Running Probabilistic Programs Backwards

Neil Toronto * Jay McCarthy † David Van Horn *

* University of Maryland † Vassar College

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Roadmap

- Probabilistic inference, and why it's hard

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(let ([x  (flip 0.5)]
      [y  (flip 0.5)])
  (cons x y))
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(let* ([x  (flip 0.5)]
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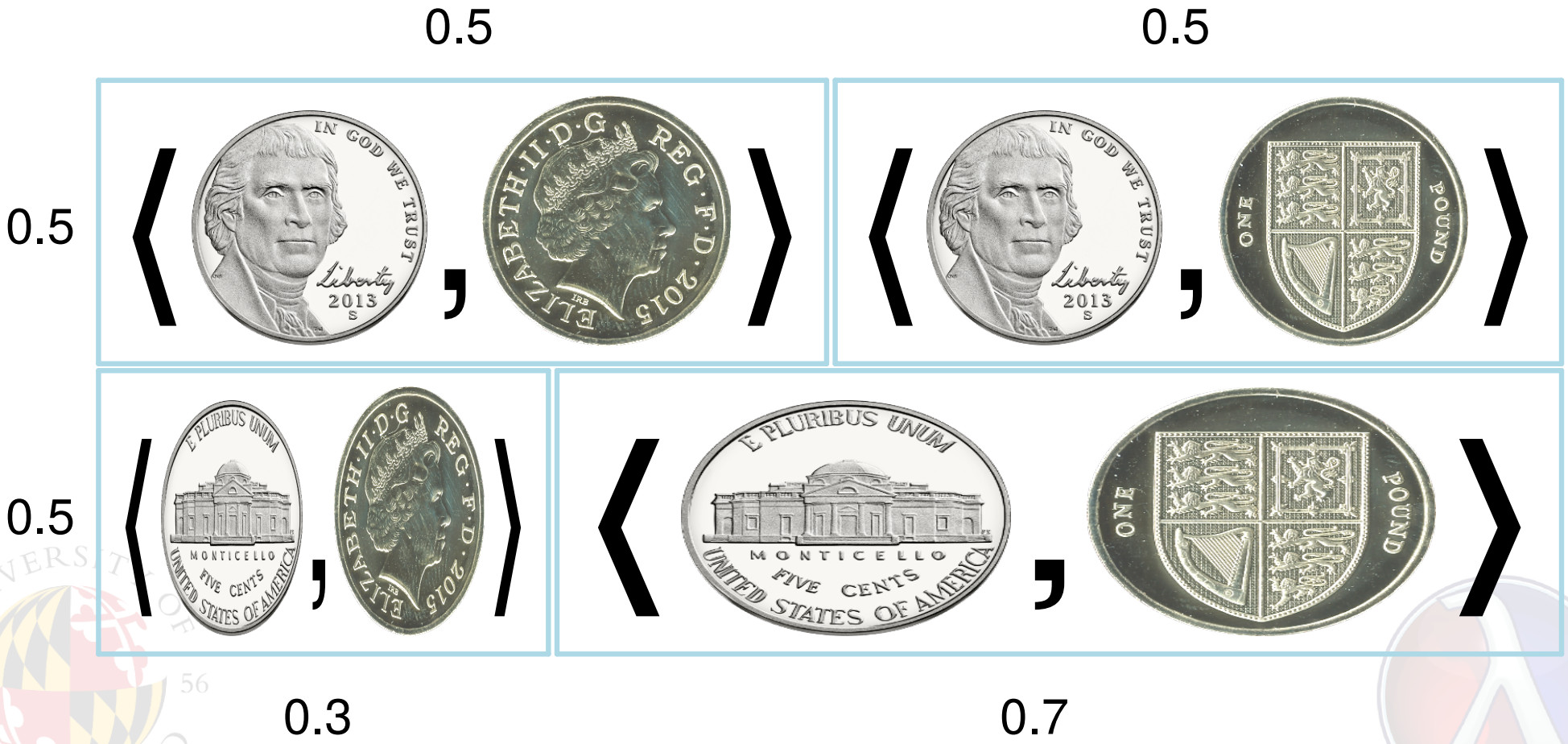
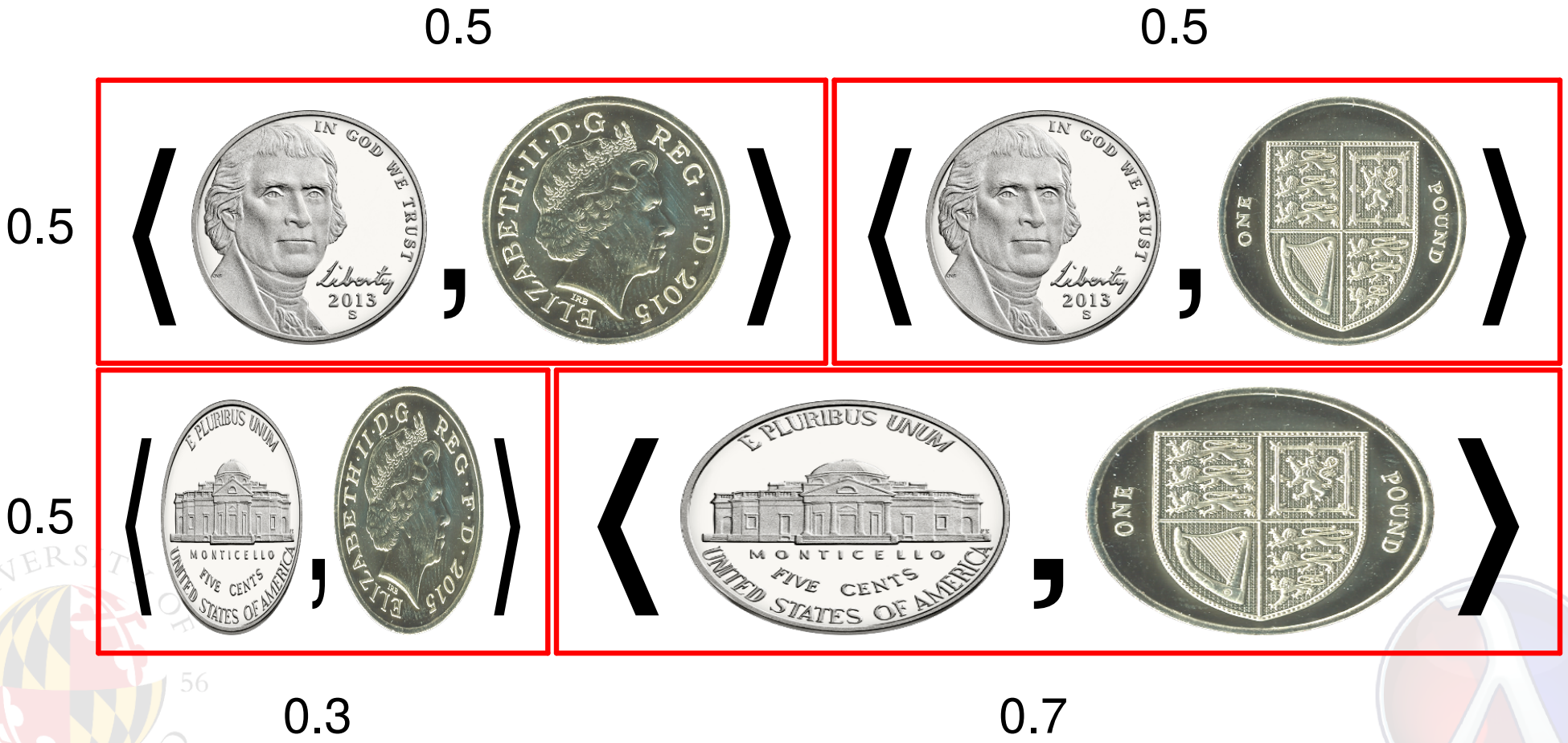


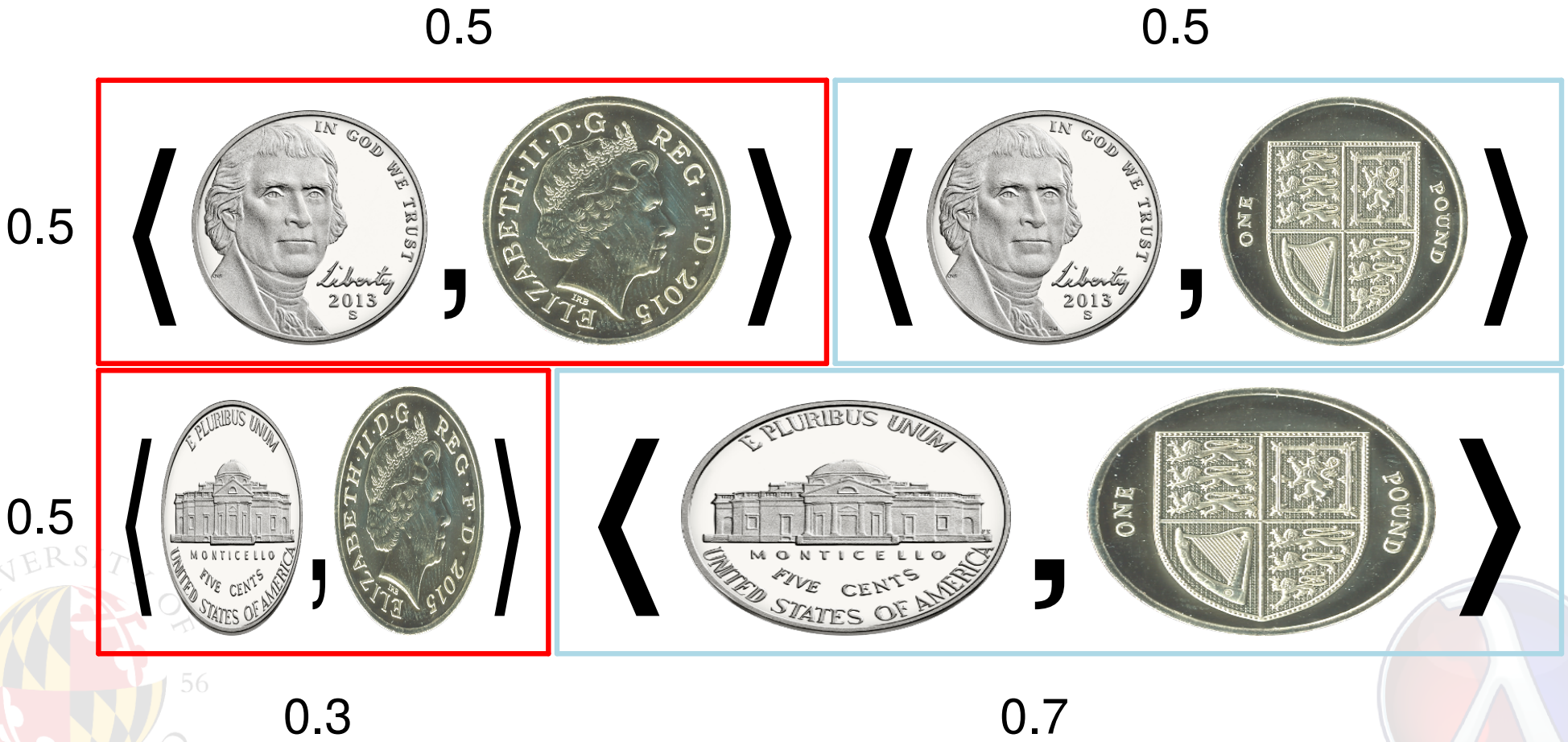
Programming Coin Flips

$$\Pr[\texttt{true}] = 0.5 \cdot 0.5 + 0.5 \cdot 0.5 + 0.5 \cdot 0.3 + 0.5 \cdot 0.7 = 1$$



Programming Coin Flips

$$\Pr[\mathrm{y} = \mathrm{heads}] = 0.5 \cdot 0.5 + 0.5 \cdot 0.3 = 0.4$$



Programming Coin Flips

$$\Pr[\mathrm{x} = \mathrm{heads} \mid \mathrm{y} = \mathrm{heads}]$$
$$= \Pr[\langle \mathrm{x}, \mathrm{y} \rangle = \langle \mathrm{heads}, \mathrm{heads} \rangle] / \Pr[\mathrm{y} = \mathrm{heads}]$$
$$= 0.25/0.4 = 0.625$$

0.5

0.5

0.5

0.3

Stochastic Ray Tracing

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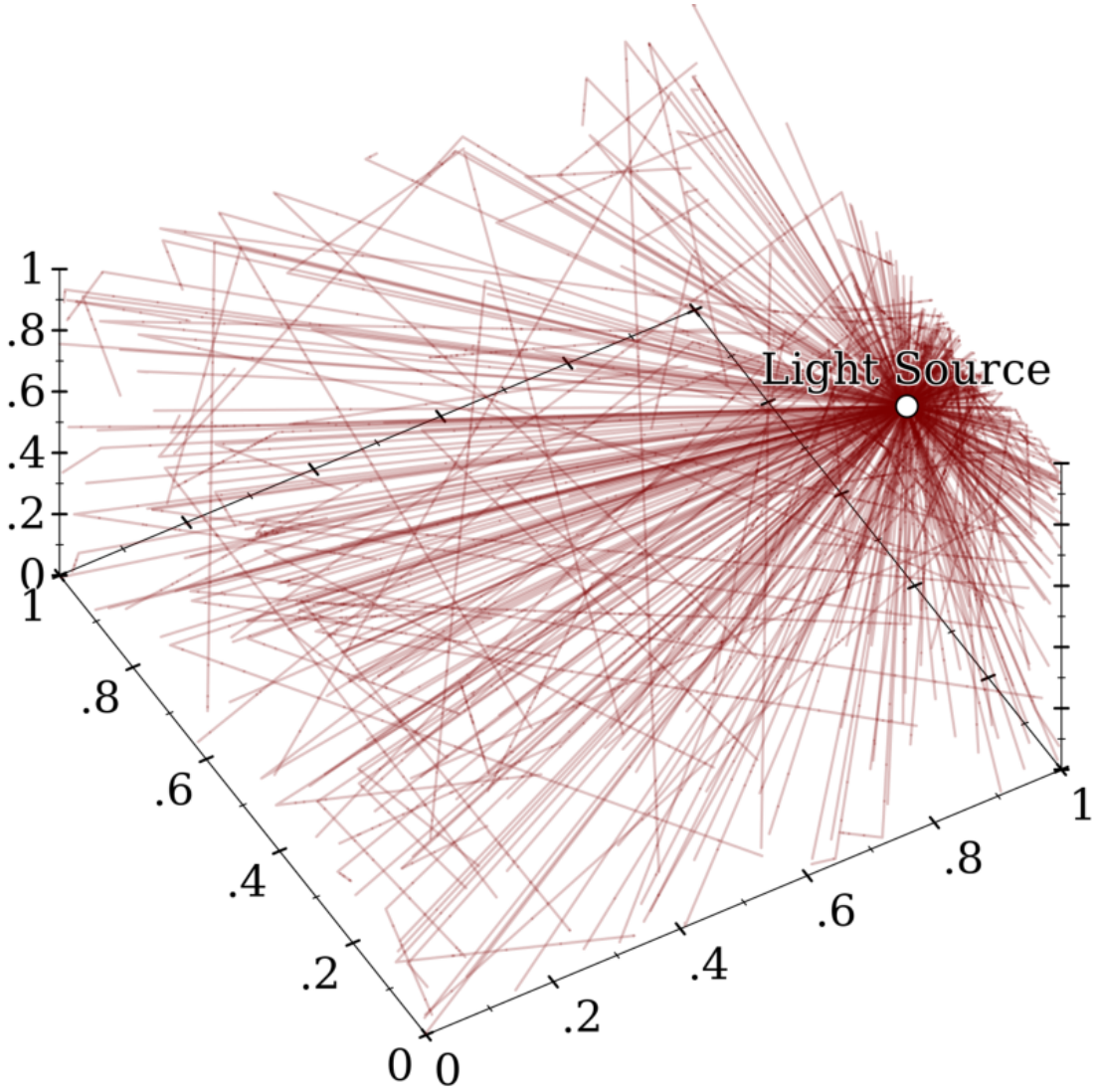

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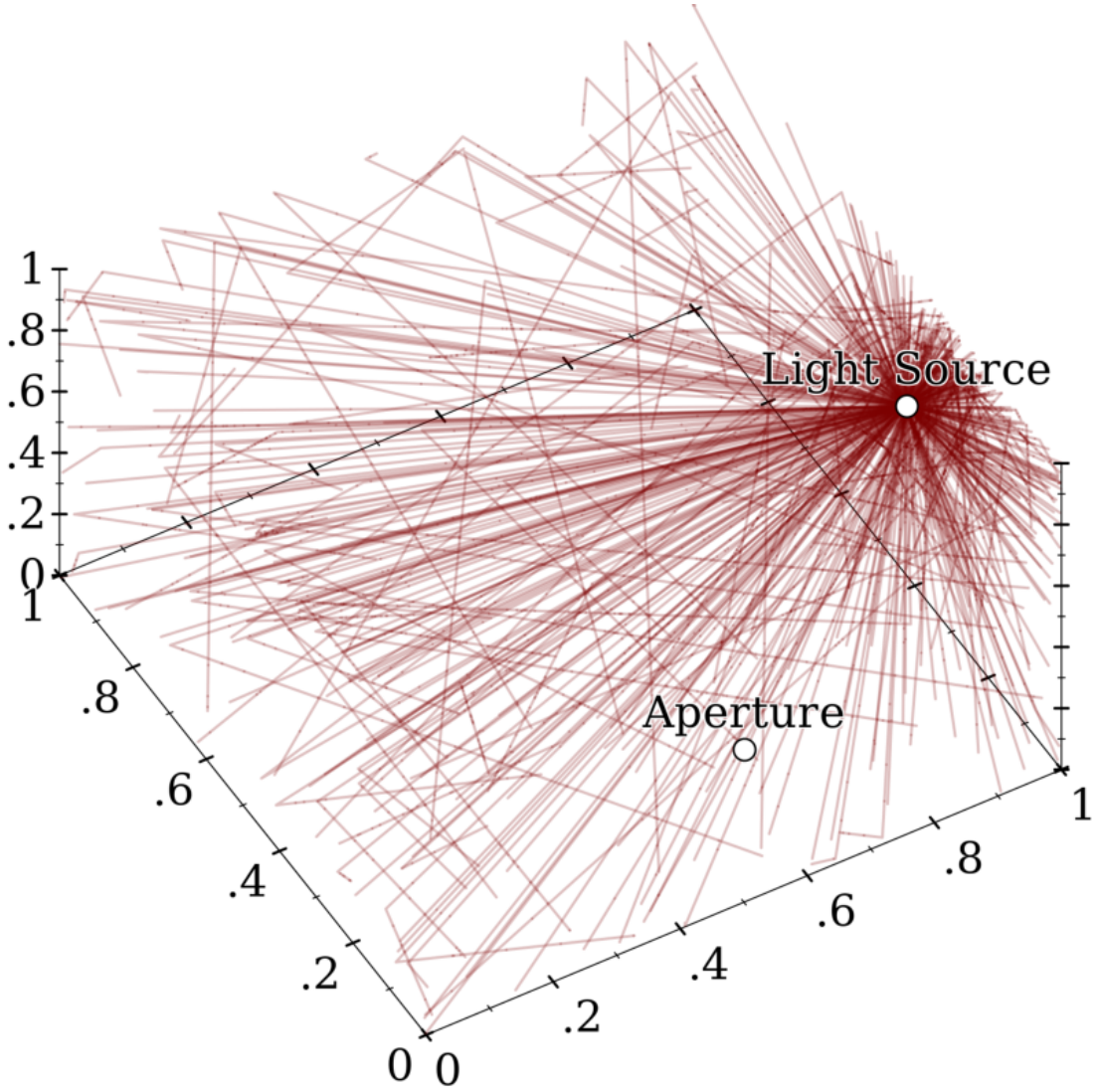

