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CONSTRAINED MATRIX SYLVESTER EQUATIONS*

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Dedicated to Gene Golub on the occasion of his 60th birthday, with gratitude for his tradition of fruitful research in linear algebra, inspired by applications.

Abstract. The problem of finding matrices L and T satisfying $TA - FT = LC$ and $TB = 0$ is considered. Existence conditions for the solution are established and an algorithm for computing the solution is derived. Conditions under which the matrix $[C^T, T^T]$ is full rank are also discussed. The problem arises in control theory in the design of reduced-order observers that achieve loop transfer recovery.

Key words. Sylvester operator, matrix Liapunov equation, loop transfer recovery

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1. Introduction. In this paper we consider the following problem: Let n, m, and p be given integers. Given $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times p}$, $C \in \mathcal{R}^{m\times n}$, and $F \in \mathcal{R}^{(n-m)\times(n-m)}$, find $L \in \mathcal{R}^{(n-m)\times m}$ and $T \in \mathcal{R}^{(n-m)\times n}$ such that

$$TA - FT = LC, \tag{1}$$

$$TB = 0, \tag{2}$$

$$\text{and} \quad \begin{bmatrix} T \\ C \end{bmatrix} \quad \text{is full rank.} \tag{3}$$

Sylvester [10] considered the homogeneous version of (1) in an 1884 paper. For this reason, (1) is often called a matrix Sylvester equation. Liapunov considered (1) with $A^T = F$ and $LC = I$ in an 1892 monograph [6].

The constrained Sylvester problem (1), (2), and (3) arises in control theory, in the design of reduced-order observers that achieve precise loop transfer recovery [11], [7]. Here, the state model of the system is

$$\dot{x} = Ax + Bu,$$
$$y = Cx,$$

and the observer $z \in \mathcal{R}^{(n-m)\times 1}$ satisfies

$$\dot{z} = Fz + (TB)u + Ly.$$

Tsui [11] has shown that the constrained Sylvester problem is the relevant one to consider in the design of L and T.

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In §2 we discuss existence and uniqueness of solutions to matrix Sylvester equations. The section following that concerns existence and uniqueness of solutions to the constrained Sylvester problem (1) and (2). The computational algorithm developed in that section is summarized in §4. In §5 we consider conditions under which that algorithm produces a solution to the full problem (1), (2), and (3).

2. Existence of solutions to matrix Liapunov equations. It is well known that a matrix Liapunov equation

$$T\mathcal{A} - \mathcal{B}T = \mathcal{C}$$

has a unique solution T for every choice of $\mathcal{C}$ if and only if $\mathcal{A} \in \mathcal{R}^{n\times n}$ and $\mathcal{B} \in \mathcal{R}^{m\times m}$ have no common eigenvalues. In this section we briefly review this result and related results for the case of common eigenvalues. Our purpose is merely to establish enough notation to discuss conditions under which the full rank condition (3) is violated. Therefore, to simplify the discussion, we consider only the case in which $\mathcal{A}$ and $\mathcal{B}$ each have a complete set of eigenvectors. The general case is studied using the Jordan canonical forms of these matrices (see, for example, [4, Chap. 8]) but leads to the same conclusions.

Let the eigendecomposition of $\mathcal{A}$ be $\mathcal{A} = U_\mathcal{A} D_\mathcal{A} U_\mathcal{A}^{-1}$, where $D_\mathcal{A}$ is diagonal with elements α_j, $j = 1, \cdots, n$. Similarly, let $\mathcal{B} = U_\mathcal{B} D_\mathcal{B} U_\mathcal{B}^{-1}$, where $D_\mathcal{B}$ is diagonal with elements β_i, $i = 1, \cdots, m$.

