Comparing Minimax and Product in a Variety of Games

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Abstract

This paper describes comparisons of the minimax back-up rule and the product back-up rule on a wide variety of games, including P-games, G-games, three-hole kalah, othello, and Ballard's incremental game. In three-hole kalah, the product rule plays better than a minimax search to the same depth. This is a remarkable result, since it is the first widely known game in which product has been found to yield better play than minimax. Furthermore, the relative performance of minimax and product is related to a parameter called the rate of heuristic flaw (rhf). Thus, rhf has potential use in predicting when to use a back-up rule other than minimax.

I. Introduction

The discovery of pathological games [Nau, 1980] has sparked interest in the possibility that various alternatives to the minimax back-up rule might be better than minimax. For example, the product rule (originally suggested by Pearl [1981, 1984]), was shown by Nau, Purdom, and Tzeng [1985] to do better than minimax in a class of board splitting games.

Slagle and Dixon [1970] found that a back-up procedure called "M & N" performed significantly better than minimax. However, the M & N rule closely resembles minimax. Until recently, poor performance of minimax relative to back-up rules significantly different from minimax has not been observed in commonly known games such as kalah.

This paper presents the following results:

(1) For a wide variety of games, a parameter called the rate of heuristic flaw appears to be a good predictor of how well minimax performs against the product rule. These games include three-hole kalah, othello, P-games, G-games, and possibly others. This suggests that rhf may serve not only as a guideline for whether it will be worthwhile to consider alternatives to minimax, but also as a way to relate other characteristics of game trees to the performance of minimax and other back-up rules.

(2) In studies of three-hole kalah, the product rule played better than a minimax search to the same search depth. This is the first widely known game in which product has been found to play better than minimax. The product rule still has a major drawback: no tree-pruning algorithm has been developed for it, and no correct pruning algorithm for it can conceivably do as much pruning as the various pruning algorithms that exist for minimax. However, the performance of the product rule in kalah suggests the possibility of exploiting non-minimax back-up rules to achieve better performance in other games.

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II. Definitions

By a *game*, we mean a two person, zero sum, perfect information game having a finite game tree. All of the games studied in this paper satisfy this restriction.

Let G be a game between two players called *max* and *min*. To keep the discussion simple, we assume that G has no ties, but this restriction could easily be removed. If n is a board position in G, let u(.) be the utility function defined as

$$u(n) = \begin{cases} 1 & \text{if n is a forced win node} \\ 0 & \text{if n is a forced loss node.} \end{cases}$$

We consider an evaluation function to be a function from the set of all possible positions in G into the closed interval [0,1]. If e is an evaluation function and n is a node of G, then the higher the value e(n), the better n looks according to e. We assume that every evaluation function produces perfect results on terminal game positions (i.e., e(n) = u(n) for terminal nodes).

If m is a node of G, then the depth d minimax and product values of m are

$$M(m,d) = \begin{cases} e(m) & \text{if depth(m)= d or m is terminal} \\ \min_n M(n) & \text{if min has the move at m} \\ \max_n M(n) & \text{if max has the move at m} \end{cases}$$

$$P(m,d) = \begin{cases} e(m) & \text{if depth(m)=d or m is terminal} \\ \Pi_n M(n) & \text{if min has the move at m} \\ 1 - \Pi_n (1-M(n)) & \text{if max has the move} \end{cases}$$

where n is taken over the set of children of m.

Let m and n be any two nodes chosen at random from a uniform distribution over the nodes at depth d of G. Let $\uparrow_e(m,n)$ (and $\downarrow_e(m,n)$) be whichever of m and n looks better (or worse, respectively) according to e. Thus if $e(m) > e(n)$, then $\uparrow_e(m,n) = m$ and $\downarrow_e(m,n) = n$. If $e(m) = e(n)$, we still assign values to $\uparrow_e(m,n)$ and $\downarrow_e(m,n)$, but the assignment is at random, with the following two possibilities each having probability 0.5:

(1) $\uparrow_e(m,n) = m$ and $\downarrow_e(m,n) = n$

(2) $\uparrow_e(m,n) = n$ and $\downarrow_e(m,n) = m$.