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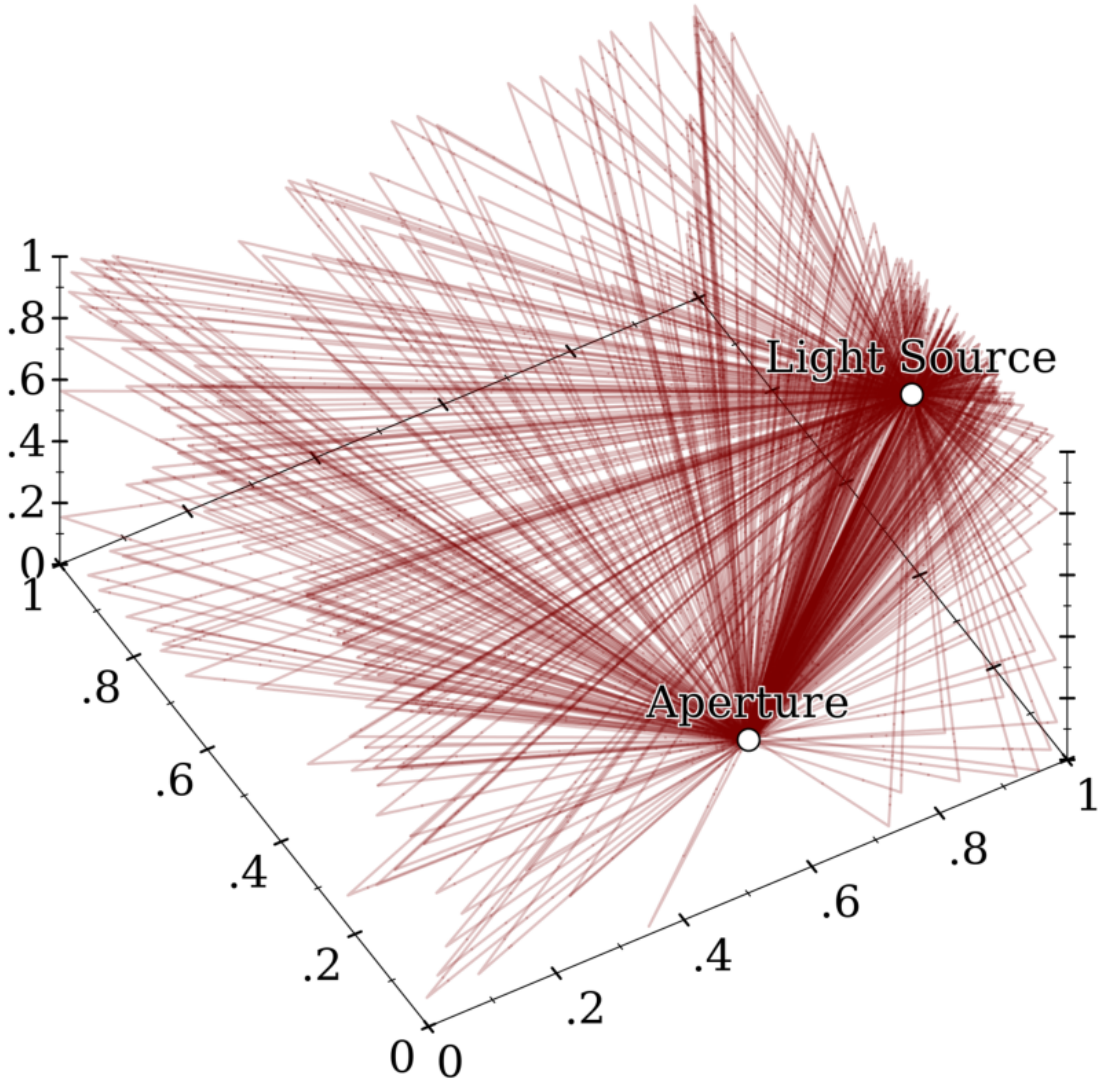
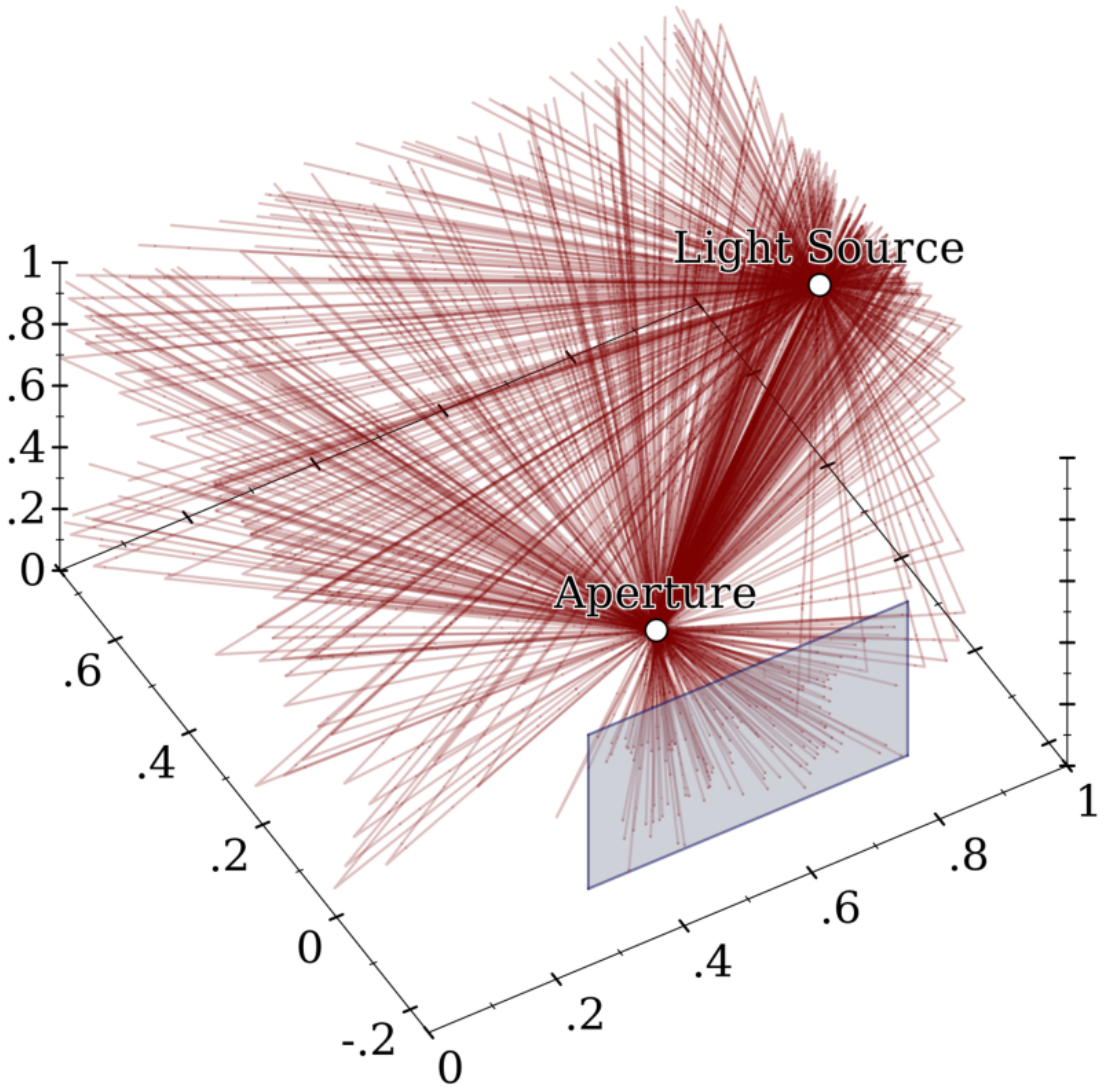


Stochastic Ray Tracing

Simulate projecting rays onto a sensor...



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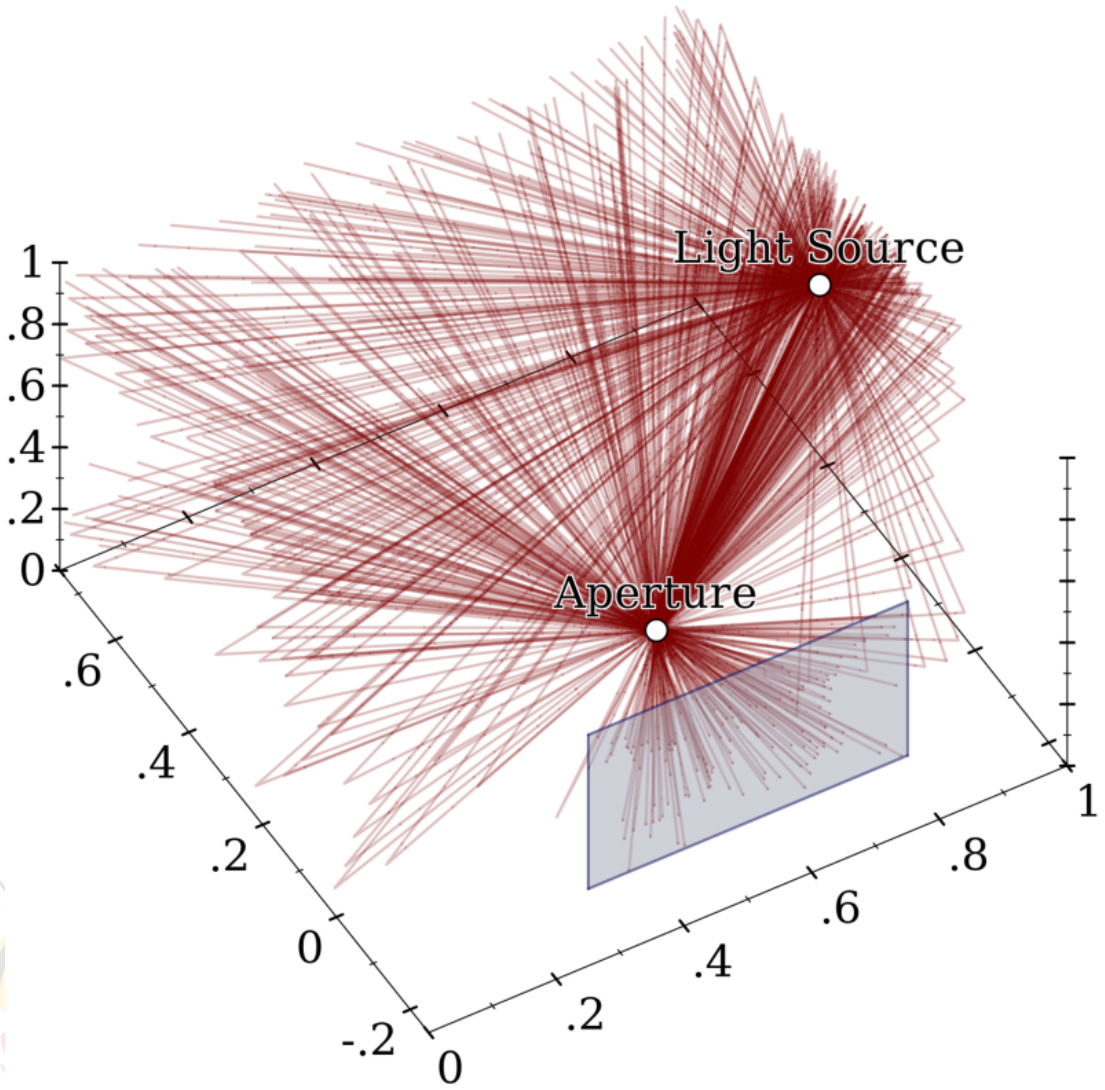
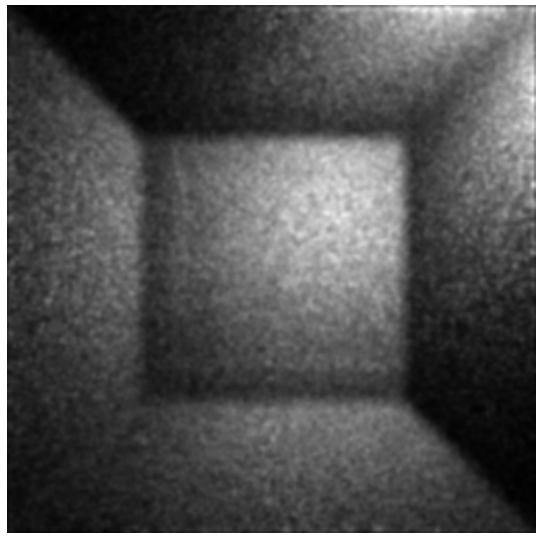
... and collect them to form an image





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(define/drbayes (ray-plane-intersect p0 v n d)
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- The issue is one of theory, not engineering effort

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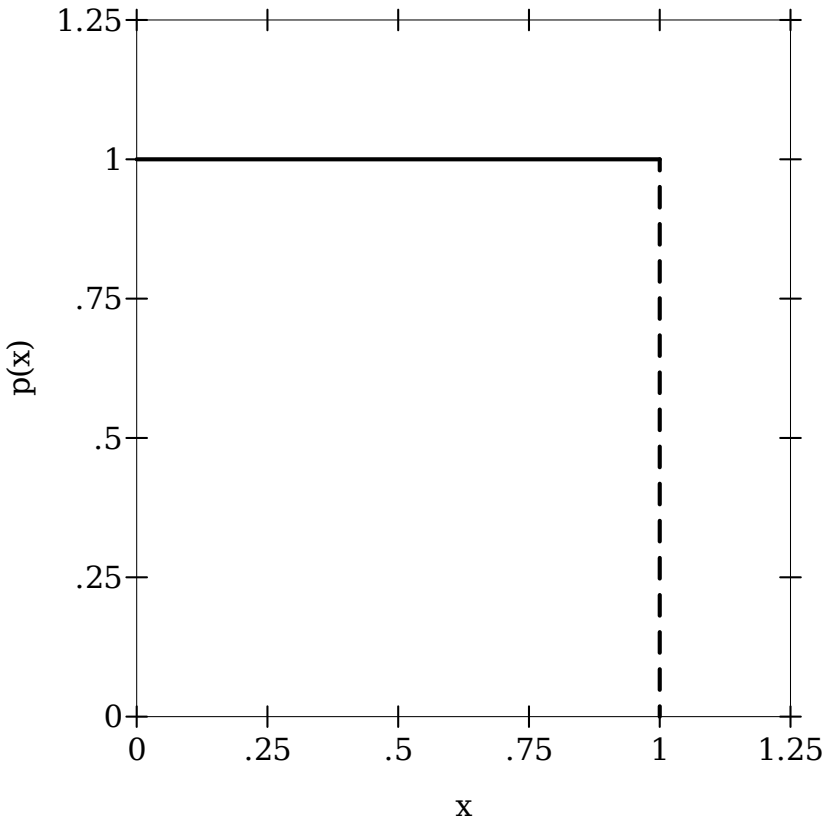
Density function p for value of **`(random)`**:



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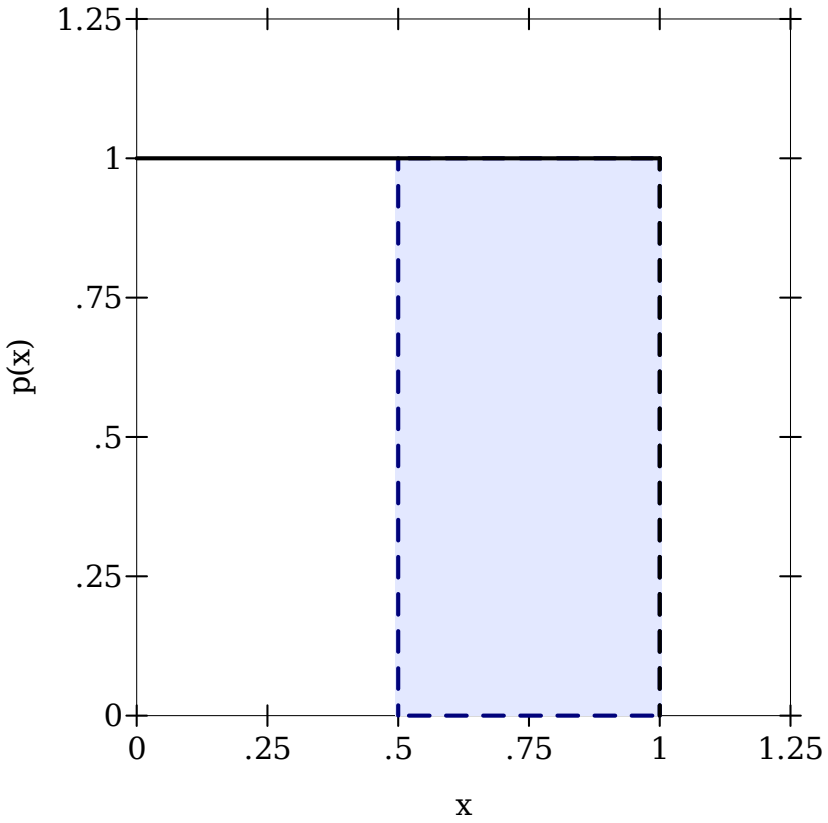

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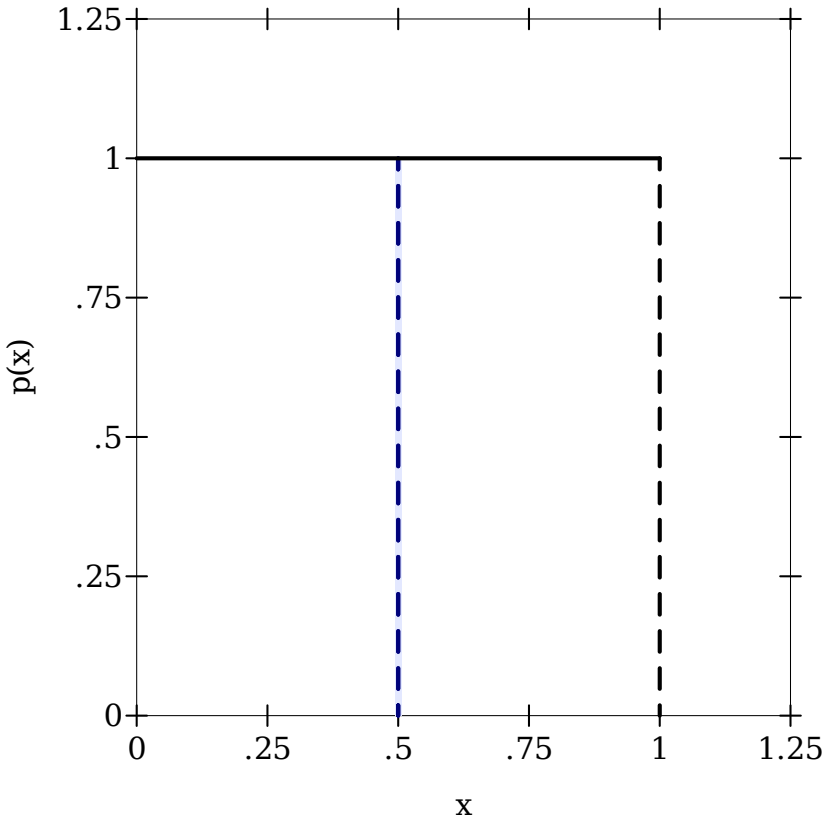

