Answers to Exercises for Part 5

Exercises

0.1. Recall that Euler's method is

$$y_{n+1} = y_n + hf(t_n, y_n)$$

and Backward Euler is

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$$

$$\begin{aligned} P &: y = 1 + .1(1^2) = 1.1 \\ E &: f = (1.1)^2 - .5 = .71 \\ C &: y = 1 + .1 * .71 = 1.071 \\ E &: f = (1.071)^2 - .5 \end{aligned}$$

0.2.

$$\|y_4 - y_{true}\| \approx \|y_4 - y_5\| \equiv \delta$$

where y_4 is the result of the predictor and y_5 is the result of the corrector. If $\delta > \tau$, reduce h and retake the step:

- perhaps $h = h/2$.
- perhaps $h = h/2^p$ where, since we need $\delta 2^{-4p} \approx \tau$, we define $p = (\log \delta - \log \tau)/(4 \log 2)$.

0.3. Let $y_{(1)} = a$, $y_{(2)} = a'$. Then our system is

$$\mathbf{y}' = \begin{bmatrix} y_{(2)} \\ y_{(1)}^2 - 5y_{(2)} \end{bmatrix}$$

```
function yp = yprime(t,y)
yp = [y(2); y(1)^2 - 5 * y(2)];
function f = fvalue(z)
[t,y] = ode45('yprime',[0 1],[5,z]);
f = y(2,1)-2;
(main program)
z = fzero('fvalue',2)
% now the solution can be obtained by using ode45 with
% initial conditions [5,z].
% For example,
[t,y] = ode45('yprime',[0:.1:1],[5 z]);
```

0.4.

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{f} = \begin{bmatrix} 7y_{(1)} - 6y_{(2)} + .4t \\ 4y_{(1)} - 2y_{(1)}y_{(2)} \\ y_{(1)} + y_{(2)} + y_{(3)} - 24 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

0.5. Subtracting, we obtain

$$y^{ab}_{n+1} - y^{am}_{n+1} = \frac{3h^4}{8} y^{(4)}(\eta) - (-\frac{h^4}{24} y^{(4)}(\nu))$$

where η, ν are in the interval containing y^{ab}_{n+1}, y^{am}_{n+1}, and the true value. Since $3/8+1/24 = 10/24$, the error in AM can be estimated as $\epsilon = |y^{ab}_{n+1} - y^{am}_{n+1}|/10$. Now, if $\epsilon > \tau$, we might reduce h by a factor of 2 and retake the step. If $\epsilon << \tau$, we might double h in preparation for the next step (expecting that the local error might increase by a factor of 2^4).

0.6. For $i = 1, \ldots, 99$,

$$\mathbf{F}_i(y) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - 6\frac{y_{i+1} - y_{i-1}}{2h} + ihy_i - y_i^2 \,,$$

where $y_0 = 5$ and $y_{100} = 0$.

0.7. Let y_j approximate $y(jh)$. Then

$$\begin{aligned} y_0 &= 2 \\ \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} &= \frac{y_{j+1} - y_{j-1}}{2h} + 6y_j \\ y_5 &= 3 \end{aligned}$$

where $j = 1, 2, 3, 4$.

0.8.

```
initval = fzero(g, 2); % finds the missing initial value
[t,y] = ode45(f,[0,1],[2,initval]); % finds the solution to the
ode
function fval = f(t,y)
fval = [6*y(2) - y(1)
y(1)^2 - y(2)];
function gval = g(z) % assuming z is a scalar value
[t,y] = ode45(f,[0,1],[2,z]);
gval = y(end,1) - 3;
```

0.9.

$$\|u\|^2_{L_2} = \int_0^1 u(x)^2 dx = \int_0^1 (e^{5x} + x^2)^2 dx = \int_0^1 e^{10x} + x^4 + 2x^2 e^{5x} dx$$