Since e may make errors, exhaustive search of the game tree may reveal that $\uparrow_e(m,n)$ is worse than $\downarrow_e(m,n)$, i.e., that

$$u(\uparrow_e(m,n)) < u(\downarrow_e(m,n)).$$

In this case, a *heuristic flaw* has occurred: the evaluation function has failed to give a correct opinion about m and n. The rate of heuristic flaw at depth d, denoted by rhf(d), is defined to be the quantity

$$\Pr[u(\uparrow_e(m,n)) < u(\downarrow_e(m,n))].$$

III. Theoretical Considerations

A. When Rhf is Low

Consider a minimax search terminating at depth d of a game tree. If rhf(d) is small, it is intuitively apparent that this search should perform quite well. The question is whether it will perform better than some other back-up rule.

For simplicity, assume that the game tree is binary. Assume further that it is max's move at some node c, and let m and n be the children of c. Let d be the depth of m and n. Then

$$\begin{aligned} (1) \quad \Pr[u(c)=1] &= \Pr[u(\uparrow_e(m,n))=1 \text{ or } u(\downarrow_e(m,n))=1] \\ &= \Pr[u(\uparrow_e(m,n))=1] + \Pr[u(\downarrow_e(m,n)) > u(\uparrow_e(m,n))] \\ &\approx \Pr[u(\uparrow_e(m,n))=1] + \text{rhf}(d). \end{aligned}$$

The smallest possible value for rhf(d) is zero. If rhf(d) is close to zero, then from (1) we have

$$\Pr[u(c)=1] \approx \Pr[u(\uparrow_e(m,n))=1],$$

which says that the utility value of c is closely approximated by the utility value of its best child. But according to the minimax rule, the minimax value of c is the minimax value of the best child. This suggests that in this case one might prefer the minimax back-up rule to other back-up rules.

More specifically, consider the extreme case where rhf(d)=0. In this case, whenever m and n are two nodes at depth d of G,

$$\Pr[u(\uparrow_e(m,n)) < u(\downarrow_e(m,n))] = 0.$$

Therefore, since there are only a finite number of nodes at depth d, there is a value $k \in (0,1)$ such that for every node m at depth d,

$$u(m) = 1 \text{ if and only if } e(m) \geq k.$$

By mathematical induction, it follows that forced win nodes will always receive minimax values larger than forced loss nodes, so a player using a minimax search will play perfectly.

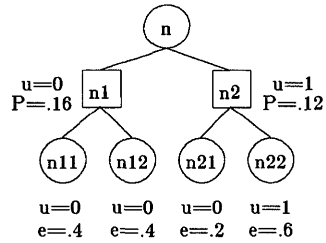

FIGURE 1: A case where product makes the wrong choice.

But if the search is a product rule search rather than a minimax search, then the search will not always result in perfect play. For example, consider the tree shown in Figure 1. By looking at the four leaf nodes, it is evident that rhf=0 with k=0.5. Thus, a minimax search at node n must result in a correct decision. However, a product rule search would result in incorrectly choosing the forced loss node n1. This suggests that when rhf is small, the minimax rule should perform better than the product rule.

B. When Rhf is Large

Let m and n be any two nodes at depth d. In general, rhf can take on any value between 0 and 1. But if e is a reasonable evaluation function, and if $\uparrow_e(m,n)$ is a forced loss, this should make it more likely that $\downarrow_e(m,n)$ is also a forced loss. Thus, we assume that

$$\Pr[u(\downarrow_e(m,n))=1 \mid u(\uparrow_e(m,n))=0] \leq \Pr[u(\downarrow_e(m,n))=1].$$

Thus since u(.) must be either 0 or 1,

$$\begin{aligned} \text{rhf} &= \Pr[u(\downarrow_e(m,n))=1 \ \&\ u(\uparrow_e(m,n))=0] \\ &\leq \Pr[u(\uparrow_e(m,n))=0]\,\Pr[u(\downarrow_e(m,n))=1]. \end{aligned}$$