Then $T\mathcal{A} - \mathcal{B}T = \mathcal{C}$ is equivalent to

$$U_\mathcal{B}^{-1} T\mathcal{A} U_\mathcal{A} - U_\mathcal{B}^{-1}\mathcal{B}TU_\mathcal{A} = U_\mathcal{B}^{-1}\mathcal{C}U_\mathcal{A},$$

or, with definitions $\hat{\mathcal{C}} = U_\mathcal{B}^{-1}\mathcal{C}U_\mathcal{A}$ and $\hat{T} = U_\mathcal{B}^{-1}TU_\mathcal{A}$,

$$\hat{T}D_\mathcal{A} - D_\mathcal{B}\hat{T} = \hat{\mathcal{C}}.$$

Writing this equation componentwise, we obtain

$$(\alpha_j - \beta_i)\hat{t}_{ij} = \hat{c}_{ij}, \qquad i = 1, \cdots, m, \qquad j = 1, \cdots, n. \tag{4}$$

This leads to the standard result that there is a unique solution $\hat{T}$ (and therefore a unique T) for every choice of $\mathcal{C}$ if and only if $\alpha_j - \beta_i \neq 0$ for all values of i and j.

If any $\alpha_J - \beta_I = 0$, then there is no solution to (4) if $\hat{c}_{IJ} \neq 0$, and an infinite number of solutions if $\hat{c}_{IJ} = 0$, since $\hat{t}_{IJ}$ is then arbitrary. Since $T = U_\mathcal{B}\hat{T}U_\mathcal{A}^{-1}$, each arbitrary component of $\hat{T}$ contributes a term $\hat{t}_{IJ}u_I v_J^T$ to T, where u_I is the Ith column of $U_\mathcal{B}$ and v_J^T is the Jth row of $U_\mathcal{A}^{-1}$.

In the problem of interest, the matrix $\mathcal{C}$ is LC, where L is to be determined, so it is sometimes possible to produce a solution to the Liapunov equation even if there are common eigenvalues.

Later we will be interested in conditions that ensure that a matrix related to $\hat{T}$ and $\hat{\mathcal{C}}$ be full column rank. Suppose there exists a nonzero vector u such that $\mathcal{A}u = \lambda u$ and $\mathcal{C}u = 0$. (This is equivalent to saying that the control system is *not observable.*) If λ is a simple eigenvalue of $\mathcal{A}$, then a column of $\hat{\mathcal{C}}$ and the corresponding column of $\hat{T}$ will be zero. If λ is a multiple eigenvalue, then all columns of $\hat{\mathcal{C}}$ may be nonzero, but there will be linear dependence among columns of $\hat{\mathcal{C}}$ corresponding to eigenvectors of that eigenvalue. In this case, the corresponding columns of $\hat{T}$ will have the same linear dependence, since the values α_j in (4) are all equal.

3. Development of an algorithm for the constrained problem. We examine the question of existence and uniqueness of the solution to (1) and (2), and we develop an algorithm for determining the solution if it exists.

Note that there are $(n-m)p$ equations in (2) and $(n-m)n$ equations in (1). There are $(n-m)n$ unknowns in T and $(n-m)m$ unknowns in L, so there are more equations than unknowns if $m < p$.

We may assume that B has full column rank; if not, throwing away the redundant columns does not change the problem. Therefore, the number of rows n in B must be greater than the number of columns p; otherwise, the only solution to $TB = 0$ is $T = 0$. (This is an explanation of the fact that loop transfer recovery cannot be accomplished if the circuit is broken at an "output point.")

Therefore, we may assume that $n > p$ *and* $\operatorname{rank}(B) = p$.

We can eliminate the constraint $TB = 0$ by using the QR factorization of B to define an unconstrained matrix Z. To do this, factor B as

$$B = W \begin{bmatrix} S \\ 0 \end{bmatrix}, \tag{5}$$

where $S \in \mathcal{R}^{p \times p}$ is full rank and $W \in \mathcal{R}^{n \times n}$ is an orthogonal matrix: $W^T W = I$. If we partition W into its first p columns W_1 and its remaining $n-p$ columns W_2, we have

$$TB = T[W_1, W_2] \begin{bmatrix} S \\ 0 \end{bmatrix} = TW_1 S.$$

Since the columns of W_2 form a basis for the orthogonal complement of the subspace spanned by the columns of W_1, and since $TW_1S = 0$ if and only if $TW_1 = 0$, we know that

$$T = ZW_2^T \tag{6}$$

for some matrix $Z \in \mathcal{R}^{(n-m) \times (n-p)}$.