We use integration by parts to evaluate the $2x^2e^{5x}$ term:

$$\int_0^1 x^2 e^{5x} dx = x^2 \frac{e^{5x}}{5}|_0^1 - \int_0^1 2xe^{5x}/5dx$$

$$= \frac{e^5}{5} - 2x\frac{e^{5x}}{25}|_0^1 + \int_0^1 2e^{5x}/25dx$$
$$= \frac{e^5}{5} - \frac{2e^5}{25} + \frac{2e^{5x}}{125}|_0^1$$
$$= \frac{e^5}{5} - \frac{2e^5}{25} + \frac{2e^5}{125} - \frac{2}{125}.$$

so

$$\|u\|_{L_2} = \sqrt{\frac{e^{10}}{10} - \frac{1}{10} + \frac{1}{5} + \left(\frac{2}{5} - \frac{4}{25} + \frac{4}{125}\right)e^5 - \frac{4}{125}}.$$

$\|u\|_{\mathcal{C}} = \max_{x\in[0,1]} |u(x)| = u(1) = e^5 + 1.$

$|u|_1^2 = \int_0^1 u'(x)^2 dx = \int_0^1 (5e^{5x}+2x)^2 dx$ can also be evaluated using integration by parts.

$\|u\|_1 = \sqrt{\|u\|_{L_2}^2 + |u|_1^2}.$

0.10.
- We let $a(x) = 1 > 0$, $b(x) = 8.125\pi \cot((1+x)\pi/8)$, $c(x) = \pi^2 > 0$, and $f(x) = -3\pi^2$. These are all smooth functions on $\bar{\Omega}$.
- Therefore, the Green's Function Theorem tells us that the solution exists (and is equal to the function in the theorem minus $2.0761 U_0(x)$ minus $2.2929 U_1(x)$).
- Cor 2.2a says that the solution is unique.
- Since $f(x) < 0$, the Maximum Principle tells us that $\max_{x\in\Omega} u(x) \le \max(-2.0761, -2.2929, 0) = 0$.
- Letting $v(x) = -3$, we see that

  $$-v'' + 8.125\pi \cot((1+x)\pi/8)v' + \pi^2 v = -3\pi^2$$

  and $v(0) = v(1) = -3$. Therefore the Monotonicity Theorem (Cor 2.2c in the notes) says that $u(x) \ge v(x)$ for $x \in \bar{\Omega}$.
- Therefore we conclude $-3 \le u(x) \le 0$ for $x \in \bar{\Omega}$.

Note on how I constructed the problem: The true solution to the problem is $u(x) = \cos((1+x)\pi/8) - 3$, which does indeed have the properties we proved about it. But we can obtain a lot of information about the solution (as illustrated in this problem) without ever evaluating it!

0.11. (a) From p.6,

$$|u|_{C^2(\bar{\Omega})} = \max_{x\in[0,1]} |u''(x)|.$$

(b) Taylor series tells us that there is a point $\xi \in [x, x+h]$ so that

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(\xi).$$

Therefore,

$$\left| u'(x) - \frac{u(x+h)-u(x)}{h} \right| = \frac{h}{2}|u''(\xi)| \le Ch|u|_{C^2(\bar{\Omega})},$$

at least if $x, x+h \in \bar{\Omega}$.

0.12. (a) The derivation proceeds as for the problem with the old boundary conditions:

$$\int_0^1 (-u'' + \pi u)v dx = \int_0^1 fv dx$$

$$\int_0^1 (u'v' + \pi uv)dx - u'v \,\Big|_0^1 = \int_0^1 fv dx$$

but this time the boundary term is zero since $u'(0) = u'(1) = 0$. Therefore, the weak formulation is

$$\int_0^1 (u'v' + \pi uv)dx = \int_0^1 fv dx$$

for all $v \in H^1$.

(b) As before, we reverse the argument: for all $v \in H^1$,

$$\begin{aligned}\int_0^1 fv dx &= \int_0^1 (u'v' + \pi uv)dx \\ &= \int_0^1 (-u'' + \pi u)v dx + u'v \,\Big|_0^1\end{aligned}$$

Since $H_0^1 \subset H^1$, we know that for all $v \in H_0^1$,

$$\int_0^1 fv dx = \int_0^1 (-u'' + \pi u)v dx,$$

and, as before, this shows that u satisfies the differential equation.

Once we know that u satisfies the differential equation, then for any $v \in H^1$, we have

$$\int_0^1 (-u'' + \pi u)v dx - \int_0^1 fv dx = 0,$$

so

$$u'v \,\Big|_0^1 = u'(1)v(1) - u'(0)v(0) = 0$$

for all $v \in H^1$. If we choose a v satisfying $v(1) = 0$ and $v(0) = 1$, we see that $u'(0) = 0$. Similarly choose a v so that $v(1) = 1$ to conclude that $u'(1) = 0$, so u also satisfies the boundary conditions.