Suppose rhf is large, i.e.,

$$\text{rhf} \approx \Pr[u(\uparrow_e(m,n))=0]\,\Pr[u(\downarrow_e(m,n))=1].$$

Then from (1),

$$\begin{aligned} \Pr[u(c)=1] \approx\ & \Pr[u(\uparrow_e(m,n))=1] \\ &+ \Pr[u(\uparrow_e(m,n))=0]\,\Pr[u(\downarrow_e(m,n))=1]. \end{aligned}$$

Thus, if $e(\uparrow_e(m,n))$ and $e(\downarrow_e(m,n))$ are good approximations of $\Pr[u(\uparrow_e(m,n))=1]$ and $\Pr[u(\downarrow_e(m,n))=1]$, then

$$\begin{aligned} \Pr[u(c)=1] &\approx e(\uparrow_e(m,n)) + (1 - e(\uparrow_e(m,n)))\, e(\downarrow_e(m,n)) \\ &= 1 - (1 - e(\uparrow_e(m,n)))\,(1 - e(\downarrow_e(m,n))), \end{aligned}$$

which is precisely the formula for the product rule given in Section II. This suggests that when rhf is large, the product rule might be preferable.

IV. Empirical Considerations

The arguments given in Section III suggest that minimax should do better against product when rhf is low than it does when rhf is high. To test this conjecture, we have examined five different classes of games. Space does not permit us to state the rules of each of these games here. However, detailed descriptions of these games may be found in the following references: G–games [Nau, 1983], Ballard's incremental game [Ballard, 1983], othello [Hasagawa, 1977], P–games [Nau, 1982], kalah [Slagle & Dixon, 1969].

A. G–Games

A G–game is a board–splitting game investigated in [Nau, 1983], where two evaluation functions e_1 and e_3 were used to compare the performance of minimax and product. The product rule did better than minimax when e_1 was used, and product did worse than minimax when e_3 was used.

For our purposes, the significance of this study is this: it can be proven that for every depth d, rhf(d) is higher using e_1 than it is using e_3. Thus, on G–games, product performed better against minimax when using the evaluation function having the higher rhf. This matches our conjecture.

B. Ballard's Experiments

Ballard [1983] used a class of incremental games with uniform branching factor to study the behavior of minimax and non–minimax back–up rules. One of the non–minimax back–up rules was a weighted combination of the computational schemes used in the minimax and product rules. Among other results, he claimed that "lowering the accuracy of either max's or min's static evaluations, or both, serves to increase the amount of improvement produced by a non–minimax strategy." Since low accuracy is directly related to a high rhf, this would seem to support our conjecture. But since Ballard did not test the product rule itself, we cannot make a conclusive statement.

C. Othello

Teague [1985] did experiments on the game of othello, using both a "weak evaluation" and a "strong evaluation." The weak evaluation was simply a piece count, while the strong one incorporated more knowledge about the nature of the game. According to Teague's study, minimax performed better than product 82.8% of the time with the strong evaluation, but only 63.1% of the time with the weak evaluation.

It would be difficult to measure the rhf values for othello, because of the immense computational overhead of determining whether or not playing positions in othello are forced wins. However, since rhf is a measure of the probability that an evaluation function assigns forced win nodes higher values than forced loss nodes, it seems clear that the stronger an evaluation function is, the lower its rhf value should be. Thus, Teague's results suggest that our conjecture is true for the game of othello.

D. P–Games

A P–game is a board–splitting game whose game tree is a complete binary tree with random independent assignments of "win" and "loss" to the terminal nodes. P–games have been shown to be pathological when using a rather obvious evaluation function e1 for the games [Nau, 1982]—and in this case, the minimax rule performs more poorly than the product rule [Nau, Purdom, and Tzeng, 1985]. However, pathology in P–games disappears when a stronger evaluation function, e2, is used [Abramson, 1985].