Substituting this in the Sylvester equation (1) and multiplying on the right by the nonsingular matrix W, we obtain

$$ZW_2^T A[W_1, W_2] - FZW_2^T[W_1, W_2] = LC[W_1, W_2].$$

This yields the two relations

$$ZA_2 - FZ = LC_2, \tag{7}$$

$$ZA_1 = LC_1, \tag{8}$$

where $A_1 = W_2^T A W_1 \in \mathcal{R}^{(n-p) \times p}$, $A_2 = W_2^T A W_2 \in \mathcal{R}^{(n-p) \times (n-p)}$, $C_1 = CW_1 \in \mathcal{R}^{m \times p}$, and $C_2 = CW_2 \in \mathcal{R}^{m \times (n-p)}$.

We now consider two cases, based on the relation between p, the number of controls, and m, the number of observed variables. *We assume in both cases that* C_1 *is full rank.*

3.1. Case I: $p > m$**.** As noted above, in this situation, there are more equations than unknowns, and in general, no solution exists.

The RQ factorization of the $m \times p$ matrix C_1 is

$$C_1 = [\hat{R}, 0] \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where $Q_1 \in \mathcal{R}^{m \times p}$, $Q_2 \in \mathcal{R}^{(p-m) \times p}$, $\hat{R} \in \mathcal{R}^{m \times m}$, and $\hat{R}$ has rank m.

Now (8) gives us the relation

$$ZA_1Q^T = LC_1Q^T = [L\hat{R}, 0],$$

or, letting $A_1Q^T = G = [G_1, G_2]$,

$$ZG_1 = L\hat{R}, \qquad ZG_2 = 0.$$

Using this formula in (7), we obtain

$$ZA_2 - FZ = ZG_1\hat{R}^{-1}C_2,$$

or

$$ZG_2 = 0, \tag{9}$$

$$Z(A_2 - G_1\hat{R}^{-1}C_2) - FZ = 0. \tag{10}$$

We now have a problem in exactly the same form as the original equations (2) and (1), except that the Sylvester equation (10) is homogeneous. Further reduction proceeds exactly as above in order to find a change of variables that produces an unconstrained Sylvester equation. However, unless $A_2 - G_1\hat{R}^{-1}C_2$ and F have at least one common eigenvalue, the only solution to (10) is $Z = 0$.

We will not consider this case further.

3.2. Case II: $p \leq m$**.** Consider a QR factorization of the $m \times p$ matrix C_1 as

$$C_1 = [Q_1, Q_2] \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix},$$

where $Q_1 \in \mathcal{R}^{m \times p}$, $Q_2 \in \mathcal{R}^{m \times (m-p)}$, $\hat{R} \in \mathcal{R}^{p \times p}$, and $\hat{R}$ has rank p.

Now let

$$\hat{L} = LQ = [\hat{L}_1, \hat{L}_2],$$

where $\hat{L}_1 \in \mathcal{R}^{(n-m) \times p}$ and $\hat{L}_2 \in \mathcal{R}^{(n-m) \times (m-p)}$. From (8), we have that

$$ZA_1 = LC_1 = \hat{L} \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix},$$

so

$$\hat{L}_1 = ZA_1\hat{R}^{-1}.$$

Using this formula in (7), and letting

$$Q^TC_2 = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix},$$

we obtain

$$ZA_2 - FZ = [\hat{L}_1, \hat{L}_2]Q^T C_2 = [ZA_1\hat{R}^{-1}, \hat{L}_2]\begin{bmatrix} E_1 \\ E_2 \end{bmatrix},$$

or

$$Z(A_2 - A_1\hat{R}^{-1}E_1) - FZ = \hat{L}_2 E_2. \tag{11}$$

We have succeeded in reducing the original constrained Sylvester problem (2) and (1) to an unconstrained one through the change of variables $T = ZW_2^T$. The $(n-m)(m-p)$ entries of $\hat{L}_2$ are free parameters. For each choice of $\hat{L}_2$, (11) has a unique solution, as long as the matrices $\hat{A} \equiv A_2 - A_1\hat{R}^{-1}E_1$ and F have no common eigenvalues. Section 2 discusses the existence of the solution in the case of common eigenvalues.