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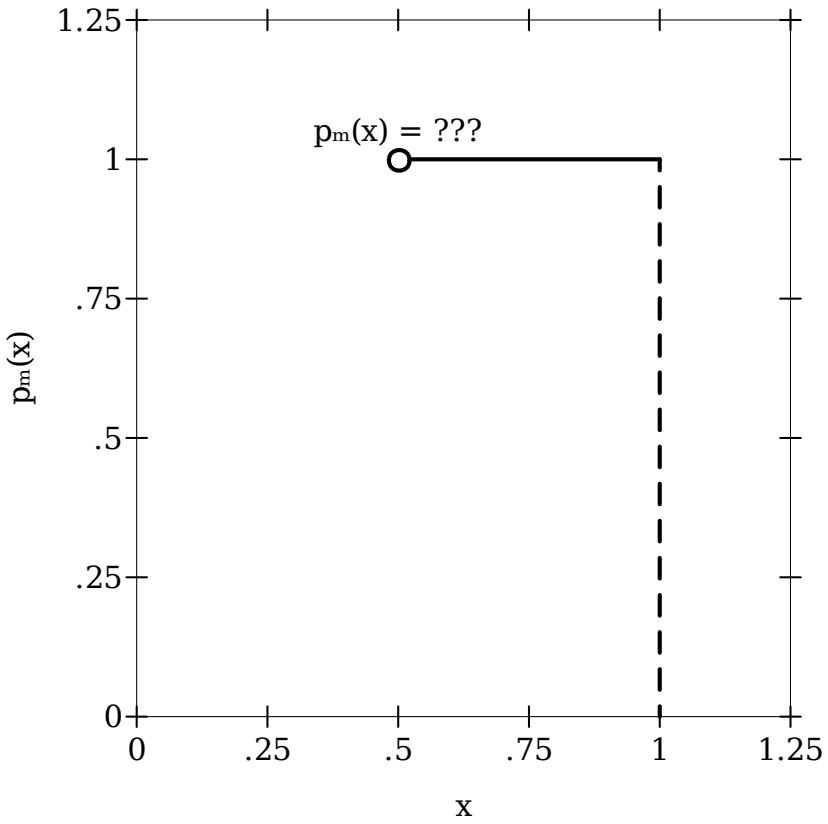

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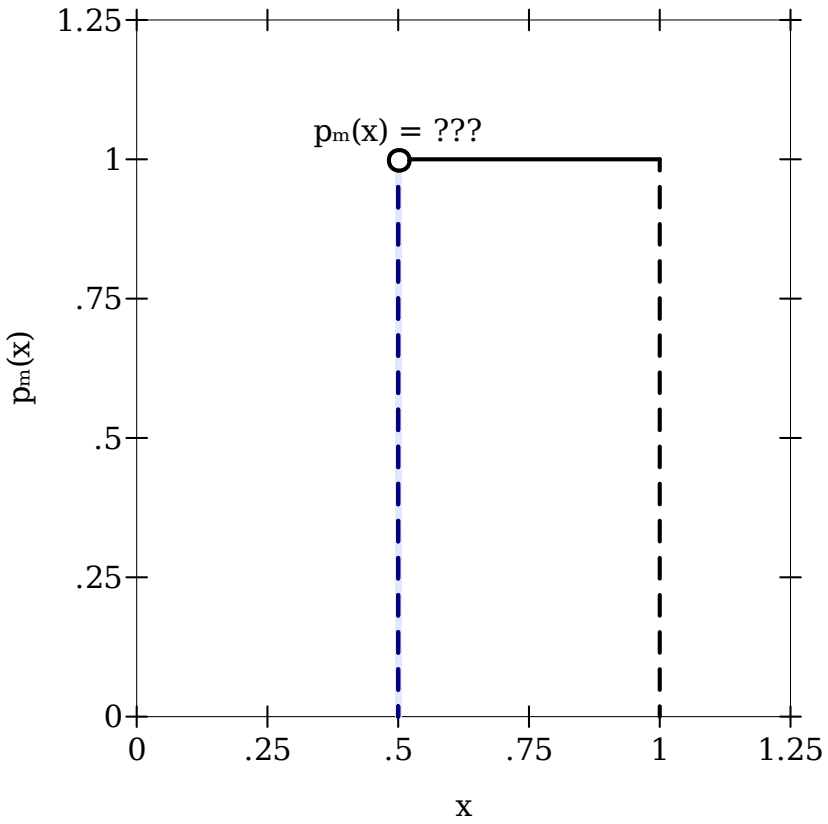

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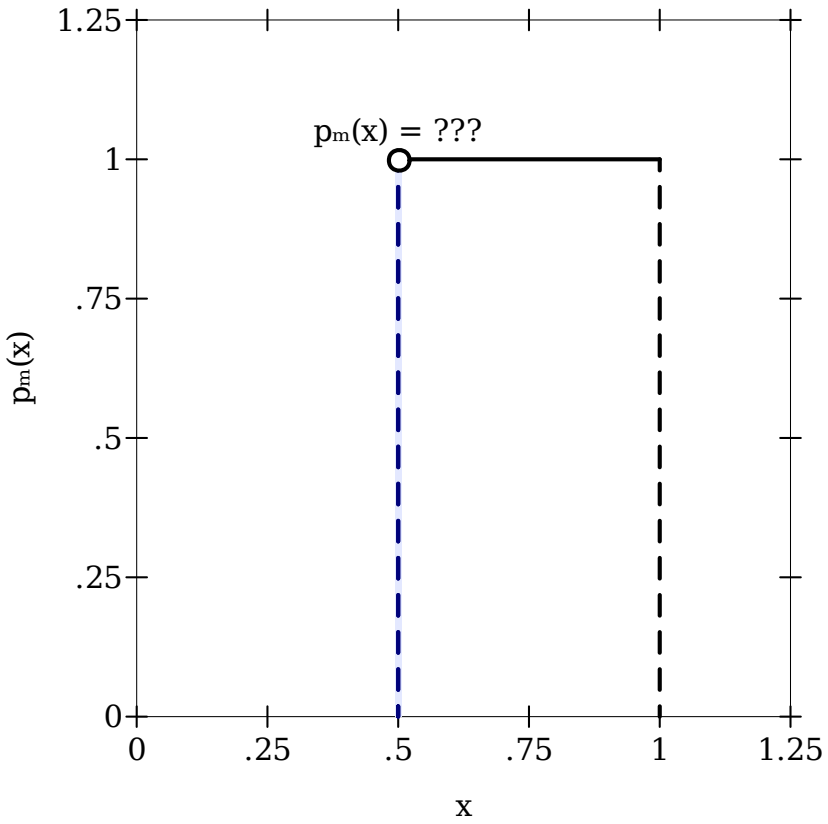

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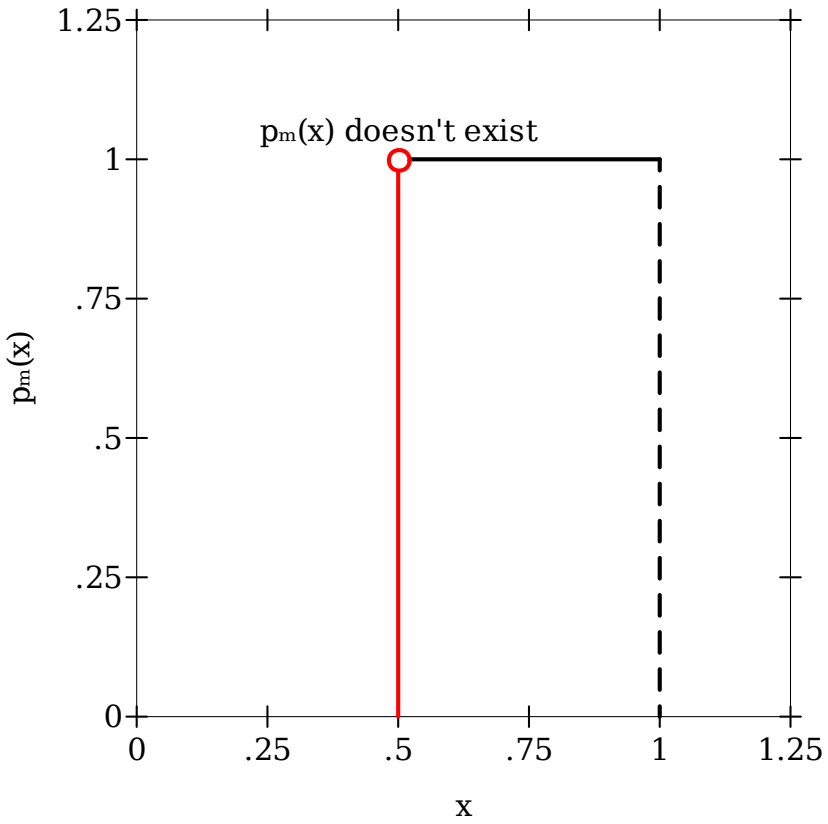

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- Infinite-dimensional things (recursion)
- In general: the distributions of program values

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- Need a way to *derive* measures from code

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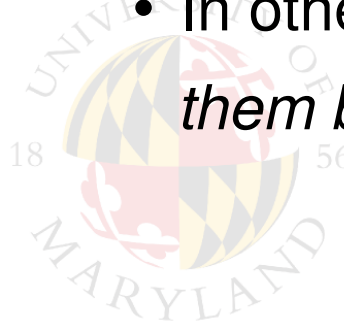
- In other words, compute measures of expressions by *running them backwards*



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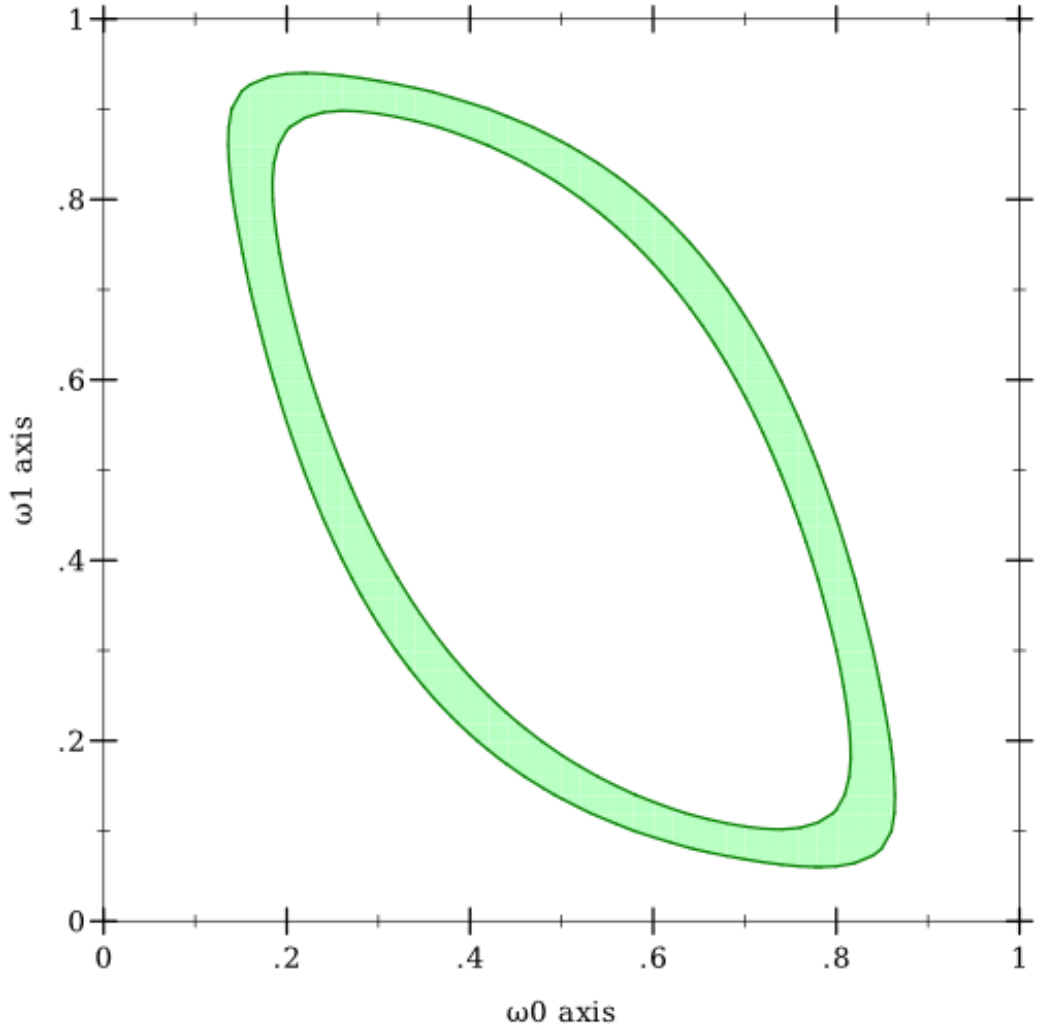
What About Approximating?

Conservative approximation with rectangles:



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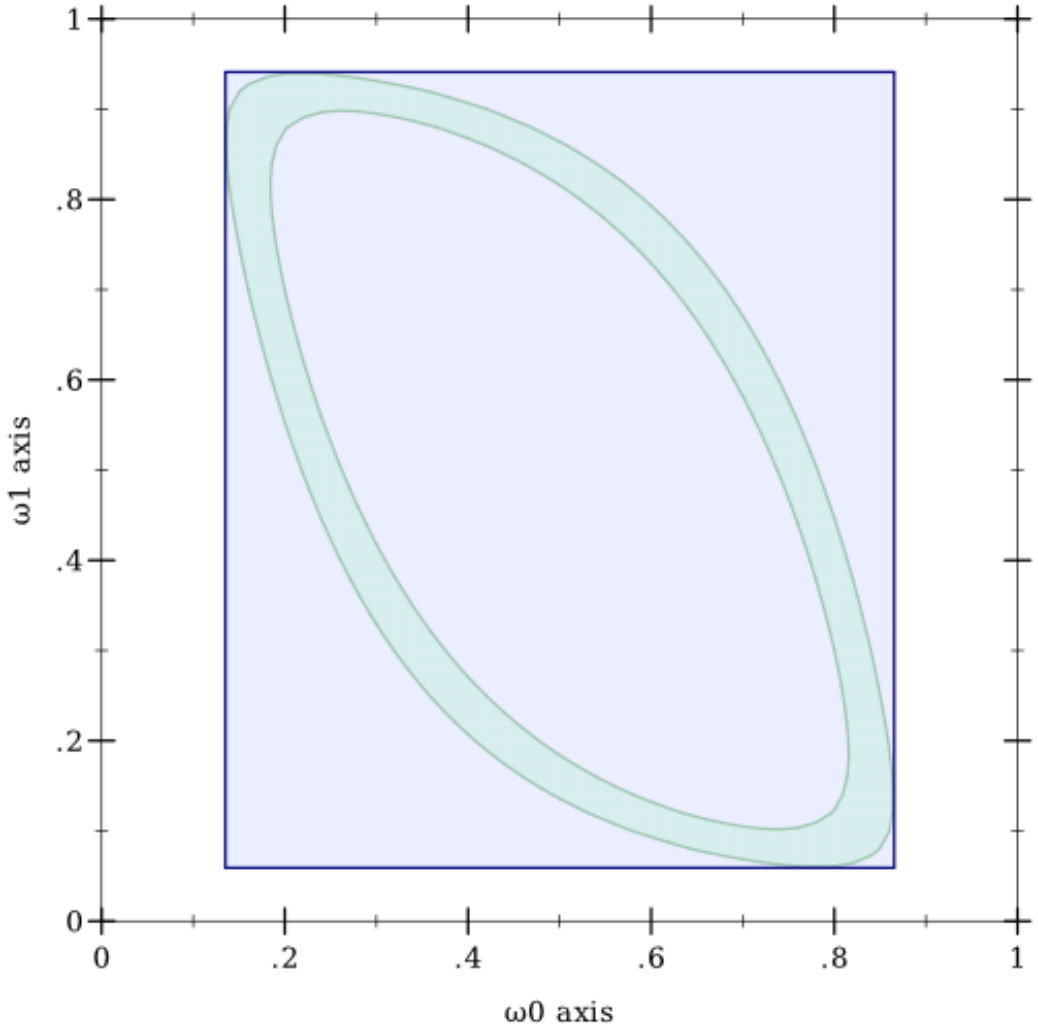

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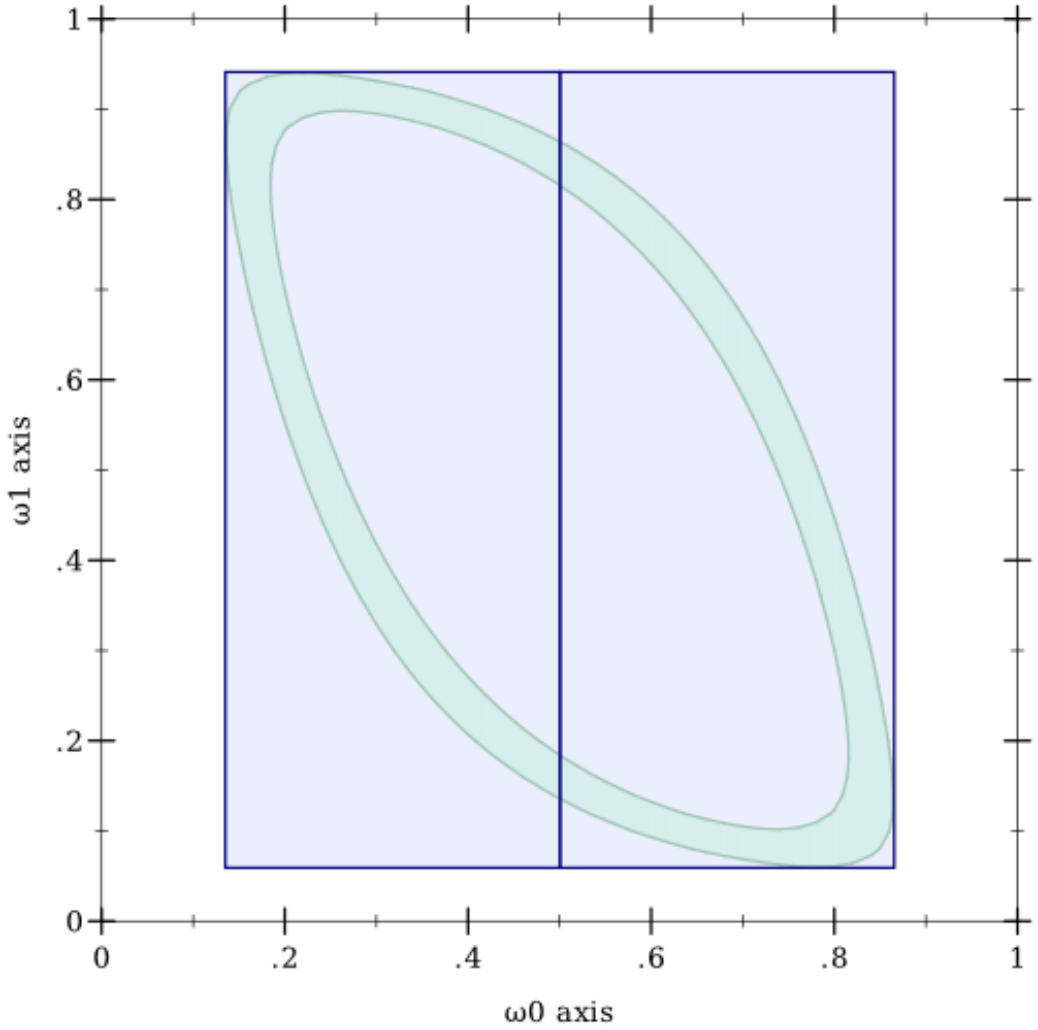

