

CMCS427 Dot product review

Computing the dot product

The dot product can be computed via

- a) Cosine rule $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$
- b) Coordinate-wise $\mathbf{a} \cdot \mathbf{b} = a_x * b_x + a_y * b_y$

Exercises:

1) If $\mathbf{a} \cdot \mathbf{b}$, $|\mathbf{a}|$ and $|\mathbf{b}|$ all equal 1, what's the angle between the vectors?

Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$ becomes $1 = \cos \theta$, then $\theta = 0$

2) If $\mathbf{a} = \langle 2, -1 \rangle$ and $\mathbf{b} = \langle 3, 2 \rangle$, what is $\mathbf{a} \cdot \mathbf{b}$ and what is the angle between the vectors?

$$\mathbf{a} \cdot \mathbf{b} = \langle 2, -1 \rangle \cdot \langle 3, 2 \rangle = 4 \quad |\mathbf{a}| = 2.2361, \quad |\mathbf{b}| = 3.6056$$

$$\theta = \arccos(4 / (2.2361 * 3.6056)) = 1.0517 \text{ radians} = 60.26 \text{ degrees}$$

3) If $\mathbf{v} \cdot \mathbf{w} = \frac{1}{2} |\mathbf{v}||\mathbf{w}|$ what is the angle between \mathbf{u} and \mathbf{v} ?

This gives $(\mathbf{v} \cdot \mathbf{w}) / (|\mathbf{v}||\mathbf{w}|) = \frac{1}{2}$, so $\arccos(1/2) = 1.0472$ radians, 60 degrees

Properties of the dot product (dot product algebra)

The dot product is

- a) Commutative $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- b) Distributive $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- c) Non-canceling $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ does not mean $\mathbf{b} = \mathbf{c}$ (it does give $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$)
- d) Magnitude squared $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- e) Linear with scalars $s\mathbf{a} \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
- f) From (b) and (d), linear $\mathbf{a} \cdot (s\mathbf{b} + r\mathbf{c}) = s(\mathbf{a} \cdot \mathbf{b}) + r(\mathbf{a} \cdot \mathbf{c})$
- g) Zero if orthogal $\mathbf{a} \cdot \mathbf{b} = 0$ if \mathbf{a} is orthogonal to \mathbf{b}

Example: Rewrite the magnitude of $|\mathbf{a} - \mathbf{b}|^2$ to eliminate $\mathbf{a} \cdot \mathbf{b}$

$$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{a} - 2(\mathbf{a} \cdot \mathbf{b}) + \mathbf{b} \cdot \mathbf{b} = |\mathbf{a}|^2 - 2(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2$$

Exercises:

1) Following the example, rewrite $|2\mathbf{a} - \mathbf{b}|^2$

$$|2\mathbf{a} - \mathbf{b}|^2 = (2\mathbf{a} - \mathbf{b}) \cdot (2\mathbf{a} - \mathbf{b}) = 2\mathbf{a} \cdot 2\mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot 2\mathbf{a} + \mathbf{b} \cdot \mathbf{b} = 2(\mathbf{a} \cdot \mathbf{a}) - 4(\mathbf{a} \cdot \mathbf{b}) + \mathbf{b} \cdot \mathbf{b} = 2|\mathbf{a}|^2 - 4(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2$$

2) If \mathbf{v} is orthogonal to \mathbf{u} and \mathbf{w} separately, is it orthogonal to $\mathbf{u} + \mathbf{w}$?

$$\text{Yes. } \mathbf{v} \cdot (\mathbf{u} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{w} = 0 + 0 = 0$$

3) Simplify the following:

$$\text{a) } (v-w) \cdot w + v \cdot w = v \cdot w - w \cdot w + v \cdot w = 2(v \cdot w) - |w|^2$$

$$\text{b) } (v+w) \cdot v - (v+w) \cdot w = v \cdot v + v \cdot w - v \cdot w - w \cdot w = v \cdot v - w \cdot w = |v|^2 - |w|^2$$

$$\begin{aligned} \text{c) } (v+w) \cdot (v+w) - 2v \cdot w &= v \cdot v + v \cdot w + w \cdot v + w \cdot w - 2v \cdot w \\ &= v \cdot v + v \cdot w + v \cdot w + w \cdot w - 2v \cdot w \\ &= v \cdot v + 2v \cdot w - 2v \cdot w + w \cdot w \\ &= |v|^2 + |w|^2 \end{aligned}$$

Using sign of the dot product for testing angles

If $a \cdot b = 0$, the angle between the vectors is zero

If $a \cdot b > 0$, the angle is acute (less than 90 degrees)

If $a \cdot b < 0$, the angle is obtuse (greater than 90 degrees)

Note: to use this test you generally need a cut off for equal to zero, so $|a \cdot b| < \delta$, δ small

Exercises:

1) Test the following pairs: $\langle 2, 1 \rangle$ and $\langle -1, 1 \rangle$; $\langle 2, 0 \rangle$ and $\langle 0, 1 \rangle$; $\langle 3.4, -3.7 \rangle$ and $\langle 3.7, 3.3 \rangle$.

Pair 1: $\langle 2, 1 \rangle \cdot \langle -1, 1 \rangle = -1$, angle is greater than 90

Pair 2: $\langle 2, 0 \rangle \cdot \langle 0, 1 \rangle = 0$, angle is 0

Pair 3: $\langle 3.4, -3.7 \rangle \cdot \langle 3.7, 3.3 \rangle = 0.37$, angle is less than 90

2) Would the length of the vectors a and b influence the numeric sensitivity of the test? Eg, if a and b are both long vectors, does it make a difference relative to your delta cut off?