It can be proven that e2 has a lower rhf than e1. Both e1 and e2 return values between 0 and 1, and the only difference between e1 and e2 is that e2 can detect certain kinds of forced wins and forced losses (in which case it returns 1 or 0, respectively).

Let m and n be any two nodes. If $e2(\uparrow_{e2}(m,n)) = 0$, then it must also be that $e2(\downarrow_{e2}(m,n)) = 0$. But it can be shown that $e2(x) = 0$ only if x is a forced loss. Thus $u(\downarrow_{e2}(n, m))=0$, so there is no heuristic flaw. It can also be shown that $e2(x) = 1$ only if x is a forced win. Thus if $e2(\uparrow_{e2}(m,n)) = 1$, then $u(\uparrow_{e2}(m,n))=1$, so there is no heuristic flaw.

Analogous arguments hold for the cases where $e2(\downarrow_{e2}(m,n)) = 0$ or $e2(\downarrow_{e2}(m,n)) = 1$.

The cases described above are the only possible cases where e2 returns a different value from e1. No heuristic flaw occurs for e2 in any of these cases, but heuristic flaws do occur for e1 in many of these cases. Thus, the rhf for e2 is less than the rhf for e1.

TABLE 1: P–game simulation results.

Search depth	% wins for minimax using e1	% wins for minimax using e2
2	51.0%	52.1%
3	52.5%	51.8%
4	49.9%	50.3%
5	50.7%	49.3%
6	46.2%	48.1%
7	46.7%	48.4%
8	44.9%	48.6%
9	47.2%	50.0%

We tested the performance of minimax against the product rule using e1 and e2, in binary P–games of depths 9, 10, and 11, at all possible search depths. For each combination of game depth and search depth, we examined 3200 pairs of games. The study showed that for most (but not all) search depths, minimax performed better against product when the stronger evaluation function was used (for example, Table 1 shows the results for P–games of depth 11). Thus, this result supports our conjecture.

E. Kalah

Slagle and Dixon [1969] states that "Kalah is a moderately complex game, perhaps on a par with checkers." But if a smaller–than–normal kalah playing board is used, the game tree is small enough that one can search all the way to the end of the game tree. This allows one to determine whether a node is a forced win or forced loss. Thus, rhf can be estimated by measuring the number of heuristic flaws that occur in a random sample of games. By playing minimax against product in this same sample of games, information can be gathered about the performance of minimax against product as a function of rhf. To get a smaller–than–normal playing board, we used three–hole kalah (i.e., a playing board with three bottom holes instead of the usual six), with each hole containing at most six stones.

One obvious evaluation function for kalah is the "kalah advantage" used by Slagle and Dixon [1969]. We let e_a be the evaluation function which uses a linear scaling to map the kalah advantage into the interval [0,1].[1] If P(m,2) is computed using $e_a(m)$, the resulting value is generally more accurate than $e_a(m)$. Thus, weighted averages of $e_a(m)$ and P(m,2) can be used to get evaluation functions with different rhf values:

$$e_a^w(m) = w\ e_a(m) + (1-w)\ P(m,2),$$

for w between 0 and 1.

We measured rhf(4), and played minimax against product with a search depth of 2, using the following values for w: 0, 0.5, 0.95, and 1. This was done using 1000 randomly generated initial game boards for three–hole kalah. For each game board and each value of w, two games were played, giving each player a chance to start first. The results are summarized in Table 2.

TABLE 2: kalah simulation results

w	rhf(4)	% games won by product	% games won by minimax
1	0.135	63.4%	36.6%
0.95	0.1115	55.5%	44.5%
0	0.08	53.6%	46.4%
0.5	0.0765	51.2%	48.8%

Note that the lowest rhf was obtained with w = 0.5. This suggests that a judicious combination of direct evaluation with tree search might do better than either individually. This idea needs to be investigated more fully.

[1] A preliminary study of rhf [Chi & Nau, 1986] compared minimax to the product rule using e_a in three different variants of kalah. This study, which used a somewhat different definition of rhf than the one used here, motivated the more extensive studies reported in the current paper.