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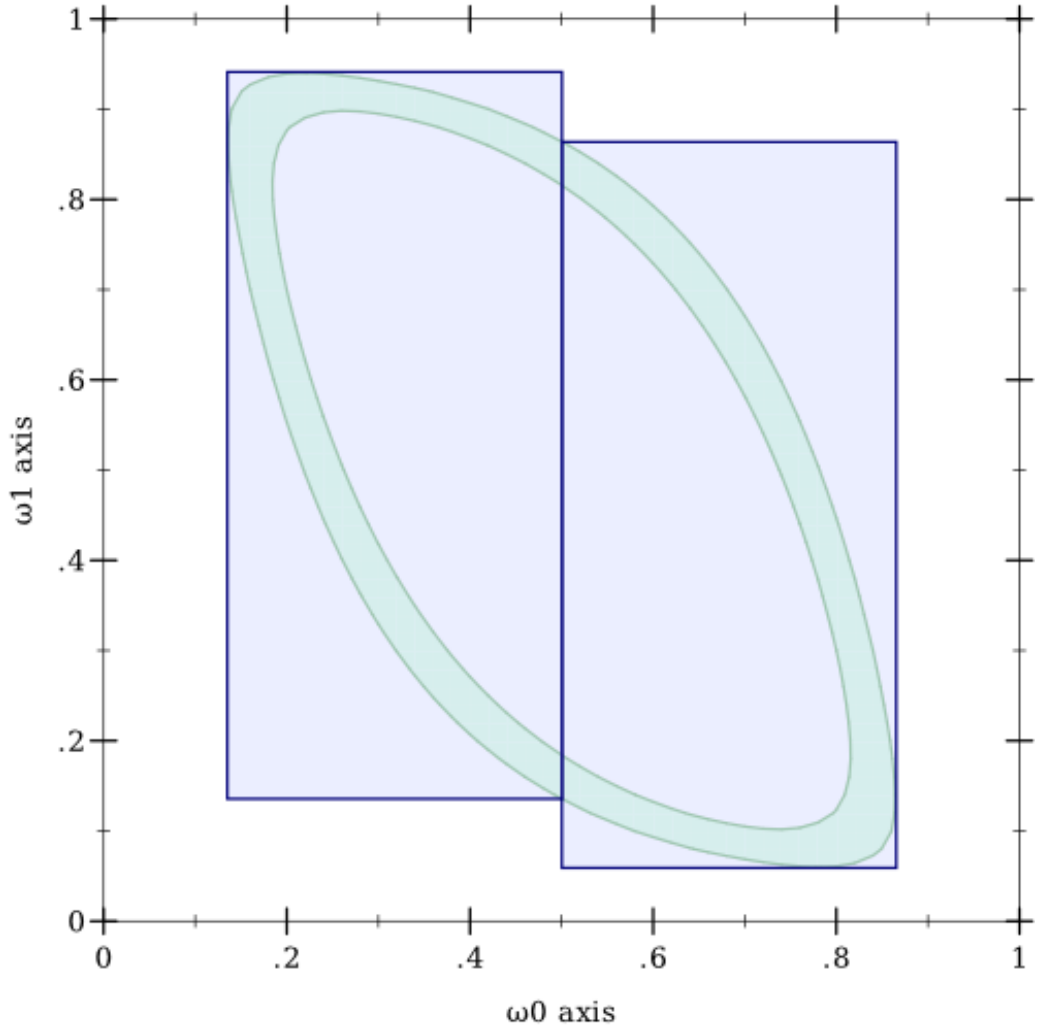

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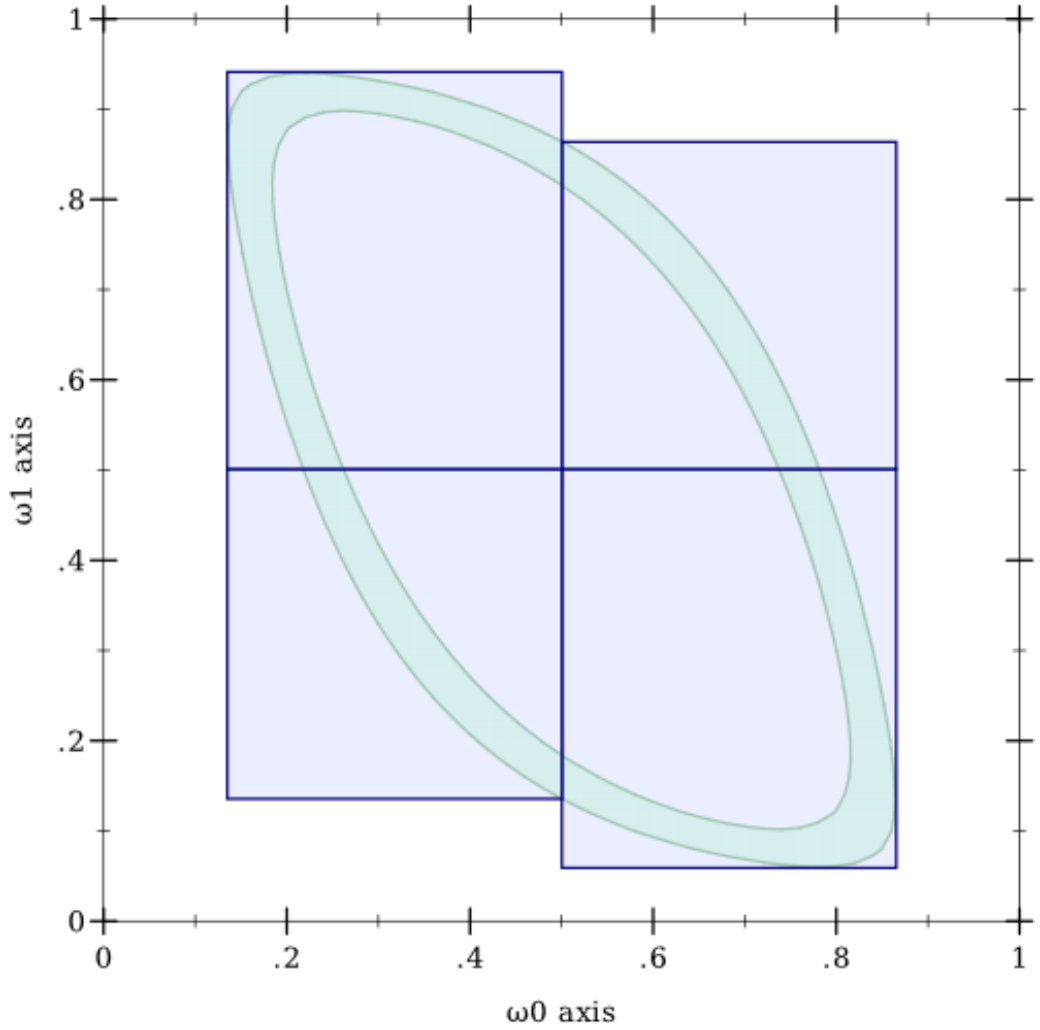

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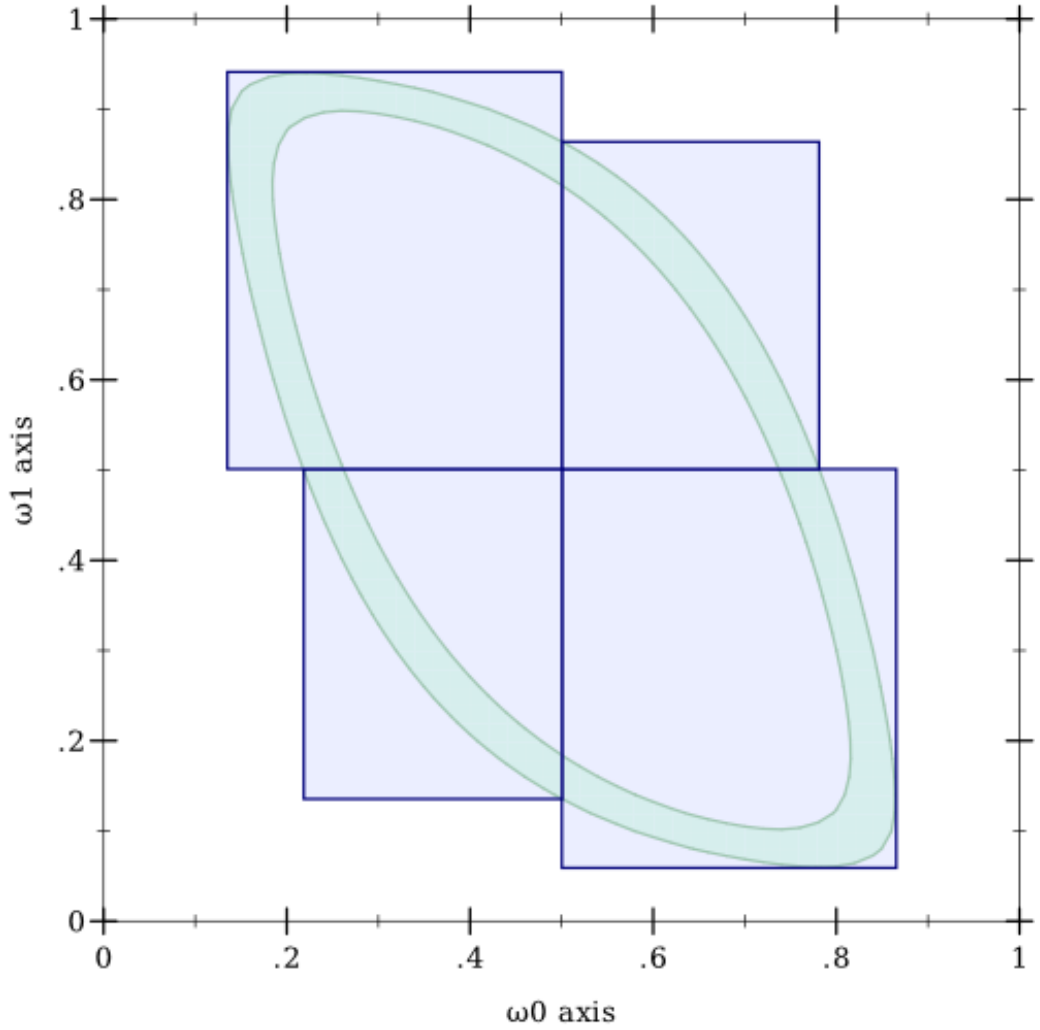

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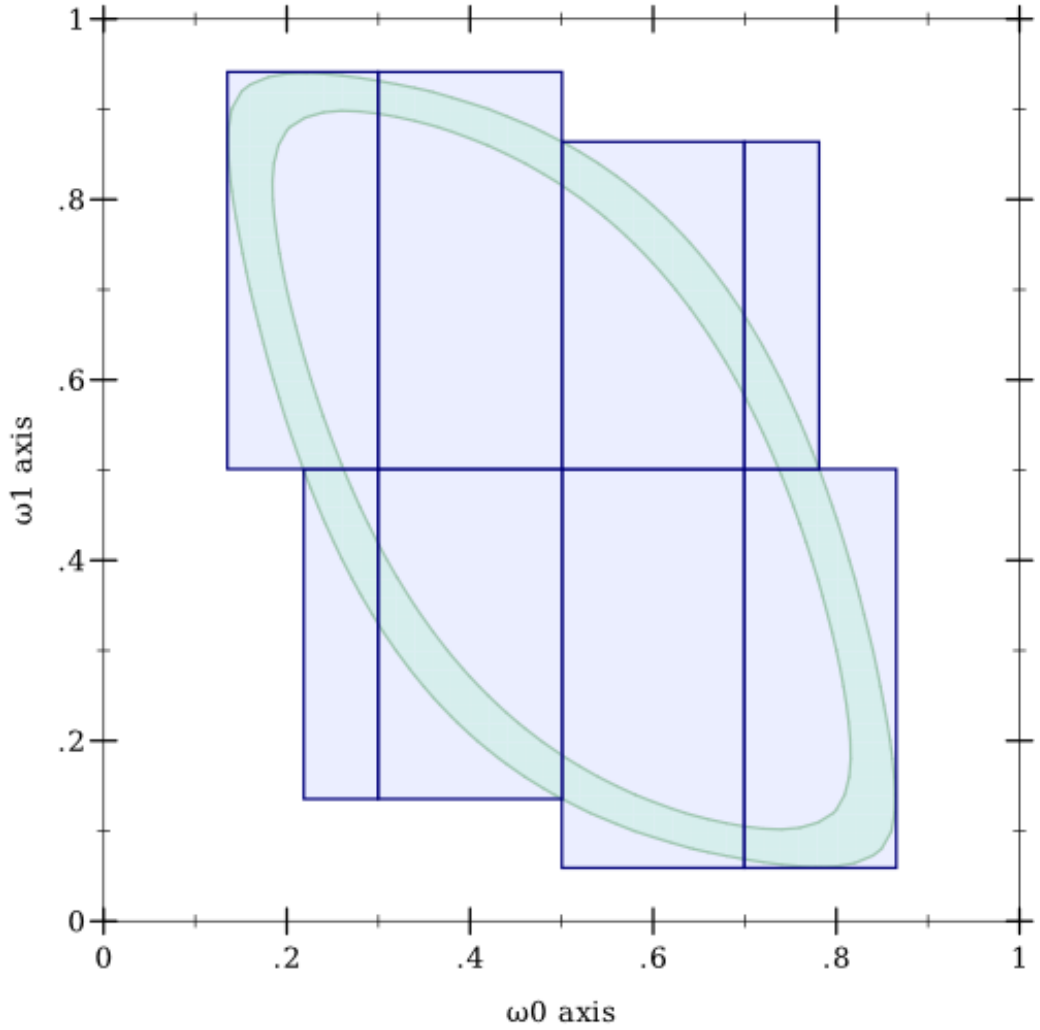

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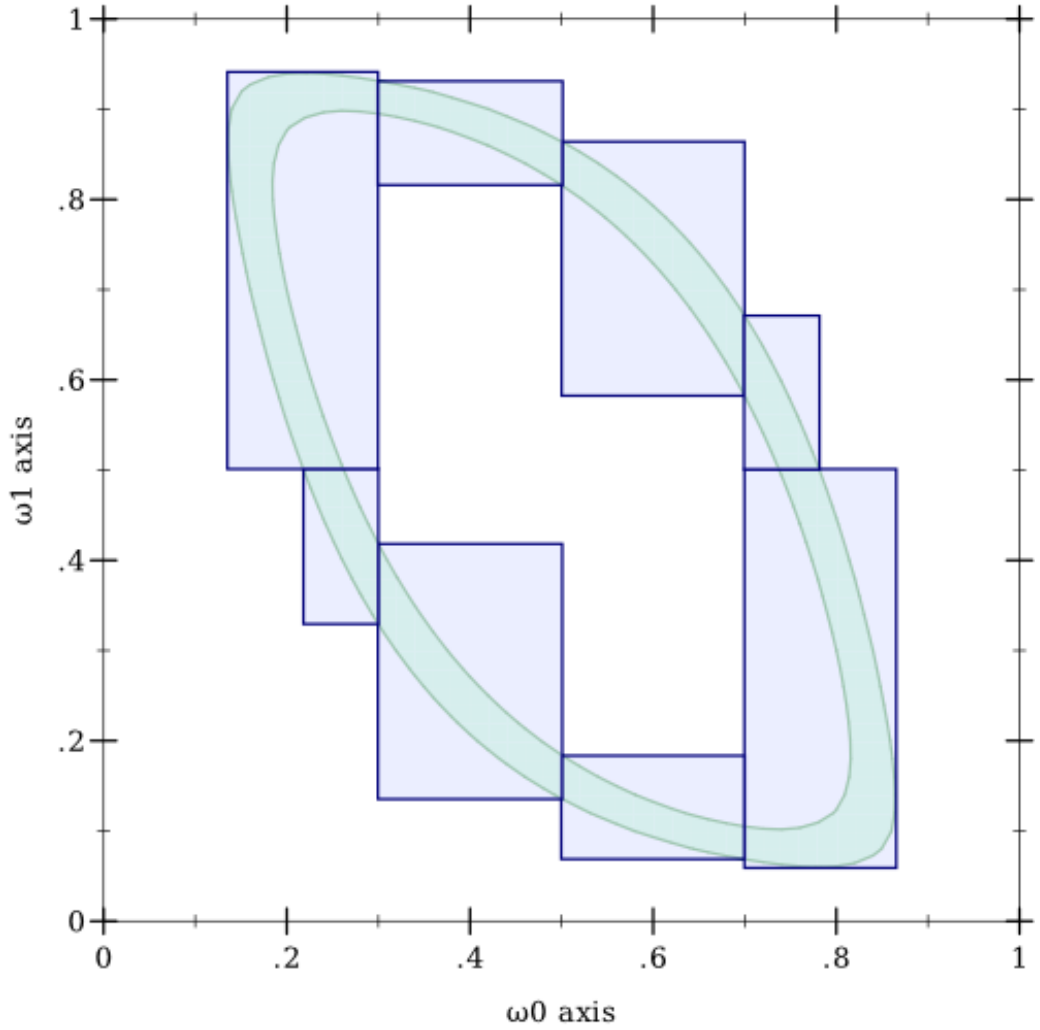

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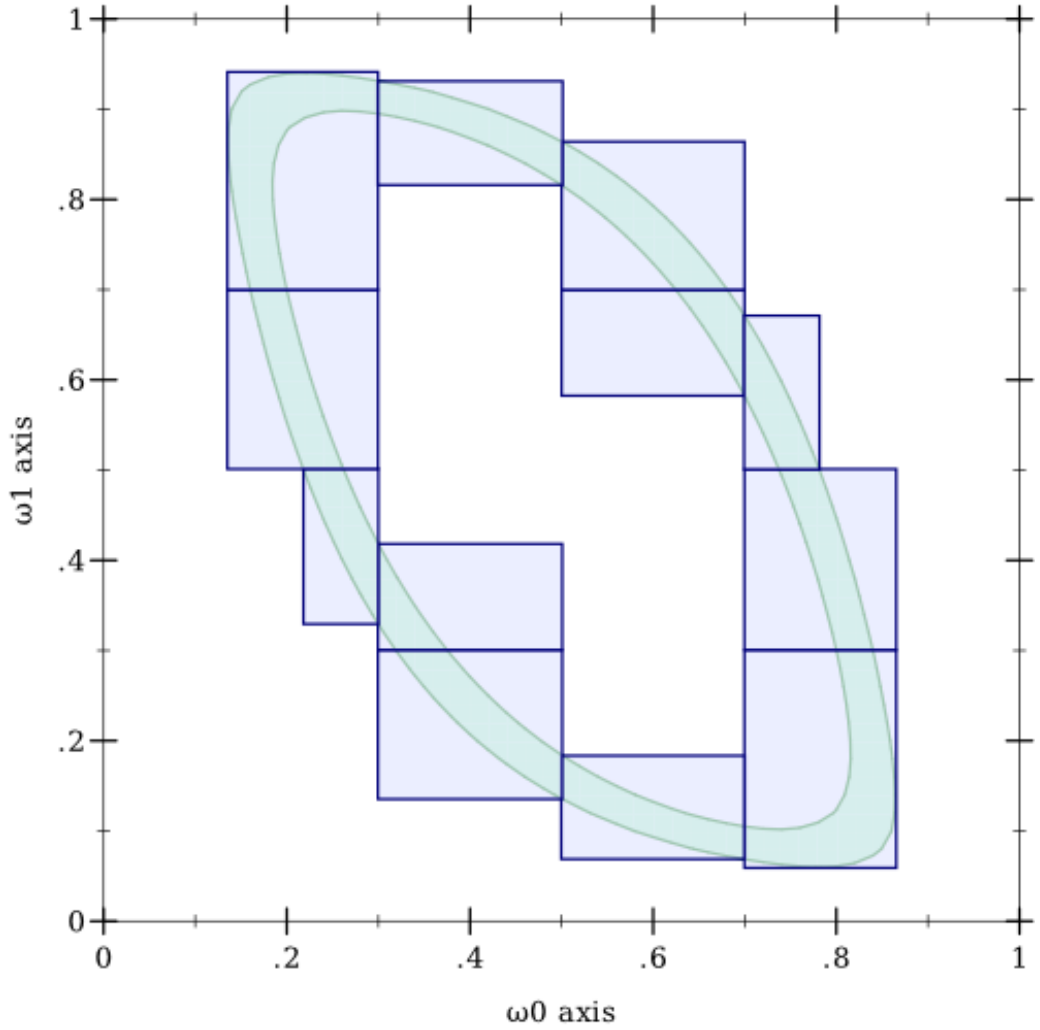

