AMSC 607 / CMSC 764 Homework 7, Fall 2010
Due November 2, before class begins.

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9. Consider using the log barrier method to solve the problem

$$\min_{\boldsymbol{x}} x_1^2 + x_2^2$$

subject to

$$x_1 + x_2 \geq 1.$$

As the barrier parameter μ is changed, we have a curve of minimizers $\boldsymbol{x}(\mu)$ and a curve of Lagrange multipliers $\lambda(\mu)$.

9a. (4) What is the solution to the constrained minimization problem?

9b. (4) What is $\boldsymbol{x}(\mu)$?

9c. (4) What is $\lambda(\mu)$?

9d. (4) Compute the Hessian matrix of the log barrier function at $\mu = 10^{-4}$ and $\boldsymbol{x} = \boldsymbol{x}(\mu)$. What is the **condition number** of the matrix, i.e., the ratio of the two eigenvalues? (If the condition number is large, then it will be difficult to compute Newton directions near this point.)

9e. (4) Given the QR factorization

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix},$$

apply the method on pp. 6-7 of the barrier notes and compute $\boldsymbol{K}$ and $\boldsymbol{G}$.

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10. In this problem, we establish various properties of the log function that will help us in studying its use as a barrier.

10a. (8) Let $f(x) = -\log(x)$ for $x > 0$. Let $\hat{f}(x) = -\log(ax+b)$ where $a, b > 0$. Prove the following properties:

- $|f'''(x)| \le 2(f''(x))^{3/2}$.
- $|\hat{f}'''(x)| \le 2(\hat{f}''(x))^{3/2}$.

10b. Let

$$f(\boldsymbol{x}) = -\sum_{j=1}^{n} \log(x_j).$$

As usual, let $\boldsymbol{H}(\boldsymbol{x})$ denote the Hessian of f.

The domain of f is $D_f = \{\boldsymbol{x} : \boldsymbol{x} > \boldsymbol{0}\}$.

Define $\|\boldsymbol{u}\|_{\boldsymbol{x}}^2 = \boldsymbol{u}^T \boldsymbol{H}(\boldsymbol{x})\boldsymbol{u}$.

Define

$$B_{\boldsymbol{x}}(\boldsymbol{y}, r) = \{\boldsymbol{z} : \|\boldsymbol{z} - \boldsymbol{y}\|_{\boldsymbol{x}} < r\}.$$

Prove the following properties:

- (5) For any $\boldsymbol{x} \in D_f$, $B_{\boldsymbol{x}}(\boldsymbol{x}, 1) \subset D_f$.
- (7) If $\boldsymbol{y} \in B_{\boldsymbol{x}}(\boldsymbol{x}, 1)$ and if $\boldsymbol{v} \ne \boldsymbol{0}$, then

$$1 - \|\boldsymbol{y} - \boldsymbol{x}\|_{\boldsymbol{x}} \le \frac{\|\boldsymbol{v}\|_{\boldsymbol{y}}}{\|\boldsymbol{v}\|_{\boldsymbol{x}}} \le \frac{1}{1 - \|\boldsymbol{y} - \boldsymbol{x}\|_{\boldsymbol{x}}}.$$

Hint: Show that $\|\boldsymbol{v}\|_{\boldsymbol{y}}^2 \le \|\boldsymbol{v}\|_{\boldsymbol{x}}^2 \max_j (x_j/y_j)^2$. Then show that $y_j/x_j > 1 - |y_j/x_j - 1|$.