Note. If C_1 fails to have full rank, then we have

$$C_1 = [Q_1, Q_2]\begin{bmatrix} \hat{R} & \hat{P} \\ 0 & 0 \end{bmatrix},$$

where $Q_1 \in \mathcal{R}^{m\times r}$, $Q_2 \in \mathcal{R}^{m\times(m-r)}$, $\hat{R} \in \mathcal{R}^{r\times r}$, $\hat{P} \in \mathcal{R}^{r\times(p-r)}$, and $\hat{R}$ has rank r. A derivation following the steps above leads to the Sylvester equation (11) but with the side constraint

$$Z(A_{12} - A_{11}\hat{R}^{-1}\hat{P}) = 0,$$

where A_{11} denotes the first r columns of A_1 and A_{12} denotes the remaining columns. Thus we reduced the problem to a smaller constrained Sylvester equation of the same form as the original, and the process needs to be repeated. □

4. The resulting algorithm. The following algorithm computes a solution to the constrained Sylvester problem (1) and (2).

$$TA - FT = LC, \tag{12}$$

$$TB = 0, \tag{13}$$

under the assumptions that $n > m > p$, $\text{rank}(CB) = p$, and (redundantly) $\text{rank}(B) = p$.

Step 1. Factor B into its QR factors

$$B = W\begin{bmatrix} S \\ 0 \end{bmatrix},$$

where $S \in \mathcal{R}^{p\times p}$ is full rank and $W \in \mathcal{R}^{n\times n}$ is an orthogonal matrix: $W^T W = I$. Partition W into its first p columns W_1 and its remaining $n-p$ columns W_2.

Step 2. Set $A_1 = W_2^T AW_1 \in \mathcal{R}^{(n-p)\times p}$, $A_2 = W_2^T AW_2 \in \mathcal{R}^{(n-p)\times(n-p)}$, $C_1 = CW_1 \in \mathcal{R}^{m\times p}$, and $C_2 = CW_2 \in \mathcal{R}^{m\times(n-p)}$.

Step 3. Perform a QR factorization of the $m \times p$ matrix C_1 as

$$C_1 = [Q_1, Q_2]\begin{bmatrix} \hat{R} \\ 0 \end{bmatrix},$$

where $Q_1 \in \mathcal{R}^{m\times p}$, $Q_2 \in \mathcal{R}^{m\times(m-p)}$, $\hat{R} \in \mathcal{R}^{p\times p}$, and $\hat{R}$ has rank p.

Step 4. Let

$$Q^T C_2 = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix},$$

where $E_1 \in \mathcal{R}^{p\times(n-p)}$ and $E_2 \in \mathcal{R}^{(m-p)\times(n-p)}$.

Step 5. Solve the Sylvester matrix equation $Z(A_2 - A_1\hat{R}^{-1}E_1) - FZ = \hat{L}_2 E_2$, where the entries of $\hat{L}_2$ are chosen randomly.

Step 6. Set $\hat{L}_1 = ZA_1\hat{R}^{-1}$ and $L = [\hat{L}_1, \hat{L}_2]Q^T$.

Step 7. Set $T = ZW_2^T$.

The software tasks needed to implement the algorithm are matrix multiplication, the QR factorization [3], and an algorithm for solving unconstrained Sylvester problems [1], [5]. The highest-order terms in the operation counts are cubic in n, m, and p, with constants depending on the specific choice of software. There is substantial potential parallelism in the computation, since there are well-known parallel algorithms for each of these basic tasks; see, for example, [9], [8], and the references therein.