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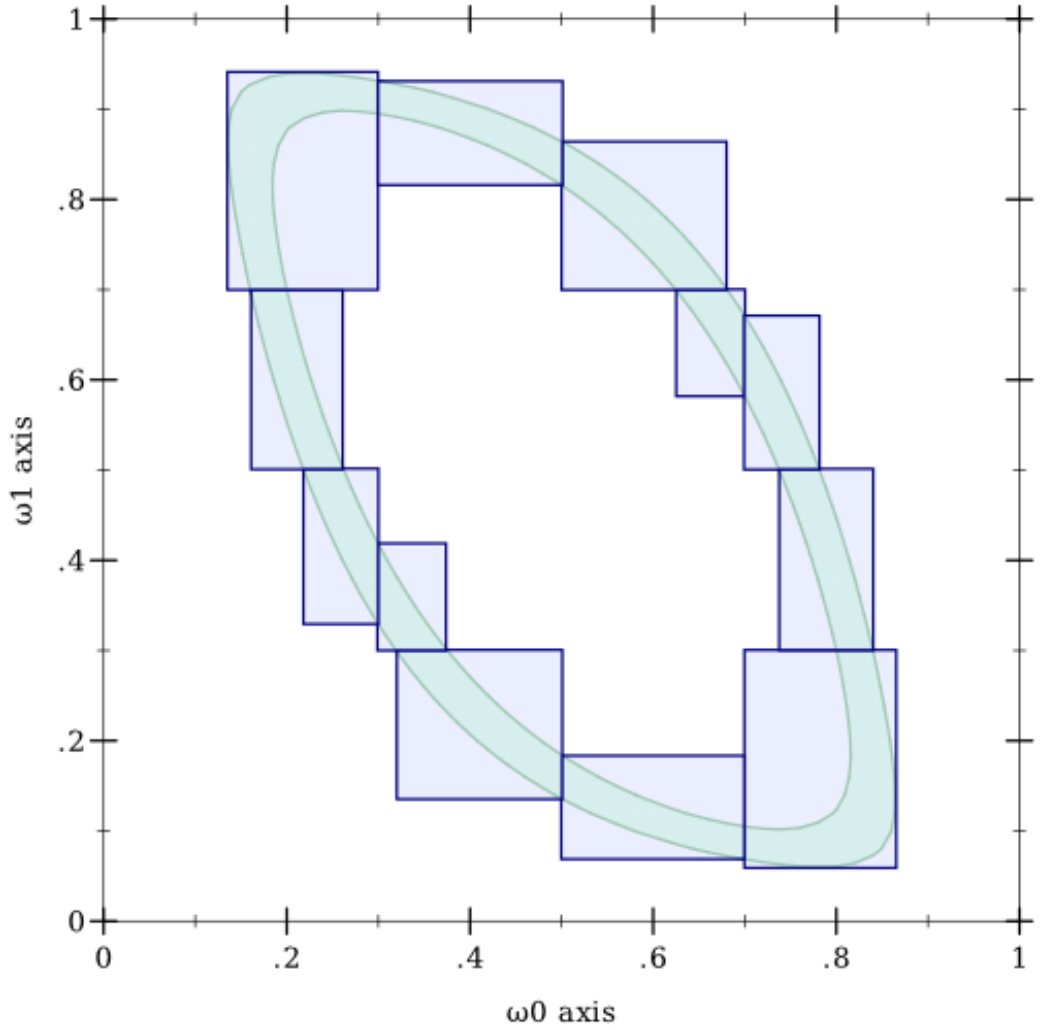

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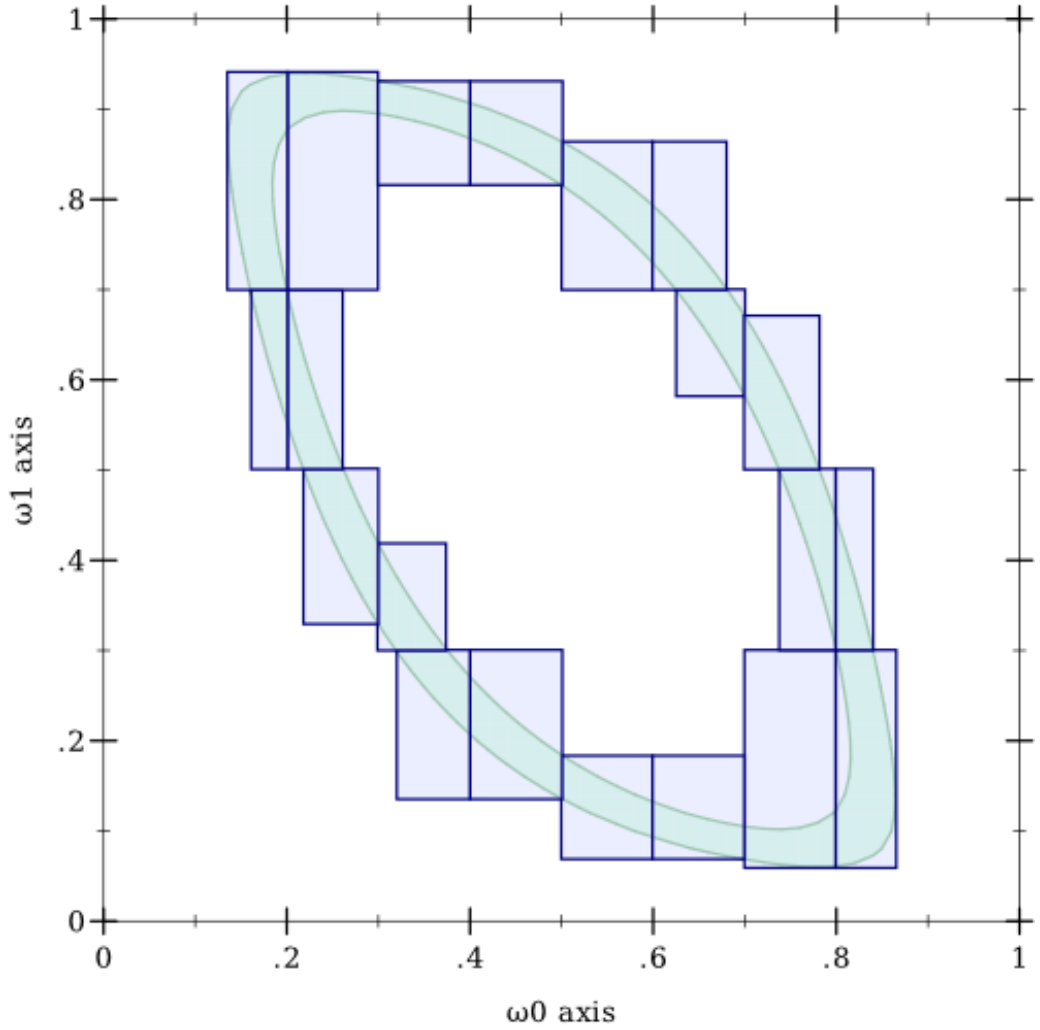

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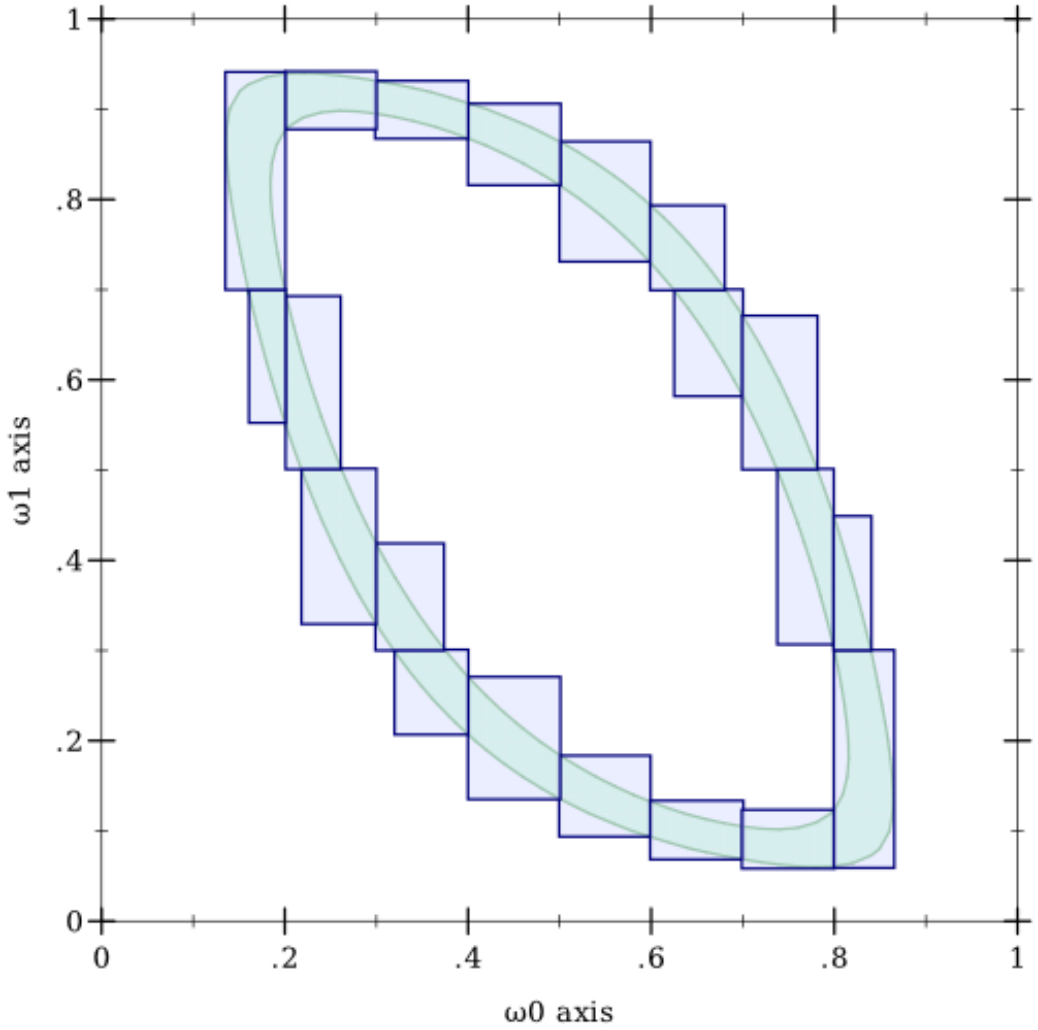

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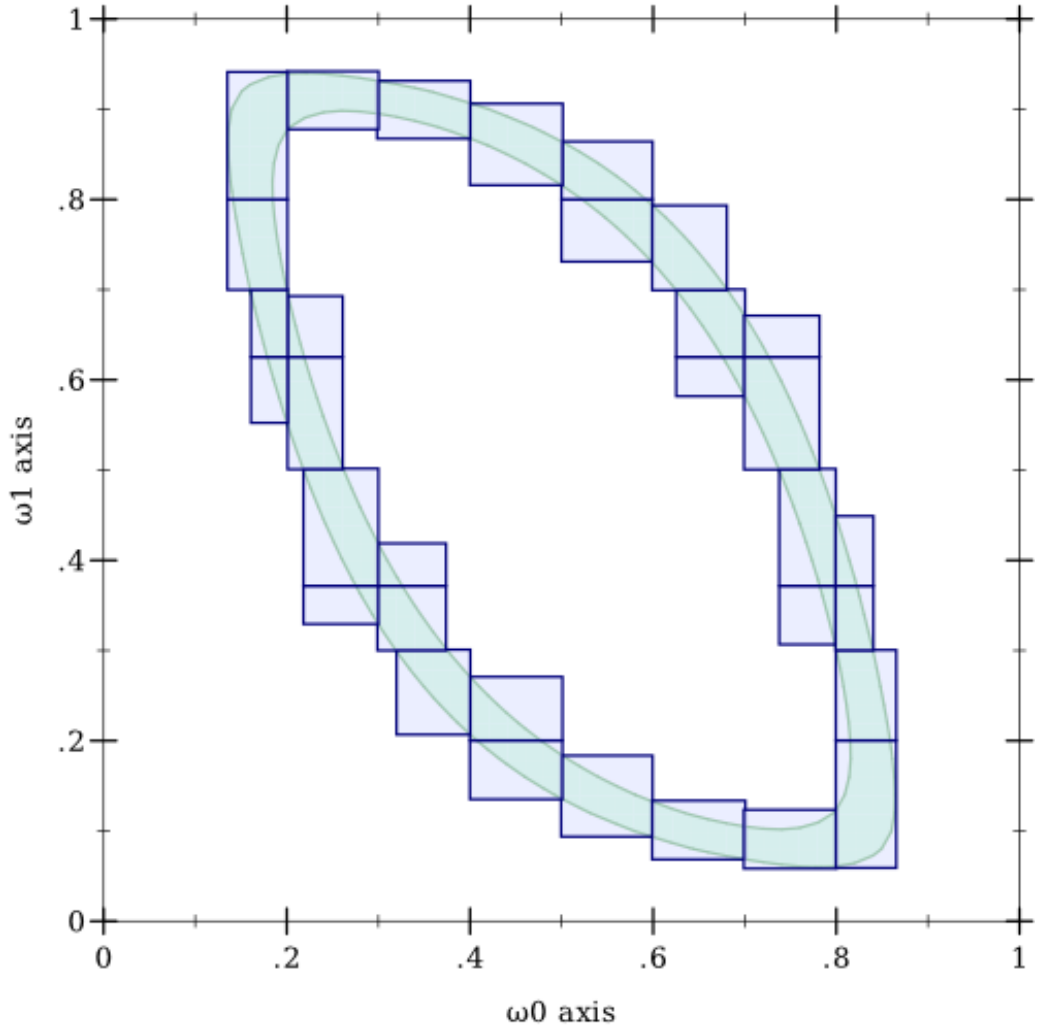

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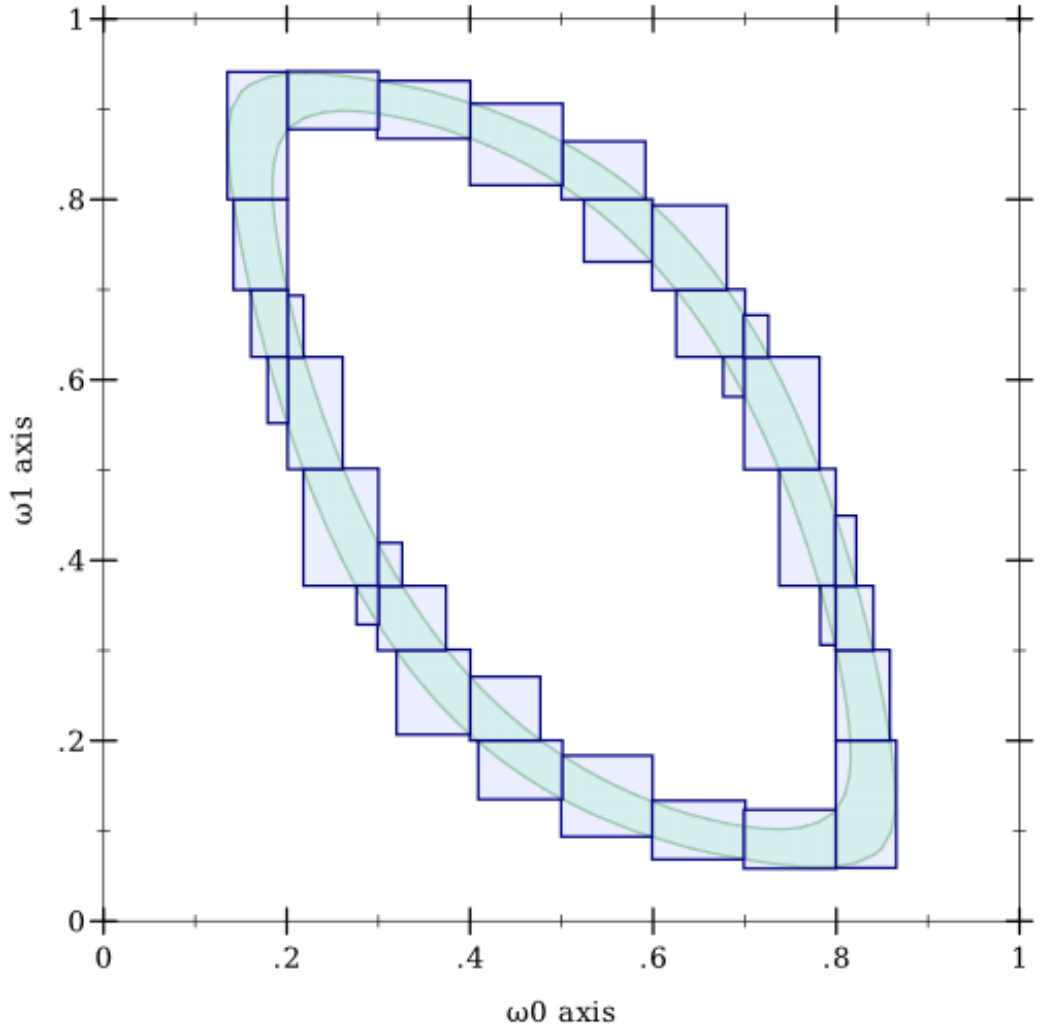

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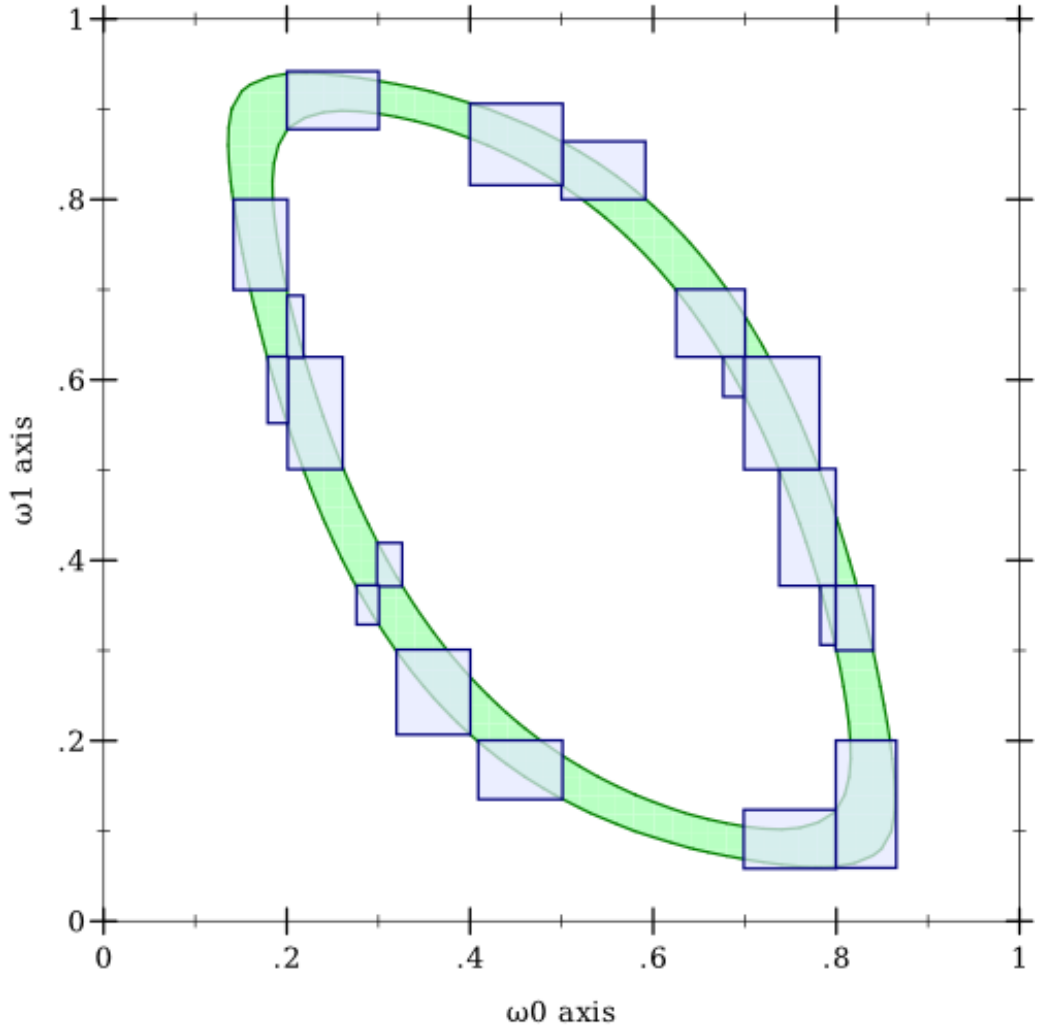

