moves followed by a loss. We want L to have a bigger advantage after the run than he had before the run.

L will bid $\frac{d}{n} + \epsilon_i\Delta$ dollars at each node i. ϵ_i is a number to be determined later that depends on i. We will have $0 \le \epsilon_i \le 1$.

Lemma 2.2 *If the token is at node i and L has $\frac{id}{n} + \Delta$ dollars, then after a run of m moves, the token will be at node $i' = i - m + 1$ and L will have $\frac{i'd}{n} + (1+\alpha)\Delta$ dollars, where $\alpha \ge \frac{1}{2^{2n-2}}$.*

Proof After a run of m moves starting from node i, we want

$$\frac{id}{n} + \Delta - (\frac{md}{n} + (\epsilon_i + \cdots + \epsilon_{i-m+1})\Delta) + \frac{d}{n} + \epsilon_{i-m}\Delta > (i-m+1)\frac{d}{n} + \Delta$$

$$(i-m+1)\frac{d}{n} + (1 - (\epsilon_i + ... + \epsilon_{i-m+1}) + \epsilon_{i-m})\Delta > (i-m+1)\frac{d}{n} + \Delta$$

which reduces to

$$\epsilon_1 > \epsilon_2 + \epsilon_3 + \epsilon_4 + \cdots + \epsilon_{2n-1}$$

$$\epsilon_2 > \epsilon_3 + \epsilon_4 + \cdots + \epsilon_{2n-1}$$

$$\epsilon_3 > \epsilon_4 + \cdots + \epsilon_{2n-1}$$

$$\cdots$$

Since at node 1, L will bet all of his money, then

$$\epsilon_1 = 1.$$

The above constraints are satisfied if we let $\epsilon_i = 1/(2^{i-1})$ for all $0 < i < 2n$. After a run, L will now have gained an advantage of $\epsilon_i\Delta$. Since $i \leq 2n-1$, L is guaranteed to gain an advantage of at least $\frac{1}{2^{2n-2}}\Delta$

Lemma 2.3 *There exists a number r such that after r runs, L will have at least $((2^{2n-2} - 1)(2d))/2^{2n-2}$ dollars.*

Proof After r runs, L will have gained an advantage of at least $r\alpha\Delta$ dollars, where $\alpha = 1/2^{2n-2}$. $r\alpha$ increases linearly with r, so when r reaches a certain number, L will have at least $((2^{2n-2} - 1)(2d))/2^{2n-2}$ dollars.

2.3 Phase Three: Steamroll into Victory

Lemma 2.4 *Once L has $((2^{2n-2} - 1)(2d))/2^{2n-2}$ dollars, he can bid exactly what R has at every turn, winning every move and finally winning the game.*

Proof If L has $((2^i - 1)(2d))/2^i$ at node i, then R has $(2d)/2^i$. L will bid $(2d)/2^i$ and R cannot outbid this. On the next turn, L will have

$$((2^i - 1)(2d))/2^i - (2d)/2^i = 2((2^{i-1} - 1)(2d))/2^i = ((2^{i-1} - 1)(2d))/2^{i-1}$$

dollars. Since $i \leq 2n-1$, L must have $((2^{2n-2} - 1)(2d))/2^{2n-2}$ dollars to guarantee the win from any node i.

3 Variant of Continuous Game in which Right has Δ More Dollars

We have shown that Left has a winning strategy for the variant of the game in which Left and Right have equal amounts of money. This is not surprising; the game is favored towards Left, since bid ties are resolved by giving Left the power to move the token. In an attempt to balance the game, we consider a variation of the game in which ties are decided as before, but Right begins the game with Δ dollars more than Left, where Δ is an arbitrarily small number. This time, Left has the advantage of the tie, but Right has the advantage of having more money. We will show that this version of the game is not fair, and that Right has a winning strategy.

4 Strategy and Analysis

Let $\gamma_k = (1/2)^{2n-k+1}$, for $k = 0, 1, \ldots, 2n$. Right will play the game using the following strategy:

(a) If Left bids $\geq \frac{d}{n} + \gamma_i\Delta$ at node i, where $0 \leq i \leq 2n-1$, Right bids 0.

(b) If Left bids $< \frac{d}{n} + \gamma_i\Delta$ at node i, where $0 \leq i \leq 2n-1$, Right bids $\frac{d}{n} + \gamma_i\Delta$.

The next lemma shows that Right can indeed play the above strategy.

Lemma 4.1 *For all i, each time the token is at position i, Right has enough money to bid $\frac{d}{n} + \gamma_i\Delta$.*

Proof: We can think of Right as having two bank accounts, A and B. The initial $d + \Delta$ dollars are distributed as follows. Bank account A has d dollars, which we think of as consisting of n *units* where a unit is $\frac{d}{n}$ dollars. Bank account B starts with Δ dollars. At a move of type (a), Right adds $\frac{d}{n}$ (i.e., a unit) to A and $\gamma_i\Delta$ to B. At a move of type (b), Right subtracts $\frac{d}{n}$ from A and $\gamma_i\Delta$ from B. We show that for all i, each time the token is at position i, Right has at least $\frac{d}{n}$ in account A and at least $\gamma_i\Delta$ in account B.

Let us first look at account A. We fix position i where $0 \leq i \leq 2n-1$. We have two cases:

Case 1: $i \geq n$ (i.e., the position i is in the right half).

The first time the token moves from i to $i+1$, Right spends for the move out of his initial n units. Let's consider the second time the token moves from i to $i+1$. There must have been an earlier stop when the token moved from $i+1$ to i. At that time, Right gained a unit, which can be spent now. The same logic applies to all subsequent moves.