For examples of applications of this algorithm to loop transfer recovery, see [7].

5. Necessary conditions and sufficient conditions for solutions to the full problem. In this section we develop some conditions that are necessary in order to obtain a solution T to the problem (1), (2), and (3). For ease of reference, we define

$$\mathcal{T} = \begin{bmatrix} T \\ C \end{bmatrix},$$

where $T \in \mathcal{R}^{(n-m)\times n}$ and $C \in \mathcal{R}^{m\times n}$.

Recall that we already assumed, without loss of generality, that $n > p$ and $\text{rank}(B) = p$. We will consider the case $p < m$, since the other case has a solution only under accidental conditions. We also restrict ourselves to the case in which F has no eigenvalues in common with the matrix $\hat{A} \equiv A_2 - A_1\hat{R}^{-1}E_1$ of (11). Under these circumstances, (1) and (2) always have a solution, and the only issue is the rank of $\mathcal{T}$.

Recall that W and Q are $n \times n$ orthogonal matrices. We note that $\mathcal{T}$ is full rank if and only if the matrices

$$\mathcal{T}W = \begin{bmatrix} T \\ C \end{bmatrix} W = \begin{bmatrix} 0 & Z \\ C_1 & C_2 \end{bmatrix}$$

and

$$Q^T \mathcal{T} W = \begin{bmatrix} 0 & Z \\ Q_1^T C_1 & E_1 \\ Q_2^T C_1 & E_2 \end{bmatrix} = \begin{bmatrix} 0 & Z \\ \hat{R} & E_1 \\ 0 & E_2 \end{bmatrix},$$

are full rank, and it is sometimes easier to work with these.

NECESSARY AND SUFFICIENT CONDITION 1. *For $\mathcal{T}$ to be full rank, it is necessary and sufficient that $Q_1^T C_1$ (or, equivalently, C_1) and $[Z^T, E_2^T]$ be full rank.*

Our goal is to express such conditions more obviously in terms of the data matrices A, B, C, and F.

NECESSARY CONDITION 1. *The matrix C must have full rank m.*

Proof. If $\alpha^T C = 0$ for some nonzero α, then $[0^T, \alpha^T]\mathcal{T} = 0$ and $\mathcal{T}$ is not full rank. □

NECESSARY CONDITION 2. *The system A, B, C must be regular* [2, p. 661], *i.e., the matrix CB must have full rank p.*

Note. This condition is also necessary and sufficient for the existence of a full rank triangular factor $\hat{R}$ for C_1. □

Proof. Recall from (5) that the first p columns of W span the range of B. For $\mathcal{T}W$ to be full rank, it is necessary that C_1 have full column rank p. Now, $C_1 = CW_1 = CBS^{-1}$, so it is necessary that CB be full rank. □

NECESSARY CONDITION 3. *The system must be observable, i.e., the only vector y satisfying $Ay = \mu y$ and $Cy = 0$ must be the vector $y = 0$.*

Note. If we add the assumption that A and F have no common eigenvalues, then this result is easy to prove. Suppose there is a nonzero y satisfying $Ay = \mu y$ and $Cy = 0$. Then, since $TA - FT = LC$,

$$TAy - FTy = (\mu I - F)Ty = LCy = 0,$$

so $Ty = 0$. This fact, along with $Cy = 0$, implies that $\mathcal{T}$ is rank deficient. If we avoid this extra assumption on the eigenvalues of A and F, then the outline of the proof is similar, but it must be done in terms of the reduced Sylvester equation (11). □