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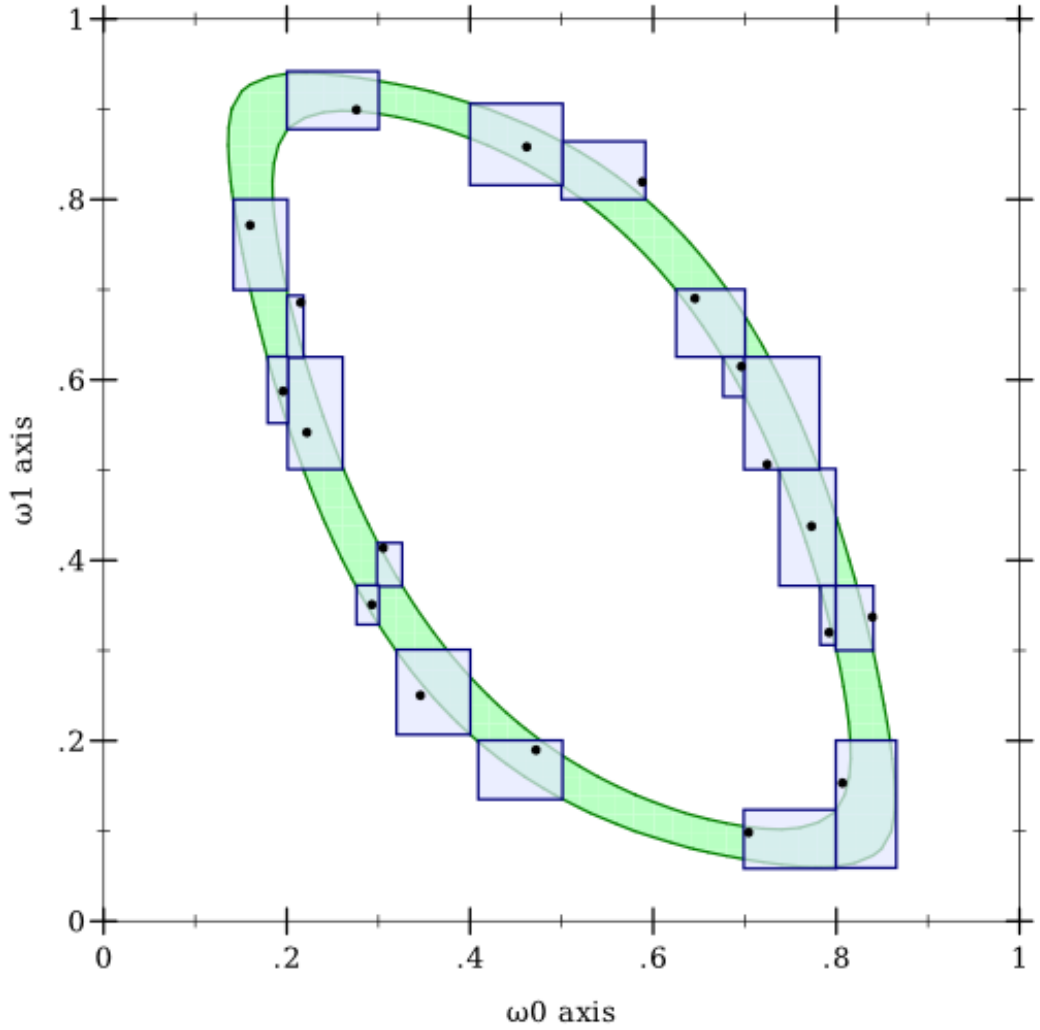

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- Standard interpretation of programs as pure functions from a random source
- Efficient way to compute preimage sets
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Tally: 3+3+2+1 = 9 semantic functions, 11 or 12 rules each

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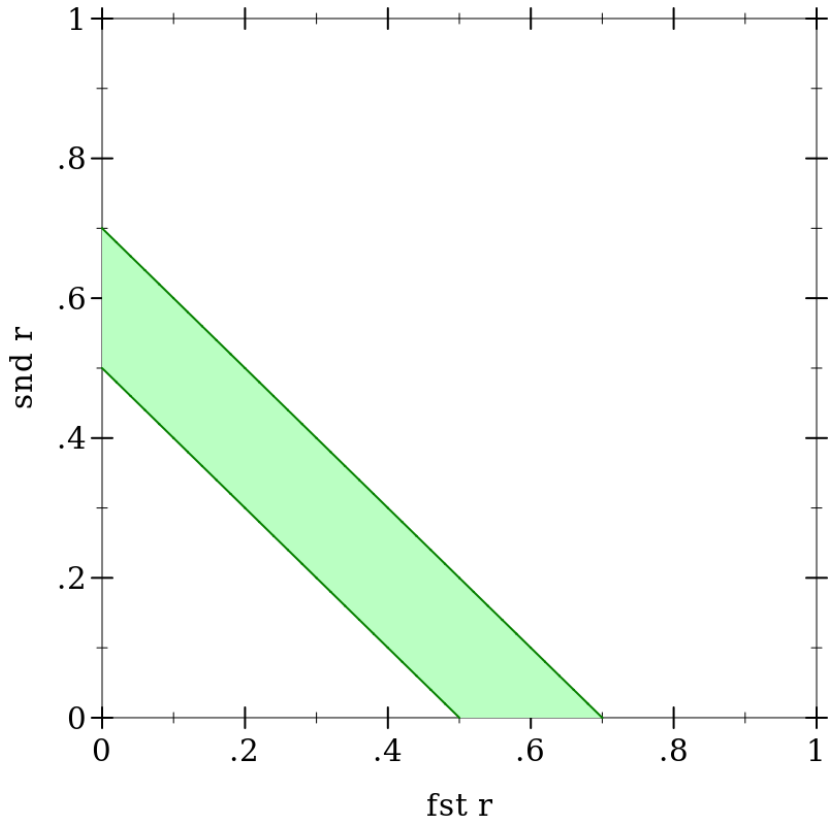

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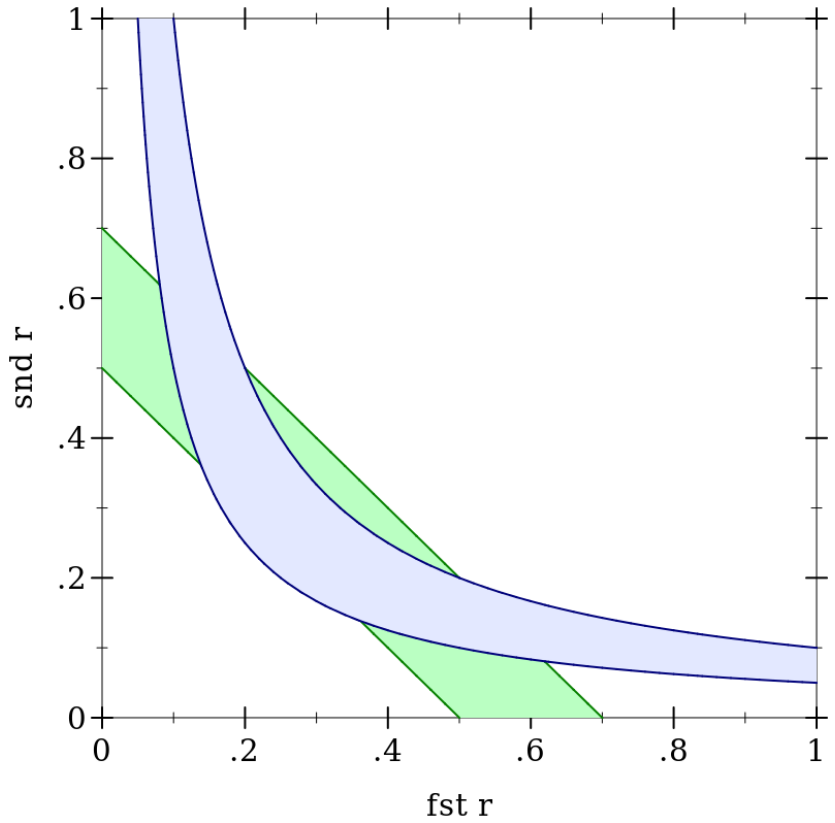

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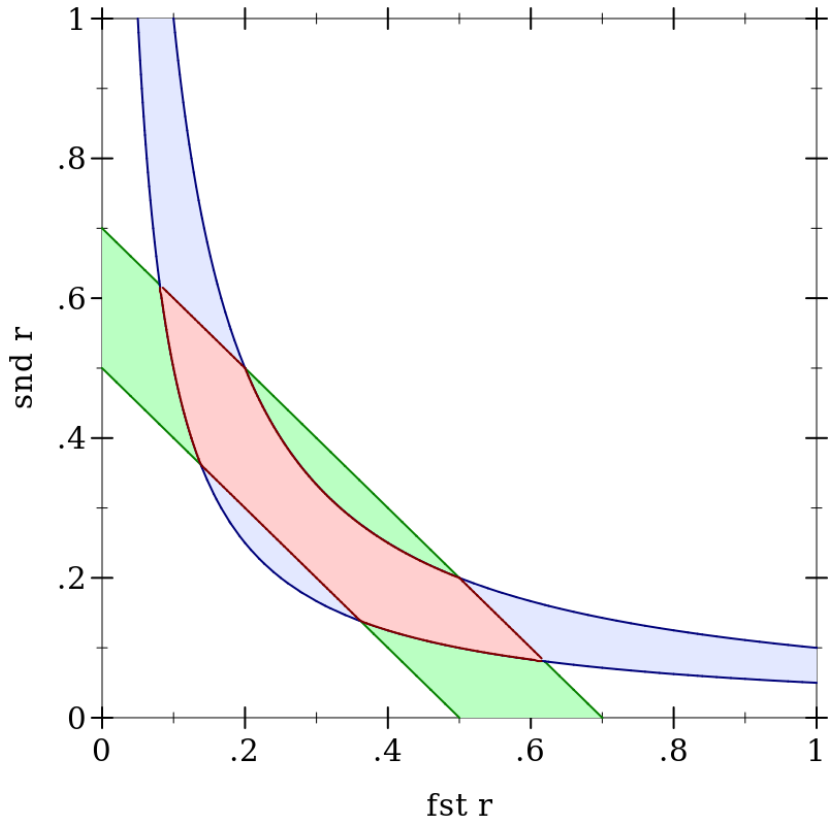

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- Can add **`(random)`** and recursion to all semantics in one shot

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- Recursion is somewhat tricky—requires fine control over recursion depth or **`if`** choices

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Theorem (monotone). $[\![\cdot]\!]_{\widehat{\mathrm{pre}}}$ is monotone.

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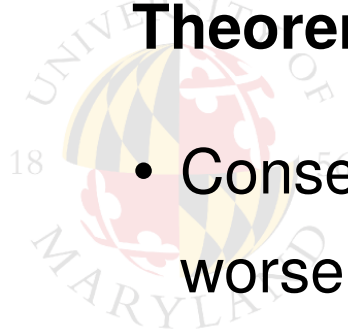

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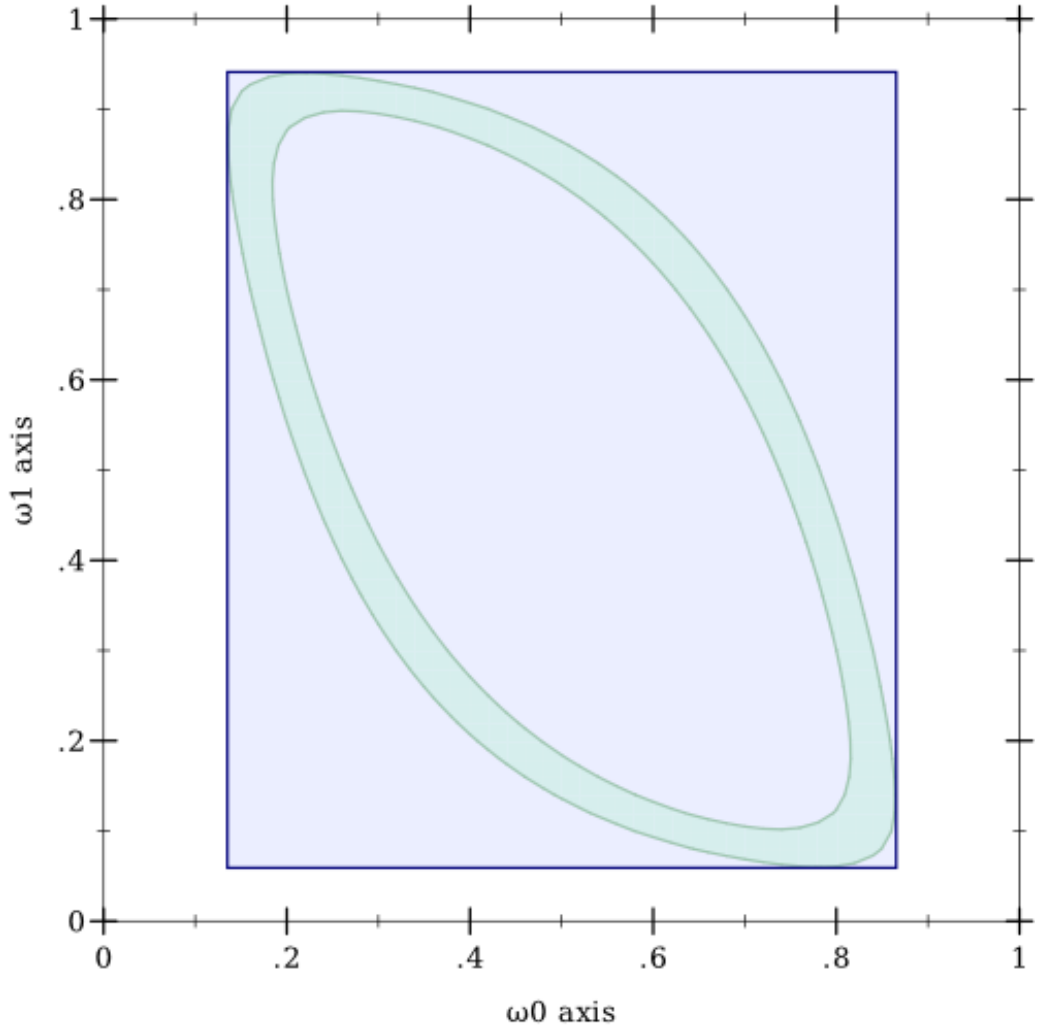

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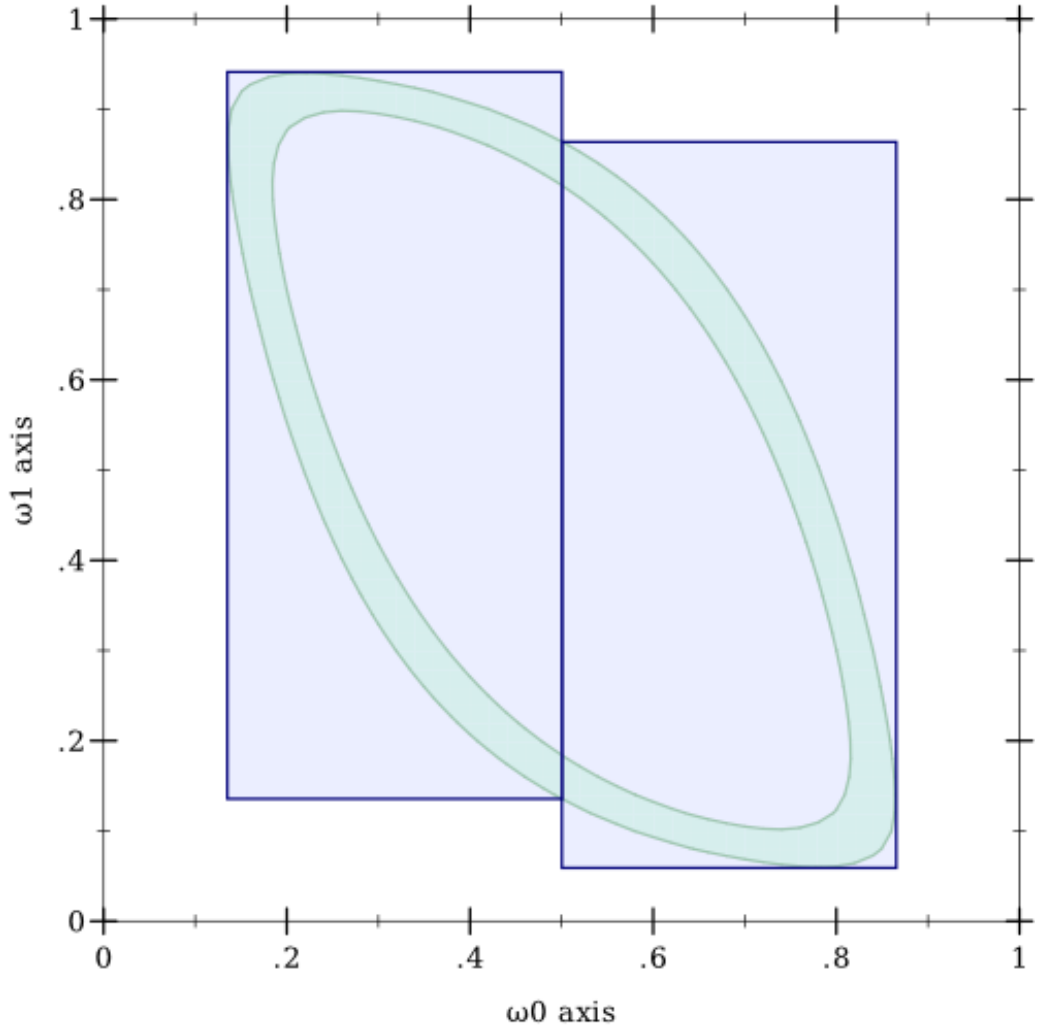