0.13. First, notice that none of the theorems in the book apply to this problem, because

- The boundary has corners, so it is not smooth. (The region is a convex polygon, though.)
- The function a in this problem is a matrix, not a scalar:

$$a = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also, some of Chapter 3 (including Thm 3.6) assumes $u = 0$ on the boundary. But we can verify that $c \geq 0$, a is positive definite (the generalization of "non-negative" for matrices), $f = -3\pi^2$ is smooth, and the boundary condition is smooth.

It is true that the solution is unique, by a generalization of Thm 3.6 or the Green's theorem.

The solution exists, by a generalization of Green's theorem or a generalization of Cor 3.2a (class notes).

A generalization of the Maximum Principle shows that u in Ω is bounded above by its maximum on the boundary, which is 2.

As in Quiz 1, notice that $v = -3$ satisfies the same differential equation, with boundary condition $v = -3$. So let $w = u - v$. Applying the minimum principle, we see that $w \geq \min(w_\Gamma, 0) = 0$, so $u \geq -3$ in Ω.

0.14. Rearranging, we get

$$(1 + ka/2)u^{n+1} = (1 - ka/2)u^n$$

so

$$u^{n+1} = \frac{1 - ka/2}{1 + ka/2} u^n$$

and this is stable when

$$\left| \frac{1 - ka/2}{1 + ka/2} \right| < 1,$$

which holds for all $k > 0$. (This is Crank-Nicolson.)

0.15.

$$\begin{aligned}
F[2,1] &= 2^2 \int_{-\infty}^{\infty} \sin(\pi x)\psi(2^2 x - 1)dx \\
&= 4 \int_{1/4}^{1/2} \sin(\pi x)\psi(4x - 1)dx \\
&= 4 \int_{1/4}^{3/8} \sin(\pi x)dx - 4 \int_{3/8}^{1/2} \sin(\pi x)dx \\
&= \frac{-4}{\pi} \cos(\pi x)|_{1/4}^{3/8} + \frac{4}{\pi} \cos(\pi x)|_{3/8}^{1/2} \\
&= \frac{4}{\pi}(-\cos(3\pi/8) + \cos(\pi/4) + \cos(\pi/2) - \cos(3\pi/8)).
\end{aligned}$$

0.16. The finite element equations are derived from the weak formulation of our problem, and when we use integration by parts, we leave off the boundary term that we would have gotten at $x = 2/3$, so our equations are wrong. Therefore, as $h \to 0$, we do not converge to this function.

0.17. We have to change to a standard form.

$$\begin{aligned} \text{Let} \quad u_1 &= y' \quad \Rightarrow \quad u_1' = y'' \\ u_2 &= y \quad \Rightarrow \quad u_2' = y' \end{aligned}$$

Then,

$$\begin{aligned} u_1' &= 6u_1 - tu_2 + u_2^2 \\ u_2' &= u_1 \end{aligned}$$

Now we know these conditions,

$$u_2(0) = y(0) = 5, \ u_2(1) = y(1) = 0$$

However we don't know $u_1(0)$! What if we want to know $u_1(0)$? Well, we can guess $u_1(0)$ and then get an estimate of $u_2(1)$ by solving the ODE using a solver. If we're lucky we can get $u_2(1) = 0$, otherwise we do this again with another guess.

Now we can realize that $u_2(1)$ is a function of our initial guess z i.e. $u_2(1) = G(z)$. Since we want $u_2(1)$ to be zero, we can define a function, $F(z) = G(z) - 0$. Thus finding $u_1(0)$ is equivalent to finding zero of F.

MATLAB Code:

```
% f.m
function fval = f(t,u)
fval = [6*u(1) - t*u(2) + u(2)^2; u(1)];
% Fun.m
function Fval = Fun(z)
Fval = zeros(size(z));
for i=1:size(z,1),
  [t, y] = ode45(@f, [0 1], [z(i) 5]);
  Fval(i) = y(end, 2);
end
% run.m
z = fzero(@Fun, 5)
```