Notes on Convergence of Stationary Iterative Methods

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Some definitions:

1. A Hermitian matrix is *positive definite* if each of its eigenvalues is positive.
2. A matrix A is *reducible* if there exists a permutation matrix P so that
$$PAP^T = \begin{bmatrix} F & G \\ 0 & H \end{bmatrix}$$
where H is a square matrix and 0 is a block of zeros. Otherwise, A is *irreducible.*
3. A matrix A is *(weakly) diagonally dominant* if
$$\begin{aligned} |a_{ii}| &\geq \sum_{j\neq i} |a_{ij}|, \ \ i = 1, \ldots, n \\ |a_{kk}| &> \sum_{j\neq k} |a_{kj}|, \ \ \text{for at least one } k \end{aligned}$$
4. A matrix A is *strongly diagonally dominant* if
$$|a_{ii}| > \sum_{j\neq i} |a_{ij}|, \ \ i = 1, \ldots, n.$$
5. A matrix A has *Property A* if there exists a permutation matrix P so that
$$PAP^T = \begin{bmatrix} D_1 & H \\ K & D_2 \end{bmatrix}$$
where D_1 and D_2 are diagonal matrices. (Young)
6. A matrix A is *consistently ordered* if there exist disjoint subsets $S_1, S_2, \ldots S_T$ of indices such that
   (a) $S_1 \cup S_2 \cup \ldots \cup S_T = \{1, 2, \ldots, n\}$.
   (b) If $a_{ij} \neq 0$ or $a_{ji} \neq 0$ and $i \in S_k$ then $j \in S_{k+1}$ if $j > i$ and $j \in S_{k-1}$ if $j < i$.

   Example: Tridiagonal matrices are consistently ordered.

   Note: Only matrices with Property A can be consistently ordered.

   An equivalent definition is that the eigenvalues of $B(\alpha) \equiv \alpha^{-1}D^{-1}L + \alpha D^{-1}U$ are independent of α for $\alpha \neq 0$, where $A = D - L - U$, D is diagonal, L is strictly lower triangular, and U is strictly upper triangular.

7. A matrix A is an *L-matrix* if $a_{ii} > 0$ for all i and $a_{ij} \leq 0$ for $i \neq j$.
8. A *Stieltjes matrix* is a real symmetric positive definite L-matrix.
9. An *M-matrix* is a real nonsingular L-matrix with $A^{-1} \geq 0$ (elementwise).

The following table gives conditions under which various iterative methods converge.

Does the Iterative Method Converge?

conditions on A	Jacobi	Gauss-Seidel	SOR
symmetric pos def	if $2D - A$ pos def (Young,p.109)	yes (Young,p.109)	if $0 < \omega < 2$ (Young,p.109)
irred. diag. dom.	yes (Young,p.107)	yes (Young,p.107)	if $0 < \omega \leq 1$ (Young,p.107)
real, sym, nonsing, $D > 0$	iff A and $2D - A$ pos def (Young,p.111)	iff A pos def (Young,p.113)	iff A pos def and $0 < \omega < 2$ (Young,p.113)
L-matrix	iff M-matrix (Young,p.120)	iff M-matrix (Young,p.120) (faster than J, p.122)	if M-matrix and $0 < \omega \leq 1$ (Young,p.120)
$D > 0$, sym, consist. ord	iff A pos def (Young,p.147)	if Jacobi does (Ortega p.134)	if Jacobi does and $0 < \omega < 2$ (Ortega p.134)
strictly diag. dom.	yes (Varga p.73)	yes (Varga p.73)	

References:

James M. Ortega, *Numerical Analysis, A Second Course*, Academic Press, 1972.

Richard S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1962, Chapters 3 and 5.

David M. Young, *Iterative Solution of Large Linear Systems*, Academic Press, New York, 1971,