Proof. Suppose there is a nonzero y satisfying $Ay = \mu y$ and $Cy = 0$, and let

$$W^T y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

so that $y = W_1 y_1 + W_2 y_2$. Then

$$Ay = A(W_1 y_1 + W_2 y_2) = \mu(W_1 y_1 + W_2 y_2).$$

Multiplying by W_2^T and using the definitions following (8), we obtain

$$W_2^T A W_1 y_1 + W_2^T A W_2 y_2 = A_1 y_1 + A_2 y_2 = \mu y_2.$$

Now, y_2 is an eigenvector of $\hat{A}$, since

$$\begin{aligned} \hat{A} y_2 &= (A_2 - A_1 \hat{R}^{-1} Q_1^T C W_2) y_2 \\ &= \mu y_2 - A_1 (y_1 + \hat{R}^{-1} Q_1^T C W_2 y_2) \\ &= \mu y_2 - A_1 (y_1 + \hat{R}^{-1} Q_1^T C (y - W_1 y_1)) \\ &= \mu y_2 - A_1 (y_1 + \hat{R}^{-1} Q_1^T C y - \hat{R}^{-1} Q_1^T C W_1 y_1) \\ &= \mu y_2, \end{aligned} \tag{14}$$

since $Cy = 0$ and $CW_1 = Q_1 \hat{R}$, so $Q_1^T C W_1 = \hat{R}$. Further,

$$E_2 y_2 = Q_2^T C W_2 y_2 = Q_2^T C (y - W_1 y_1) = Q_2^T C W_1 y_1 = 0,$$

since $Q_2^T C W_1 = 0$.

Consider the reduced Sylvester equation (11) $Z\hat{A} - FZ = \hat{L}_2 E_2$, and multiply by y_2:

$$Z\hat{A} y_2 - FZ y_2 = \mu Z y_2 - FZ y_2 = (\mu I - F) Z y_2 = \hat{L}_2 E_2 y_2 = 0,$$

so, since μ is not an eigenvalue of F, we must have $Zy_2 = 0$.

We have a vector y_2 that satisfies $E_2 y_2 = 0$ and $Zy_2 = 0$, so $Q^T TW$ is rank deficient. Thus, observability is necessary for a full rank $\mathcal{T}$. □

These necessary conditions are not sufficient, as shown by the following example.

Example. Let

$$A = \begin{bmatrix} -3 & 0 & -3 \\ 0 & 1 & 1 \\ -1 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to see that C, B, and CB are all full rank. The eigenvectors of A are the columns of the matrix

$$\begin{bmatrix} 0 & 1.0000 & 1.0000 \\ 1 & -0.0819 & 0.4523 \\ 0 & 0.4343 & -0.7676 \end{bmatrix},$$

so the system is observable. We calculate

$$E_2 = [\ 1 \quad 0\ ], \qquad \hat{A} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix},$$

so in using the decompositions in §2 to solve the reduced Sylvester equation of Step 5 we obtain

$$\hat{\mathcal{C}} = \hat{L}_2 [\ 1 \quad 0\ ].$$

Since this matrix has a zero column regardless of the choice of $\hat{L}_2$, the solution matrix will as well, and we will have a rank deficient $\mathcal{T}$. □

NECESSARY CONDITION 4. *The reduced system must be observable, i.e., the only vector y satisfying $\hat{A}y = \mu y$ and $E_2 y = 0$ must be the vector $y = 0$.*

Proof. The proof of this result, motivated by the example above, follows from the discussion at the end of §2. □

If Necessary Conditions 1–4 are satisfied, then we conjecture that there exists a choice of the matrices $\hat{L}_2$ and F so that the algorithm yields a solution to the constrained Sylvester problem (1), (2) satisfying the full rank condition (3). The first two necessary conditions guarantee the existence of full rank triangular factors for B in Step 1 and C_1 in Step 3. Some freedom in the choice of F is needed so that its eigenvalues are distinct from those of $\hat{A}$, guaranteeing the existence of a solution of the reduced Sylvester equation in Step 5. The remaining freedom in F, the other two necessary conditions, and freedom in the choice of $\hat{L}_2$ in Step 5 can be used in satisfying the condition that $[Z^T, E_2^T]$ be full rank. In practice, of course, the eigenvalues of any given matrix F will virtually always be distinct from those of A, and thus the algorithm will successfully compute a solution to the constrained problem, although it may not be possible to satisfy the full rank condition.

See [7] for numerical computations using this algorithm in control design of a flexible arm, helicopter flight, and aircraft flight dynamics.

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