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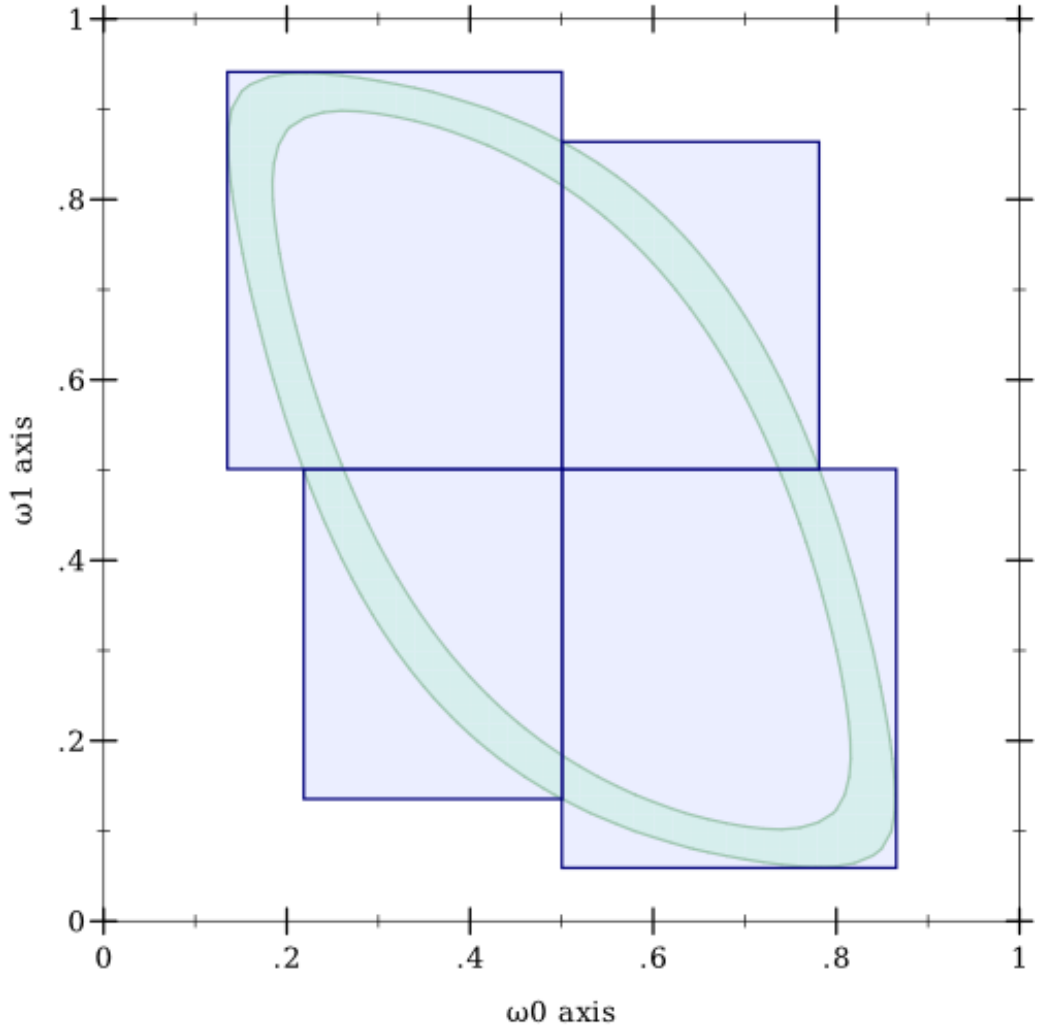

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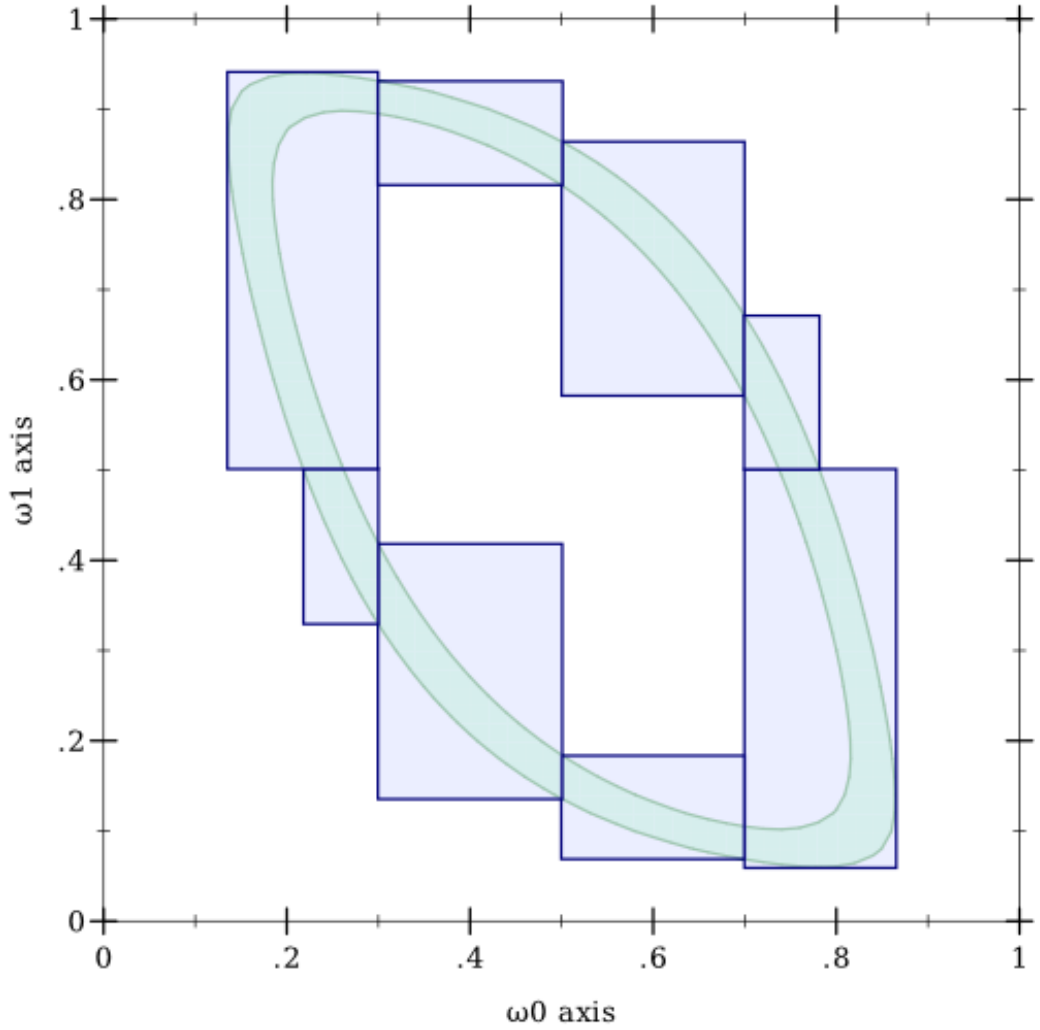

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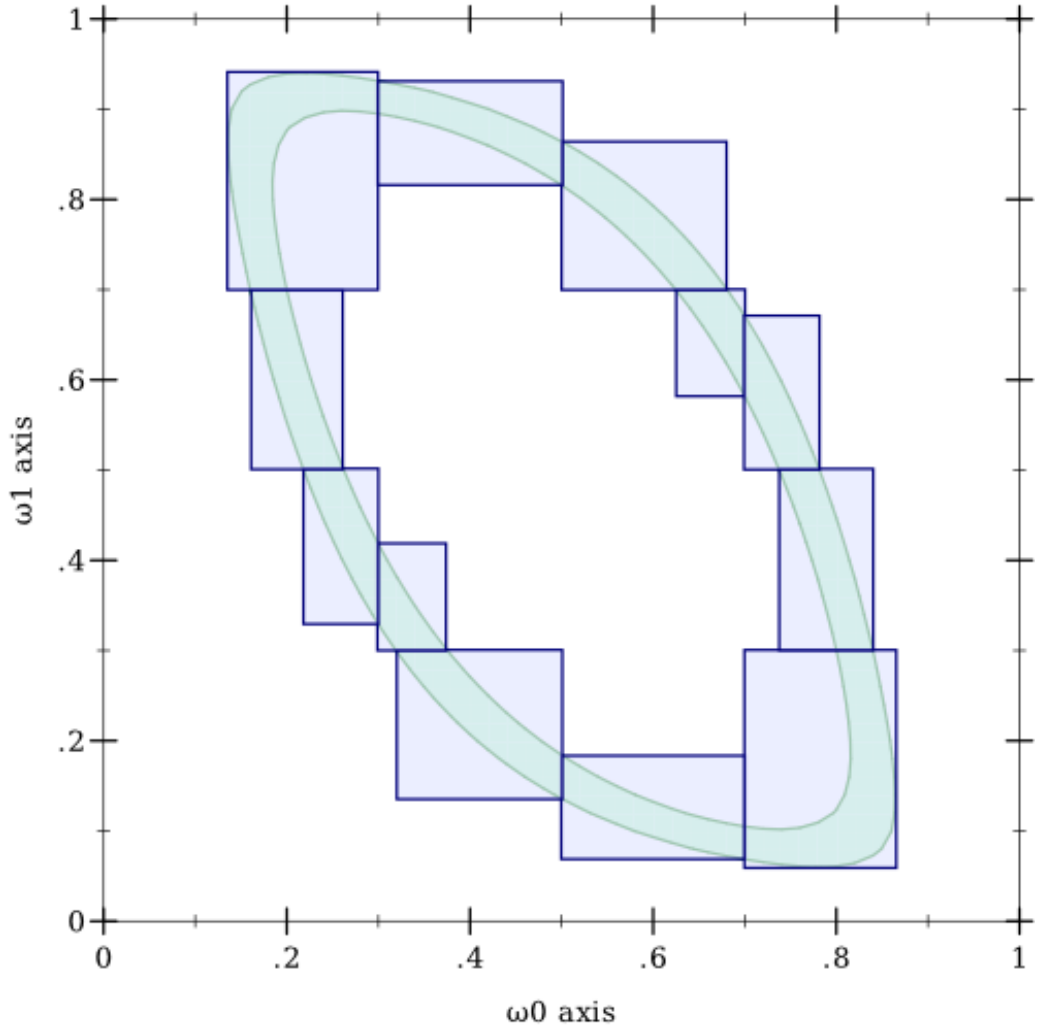

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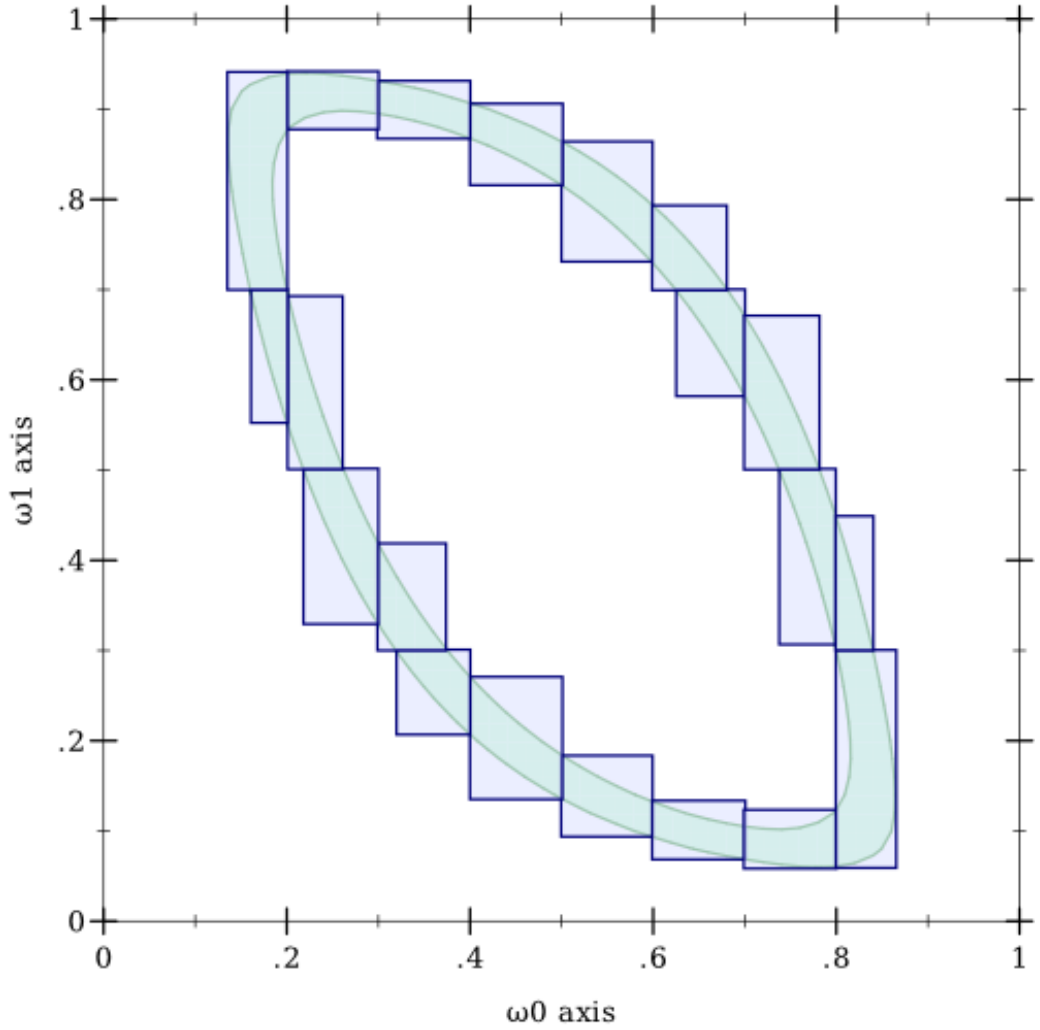

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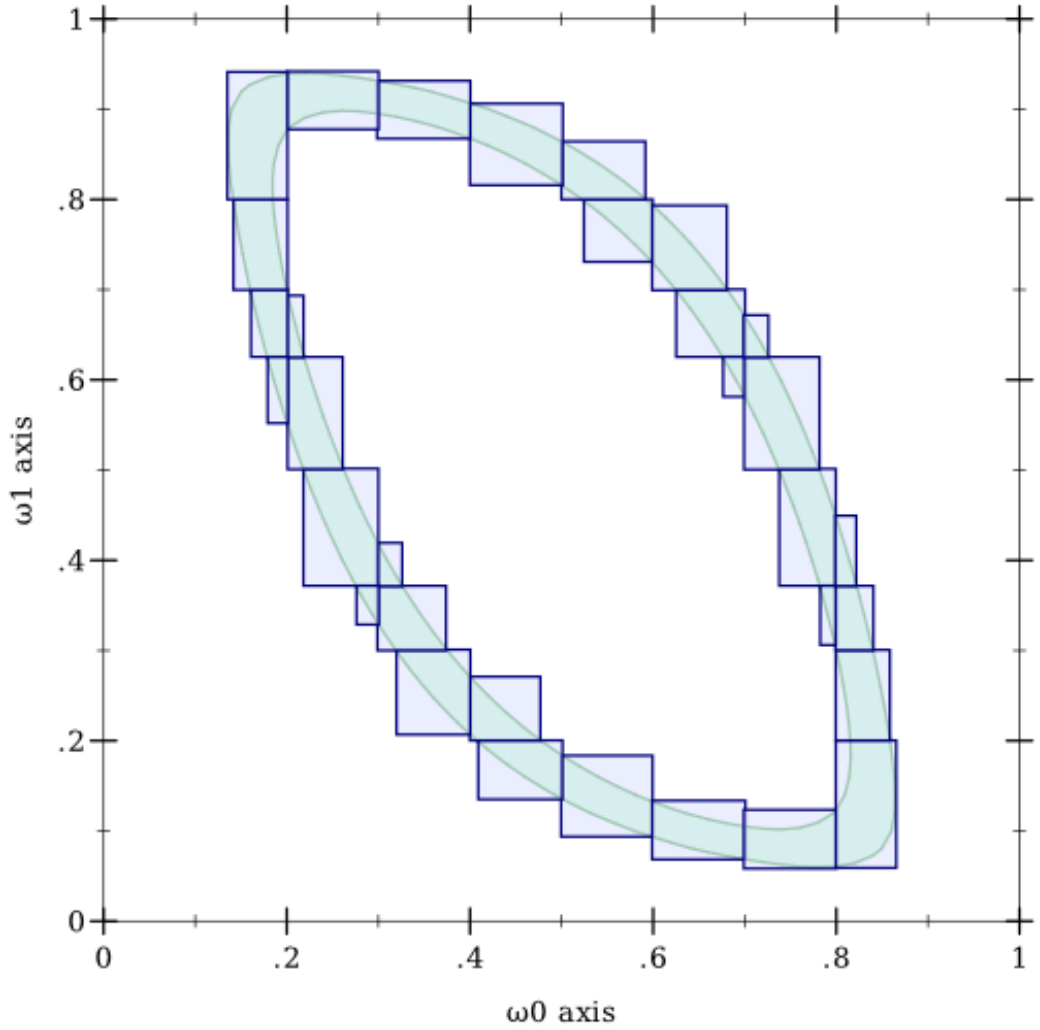

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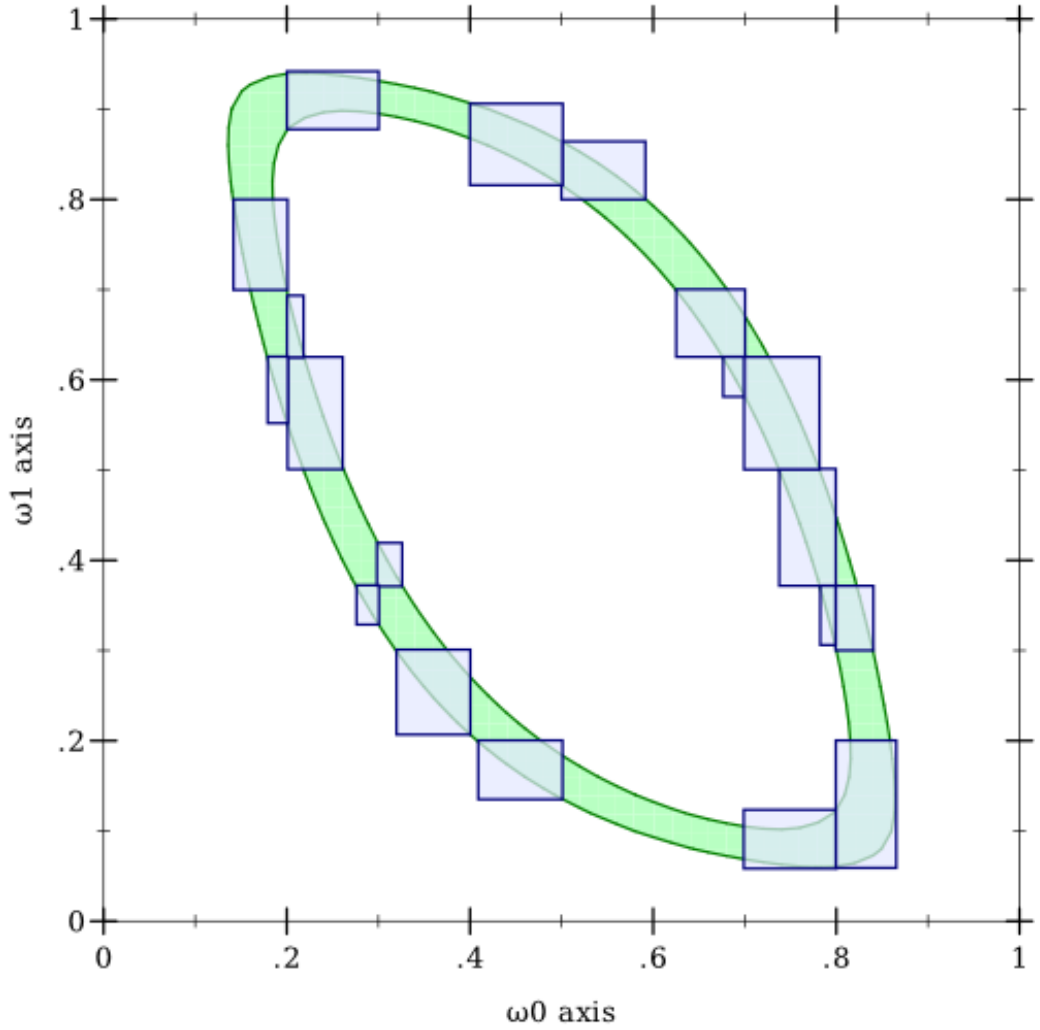