Case 2: $i < n$ (i.e., the position i is in the left half).

The first time the token is at i, it comes from $i+1$, meaning that a unit has been deposited in A. This unit is available for Right to spend as he moves right from i. The same argument applies to all subsequent times the token is at i.

Let us now consider account B. Again, we fix position i where $0 \leq i \leq 2n-1$. To move from i to $i+1$, Right needs to spend $(\frac{1}{2})^{2n-i+1} \times \Delta$ from account B. Note that $\sum_{i=0}^{2n-1}(\frac{1}{2})^{2n-i+1} \times \Delta = \Delta[(\frac{1}{2})^{2n+1} + \ldots + (\frac{1}{2})^2] < \Delta$. This implies that Right has enough money for the first move from i to $i+1$. For subsequent moves from i to $i+1$, there must have been a previous move from $i+1$ to i when Right deposited $(\frac{1}{2})^{2n-i+1} \times \Delta$ in account B. Therefore, he has this amount to pay for the move.

Next, we show that Right wins the game.

Definition 4.2 *The* advantage *at step t is (the amount of money Left has at step t) minus $\frac{d}{n}\times$ (the position of the token at step t).*

Definition 4.3 *A j-*run *is a sequence of j moves of type* (*b*) *followed by a move of type* (*a*).

Lemma 4.4 *In a j-run, the advantage decreases by $\geq (1/2)^{2n+1}\Delta$.*

Proof: Suppose that before the j-run, the token is at position i and Left has advantage S. At the beginning of the run, Left has $i \times \frac{d}{n} + S$ dollars. Let *advantage* denote the advantage, *money* denote the amount of money Left has, and *position* denote the node where the token is. There are two cases, depending on whether $j > 0$ or $j = 0$.

(*i*) If $j > 0$:

- At $t = 0$, *advantage* $= S$, *money* $= i \times \frac{d}{n} + S$, *position* $= i$.
- At $t = 1$, *advantage* $= S + i \times \frac{d}{n} + (\frac{d}{n} + \gamma_i\Delta) - (i+1) \times \frac{d}{n} = S + \gamma_i\Delta$, *money* $= S + \gamma_i\Delta + (i+1) \times \frac{d}{n}$, *position* $= i+1$.
- At $t = 2$, *advantage* $= S+\gamma_i\Delta+(i+1)\times\frac{d}{n}+(\frac{d}{n}+\gamma_{i+1}\Delta)-(i+2)\times\frac{d}{n} = S+\Delta(\gamma_i+\gamma_{i+1})$, *money* $= S+\Delta(\gamma_i+\gamma_{i+1})+(i+2)\times\frac{d}{n}$, *position* $= i+2$.

$\vdots$

- At $t = j$, $advantage = S + \Delta(\gamma_i + \gamma_{i+1} + \ldots + \gamma_{i+j-1}), money = S + \Delta(\gamma_i + \gamma_i + 1 + \ldots + \gamma_{i+j-1}) + (i+j) \times \frac{d}{n}, position = i + j$.

At $t = j + 1$, Left changes his strategy to (a). Now,

$$\begin{aligned} \text{advantage} &= S + \Delta(\gamma_i + \gamma_{i+1} + \ldots + \gamma_{i+j-1}) + (i+j) \times \tfrac{d}{n} - (\tfrac{d}{n} + \gamma_{i+j}\Delta) - (i+j-1) \times \tfrac{d}{n} \\ &= S + \Delta(\gamma_i + \gamma_i + 1 + \ldots + \gamma_{i+j-1} - \gamma_{i+j}) \\ &= S + \Delta[(\tfrac{1}{2})^{2n-i+1} + (\tfrac{1}{2})^{2n-i} + \ldots + (\tfrac{1}{2})^{2n-i+j+2} - (\tfrac{1}{2})^{2n-i-j+1}] \\ &= \ldots = S - \Delta(1/2)^{2n-i+1}. \end{aligned}$$

So the advantage decreases by $\Delta(1/2)^{2n-i+1} \geq \Delta(1/2)^{2n+1}$.

(ii) j = 0. The run consists of one move of type (a). Then the advantage decreases by $\gamma_i\Delta = (1/2)^{2n-i+1}\Delta \geq (1/2)^{2n+1}\Delta$.

Theorem 4.5 *Right has a winning strategy for this version of the Tug of War game.*

Proof: At the beginning of the game (step $t = 0$), the advantage is $d - n \times \frac{d}{n} = 0$. Lemma 4.4 proves that in one turn of the game, the advantage decreases by at least $(1/2)^{2n+1} \times \Delta$. Eventually, the advantage becomes $\leq -2d$, or Right wins before that. But the advantage is $\leq -2d$ only if Left has 0 money and the position of the token is $2n$, which implies that Right wins.

Remark: The fact that Left bids first does not give Right an advantage. Right can bid $\frac{d}{n} + \gamma_i\Delta$ when the token is at position i regardless of how much Left bids, and the above analysis is still valid.

References

[1] Lazarus, Loeb, Propp, Stromquist, and Ullman. Combinatorial games under auction play. *Games and Economic Behaviour*, 27:229–264, 1999. Available at `http://dept-info.labri.u-bordeaux.fr/~loeb/pull/a.html`.

[2] Lazarus, Loeb, Propp, and Ullman. Richman games. In *Games OF No Chance*, 1994. Available at `http://dept-info.labri.u-bordeaux.fr/~loeb/pull/a.html`.