Note also that product performs better than minimax with all four evaluation functions.[2] This suggests that product might be of practical value in kalah and other games. Also, the performance of product against minimax increases as rhf increases. This matches our conjecture about the relation between rhf and the performance of minimax and product.

V. P–Games With Varying RHF

Section IV shows that in a variety of games, minimax performs better against product when rhf is low than when rhf is high. To investigate the specific relationship between rhf and performance of minimax versus product, we did a Monte Carlo study of the performance of minimax against product on binary P–games, using an evaluation function whose rhf could be varied easily. For each node n, let r(n) be a random value, uniformly distributed over the interval [0,1]. The evaluation function e^w is a weighted average of u and r:

$$e^w(n) = w\ u(n) + (1-w)\ r(n).$$

When the weight w = 0, e^w is a completely random evaluation. When w = 1, e^w provides perfect evaluations. For $0 \le w \le 0.5$, the relationship between w and rhf is approximately linear. For $w \ge 0.5$, rhf = 0 (i.e., e^w gives perfect performance with the minimax back–up rule).

In the Monte Carlo study, 8000 randomly generated initial game boards were used, and w was varied between 0 and 0.5 in steps of 0.01. For each initial board and each value of w, two games were played: one with minimax starting first, and one with product starting first. Both players were searching to depth 2. Figure 2 graphs the fraction of games won by minimax against product, as a function of rhf. Figure 2 shows that minimax does significantly better than product when rhf is small, and product does significantly better than minimax when rhf is large.[3] Thus, in a general sense, Figure 2 supports our conjecture about rhf. But Figure 2 also demonstrates that the relationship between rhf and the performance of minimax against product is not always monotone, and may be rather complex.

[2] Table 2 shows results only for search depth 2. We examined depth 2 to 7 and product rule played better than minimax in all of them except with less statistical significance for depth 3 and 6.

[3] Furthermore, the poor performance of minimax when rhf is large corroborates previous studies which showed that product did better than minimax in P–games using a different evaluation function [Nau, Purdom, and Tzeng, 1985].

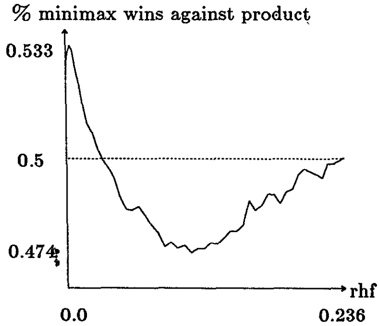

FIGURE 2: Performance of minimax against product using e W as rhf varies.

VI. Conclusions and Speculations

The results presented in this paper are summarized below:

(1) Theoretical considerations suggest that for evaluation functions with low rhf values, minimax should perform better against product than it does when rhf is high. Our investigations on a variety of games confirm this conjecture.

(2) In the game of kalah with three bottom holes, the product rule plays better than a minimax search to the same search depth. This is the first widely known game in which product has been found to yield better play than minimax.

Previous investigations have proposed two hypotheses for why minimax might perform better in some games than in others: dependence/independence of siblings [Nau, 1982] and detection/non-detection of traps [Pearl, 1984]. Since sibling dependence generally makes rhf lower and early trap detection always makes rhf lower, these two hypotheses are more closely related than has previously been realized.

One could argue that for most real games it may be computationally intractable to measure rhf, since one would have to search the entire game tree. But since rhf is closely related to the strength of an evaluation function, one can generally make intuitive comparisons of rhf for various evaluation functions without searching the entire game tree. This becomes evident upon examination of the various evaluation functions discussed earlier in this paper.

There are several problems with the definition and use of rhf. Since it is a single number, rhf is not necessarily an adequate representation for the behavior we are trying to study. Furthermore, since the definition of rhf is tailored to the properties of minimax, it is not necessarily the best predictor of the performance of the product rule. Thus, the relationship between rhf and the performance of minimax versus product can be rather complex (as was shown in Section V). Further study might lead to better ways of predicting the performance of minimax, product, and other back-up rules.

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