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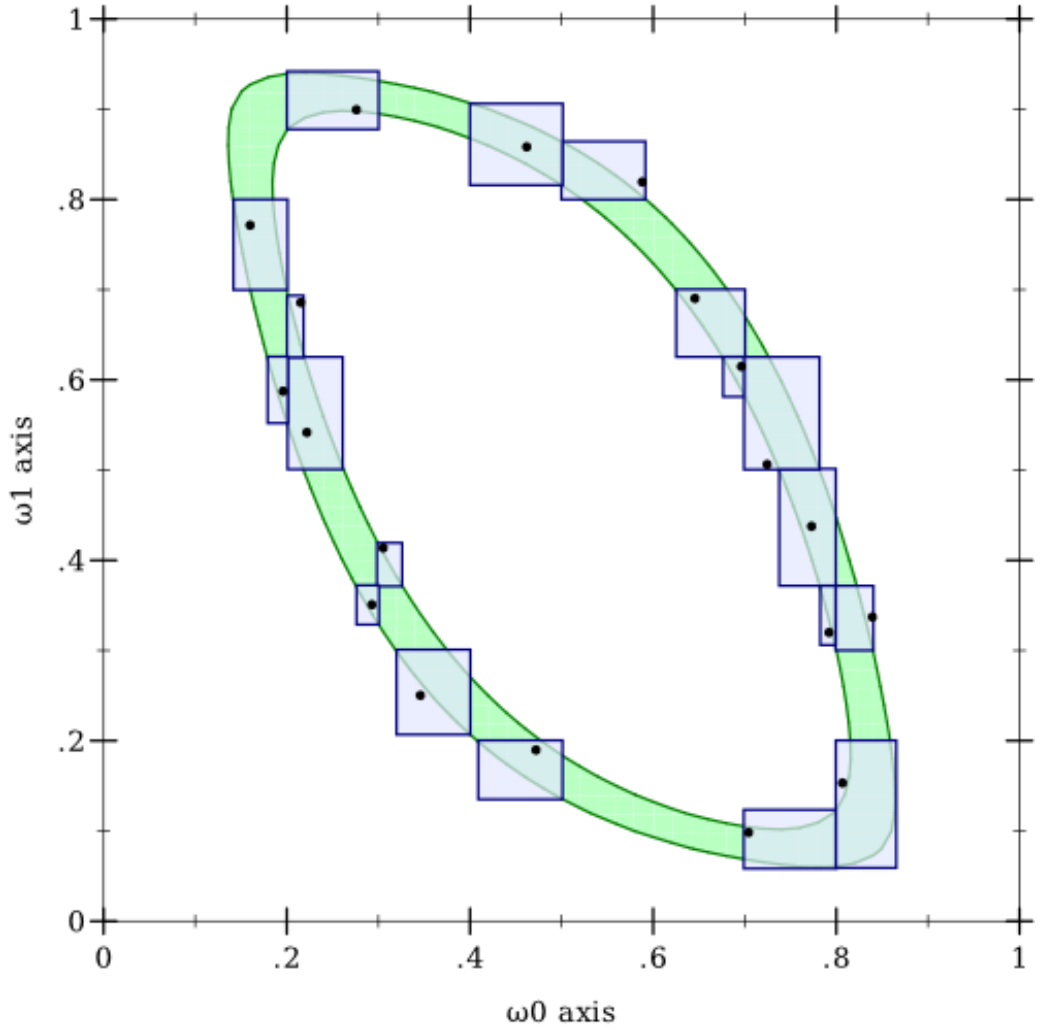

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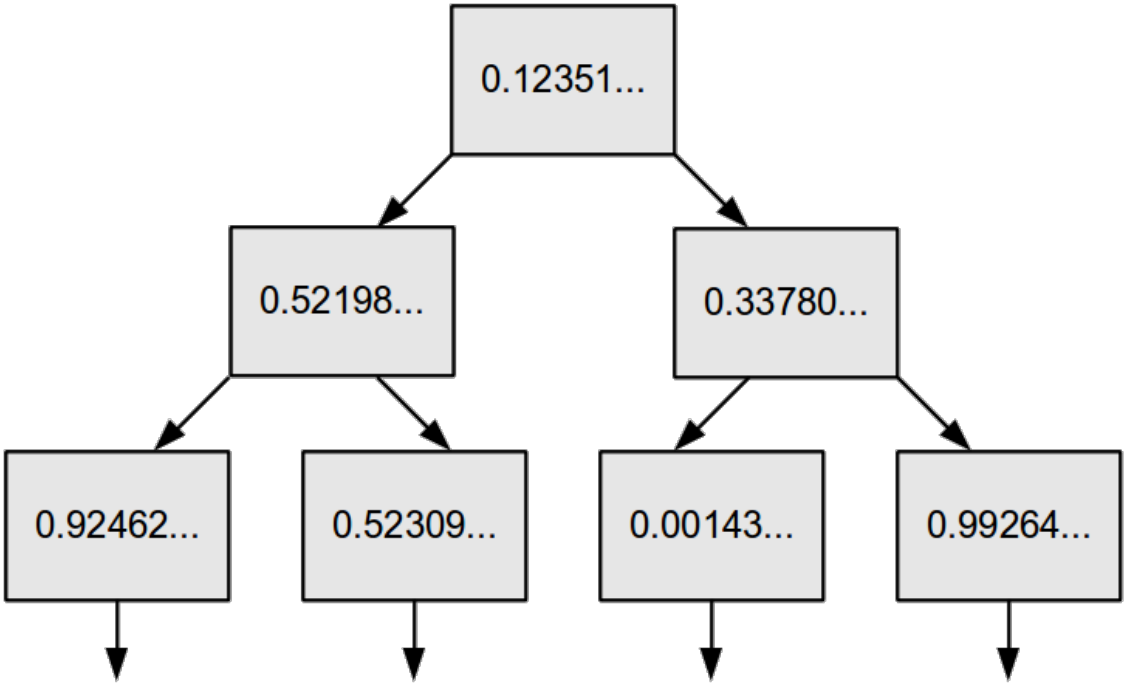

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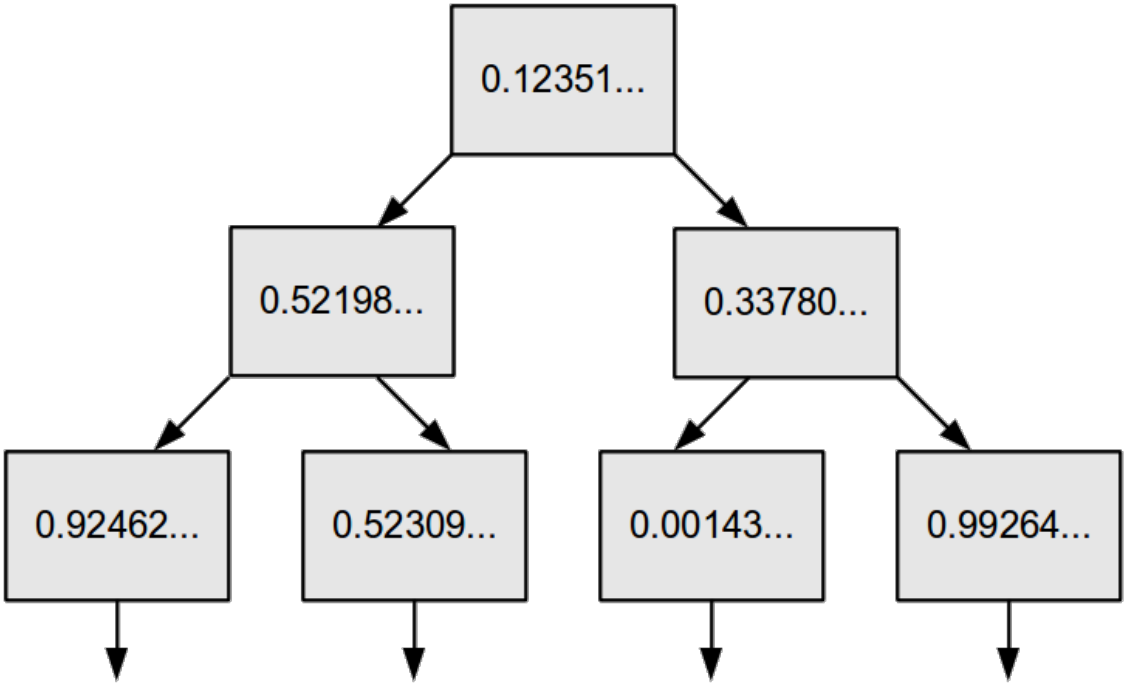

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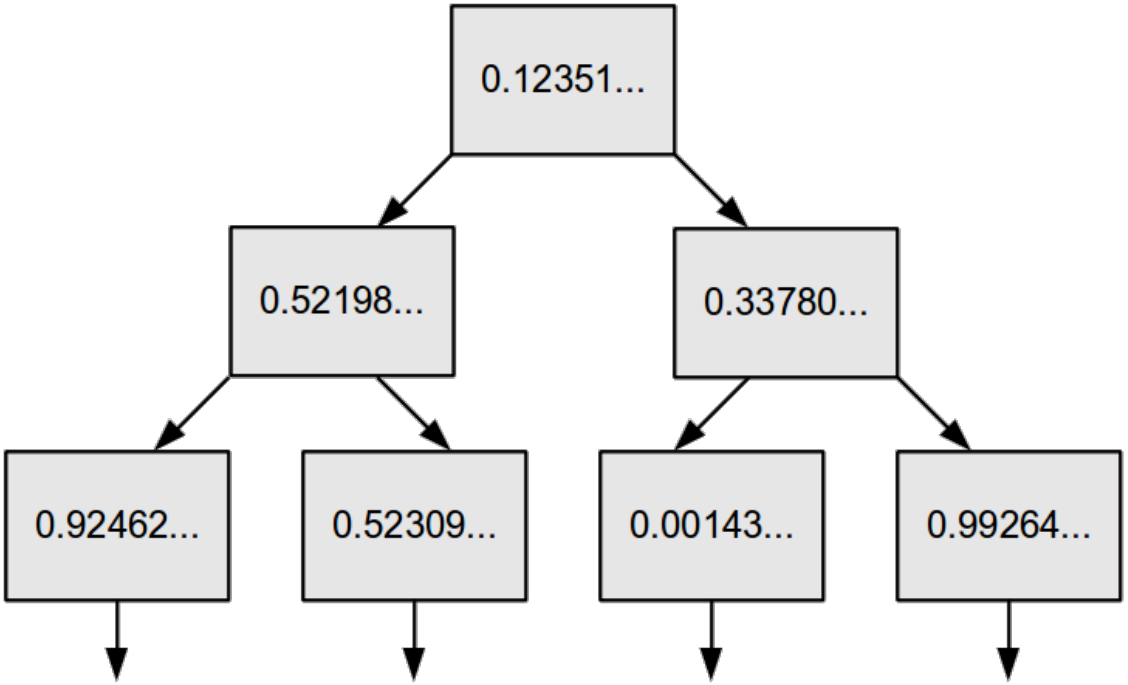

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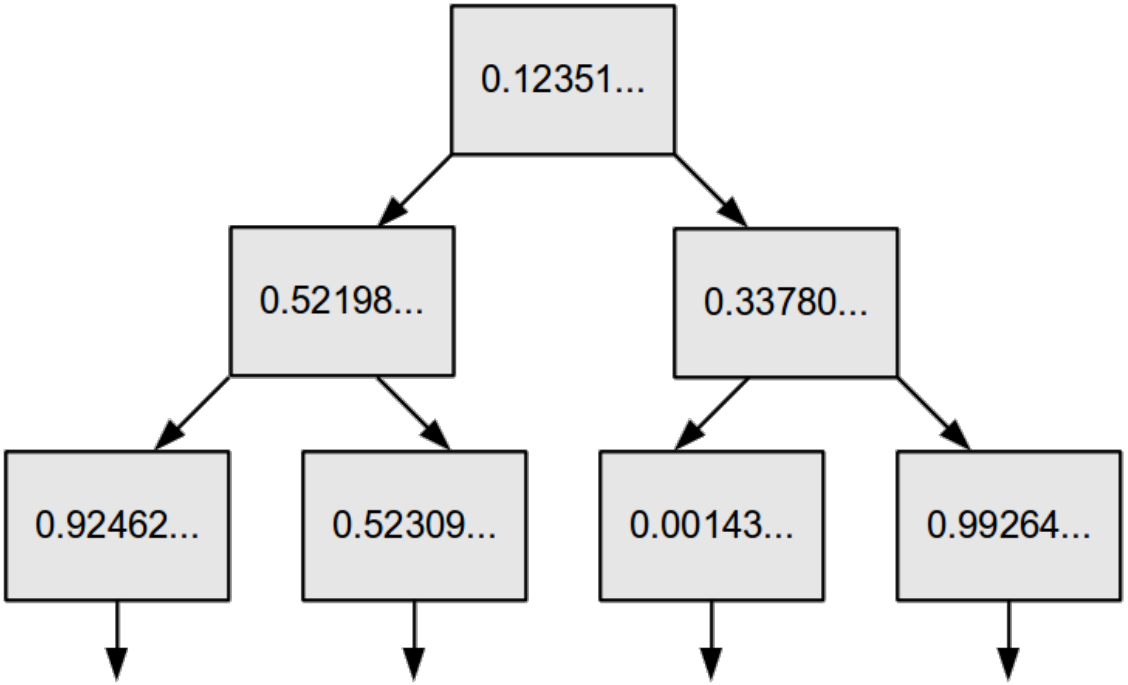
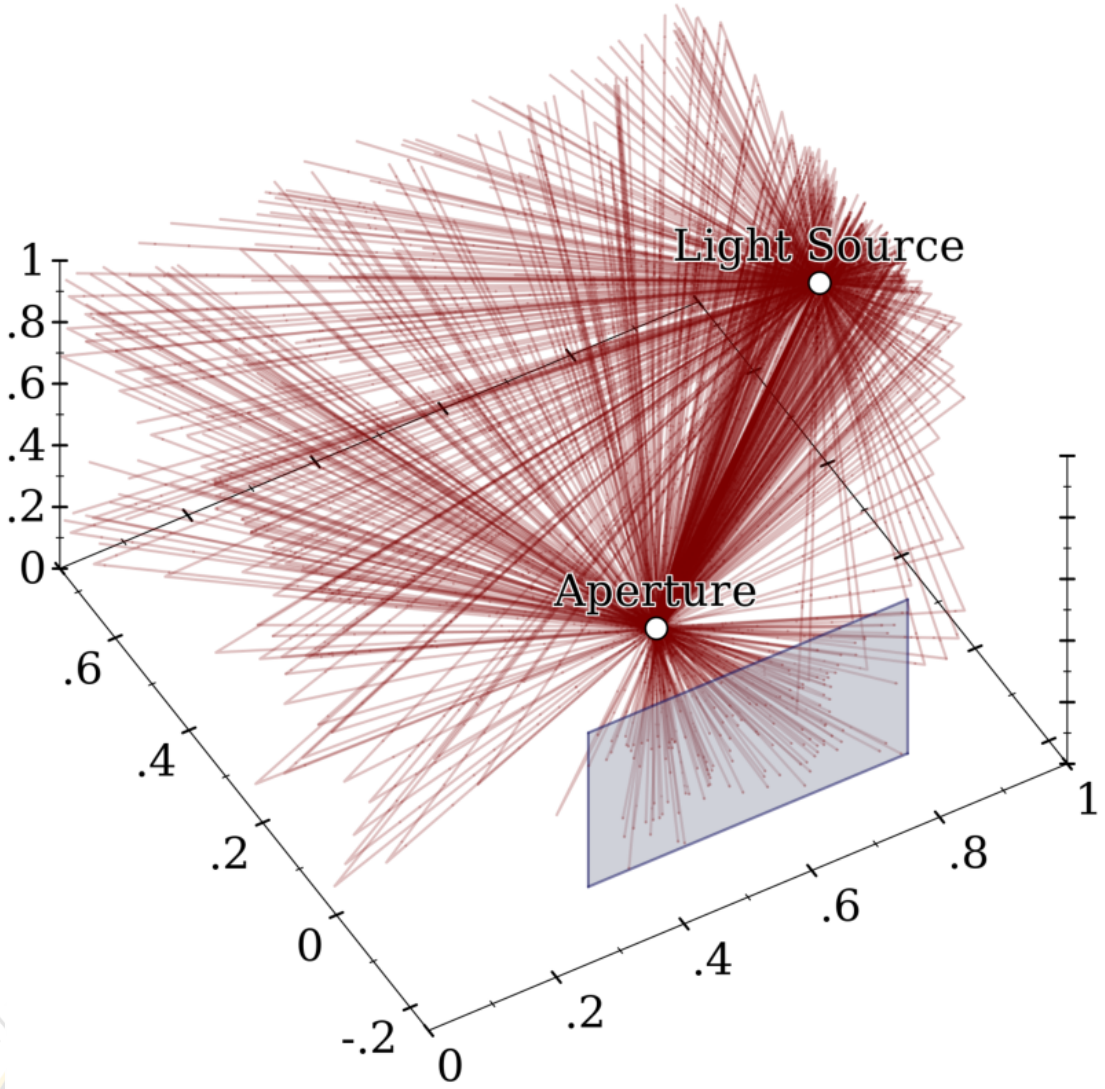
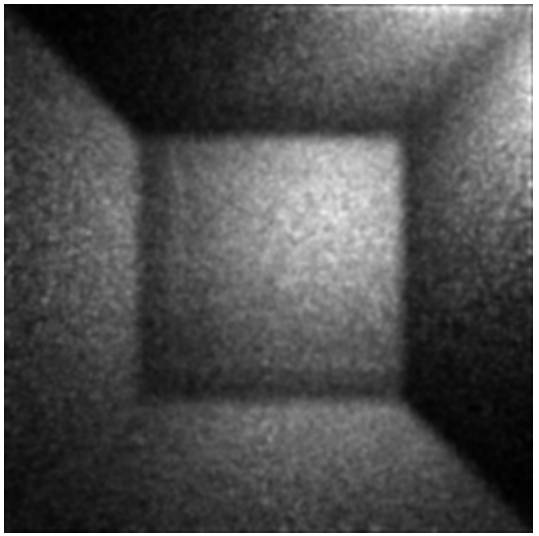


- Every expression in a program is assigned a node
- Implemented using lazy trees of random values
- No probability *density* for domain, but there is a *measure*

Example: Stochastic Ray Tracing





Example: Probabilistic Verification

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(struct/drbayes float (value error))
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  (if (float-any? x)
      x
      (let ([v  (float-value x)]
            [e  (float-error x)])
        (cond [(negative? v)  (float-any)]
              [(zero? v)      (float 0 0)]
              [else  (float (sqrt v)
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- Idea: sample `e` where `(> (float-error e) threshold)`
- Verified `flhypot`, `flsqrt1pm1`, `flsinh` in Racket's `math` library, as well as others

Examples: Other Inference Tasks

- Typical Bayesian inference
  - Hierarchical models
  - Bayesian regression
  - Model selection

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- Typical Bayesian inference
  - Hierarchical models
  - Bayesian regression
  - Model selection
- Atypical
  - Programs that halt with probability < 1, or never halt
  - Probabilistic context-free grammars with context-sensitive constraints

Summary

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- Interpreting every program requires measure theory
- Defined a semantics that computes preimages
- Measuring abstract preimages or sampling in them carries out inference
- Can do a lot of cool stuff that's normally inaccessible

https://github.com/ntoronto/drbayes