Yes, the length does influence the dot product. $(sa) \cdot b = s(a \cdot b)$, so if $s \gg 1$, the dot product is increased. (This is why we often use normalized vectors for such tests).

2) The perp vector in 2D is given by $v_{\text{perp}} = \langle -y, x \rangle$ for the vector $v = \langle x, y \rangle$.

With 3D vectors it isn't as easy to define a single perp vector - there's an infinity of directions rotating around a vector that are candidates. But, given a 3D vector $w = \langle x, y, z \rangle$, can you define a perpendicular vector by inspection that's like the perp vector? A vector w_{perp} such that $w_{\text{perp}} \cdot w = 0$?

There's multiple options. One is $\langle -y, x, 0 \rangle$ so $\langle x, y, z \rangle \cdot \langle -y, x, 0 \rangle = 0$

Dot product and projection

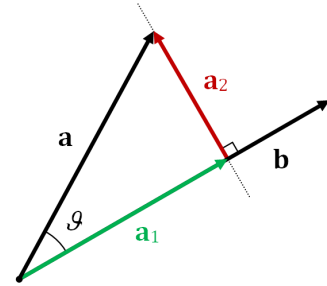
A number of critical applications of the dot product are from using it to find projections.

- The *vector projection* of a onto b is the vector a_1 which is the component of a in the direction of b .
- The *scalar projection* of a onto b is the length of a_1 .

A starting equation is $a_1 = \frac{a \cdot b}{b \cdot b} b$ and $|a_1| = \left| \frac{a \cdot b}{b \cdot b} b \right|$

From this equation we have, if $|b|=1$, that $a_1 = (a \cdot b) \hat{b}$

This gives $|a_1| = |(a \cdot b)| |\hat{b}| = |(a \cdot b)|$



Examples:

- What is the projection of $\langle 2,3,4 \rangle$ onto $\langle 0,1,0 \rangle$? $\langle 0,3,0 \rangle$ with a length of 3.
Projections onto the axes are the vector's coordinate of that axis.
- What is the projection of $\langle 1,2 \rangle$ onto $\langle 1,1 \rangle$?

$$a_1 = \frac{a \cdot b}{b \cdot b} b = \frac{\langle 1,2 \rangle \cdot \langle 1,1 \rangle}{\langle 1,1 \rangle \cdot \langle 1,1 \rangle} \langle 1,1 \rangle = \frac{3}{2} \langle 1,1 \rangle = \langle 1.5, 1.5 \rangle$$

The length is $|\langle 1.5, 1.5 \rangle| = \sqrt{2 \cdot (1.5 \cdot 1.5)} = 2.1213$

- What is the projection of $\langle 0.5, 0.5 \rangle$ onto $\langle 2, 3 \rangle$?

$$a_1 = \frac{a \cdot b}{b \cdot b} b = \frac{\langle 0.5, 0.5 \rangle \cdot \langle 2, 3 \rangle}{\langle 2, 3 \rangle \cdot \langle 2, 3 \rangle} \langle 2, 3 \rangle = \frac{2.5}{13} \langle 2, 3 \rangle = \langle 0.38462, 0.57692 \rangle$$

The vector projection has the following applications (among others):

- Vector rejection. The vector a_2 (see above) that is the component of a orthogonal to b .
- Coordinate frame resolution. The vectors a_1 and a_2 are orthogonal and represent an alternative coordinate frame to the standard x, y frame (in 2D)
- Distance from a point to a line. The distance from a point p to a line defined by point-vector.

Details of each application.

- Vector rejection

Once you have a_1 , then $a = a_1 + a_2$ so $a_2 = a - a_1$.

Example: What is the vector rejection of $\langle 1, 2 \rangle$ onto $\langle 1, 1 \rangle$? Since $a_1 = \langle 1.5, 1.5 \rangle$, then

$$a_2 = \langle 1, 2 \rangle - \langle 1.5, 1.5 \rangle = \langle -0.5, 0.5 \rangle \quad \text{and } a_1 \cdot a_2 = 0$$

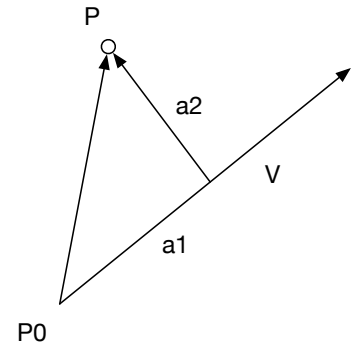
- Coordinate frame resolution

Once you have a_1 and a_2 , you have a pair of orthogonal vectors. If you normalize then you have an orthonormal basis for a new coordinate system (vectors are orthonormal if orthogonal and unit length). Now

$$\hat{a}_1 = \frac{a_1}{|a_1|} = \frac{\langle 1.5, 1.5 \rangle}{\sqrt{1.5^2 + 1.5^2}} = \langle 0.70711, 0.70711 \rangle, \text{ and } \hat{a}_2 = \langle -0.70711, 0.70711 \rangle$$

• Distance from a point to a line

If we're given a line in point-vector form with P_0 and v , then the distance from an arbitrary point P to the line is given by the length of the vector rejection a_2 , with $a = P - P_0$



$$d = |a_2| = \left| (P - P_0) - \frac{(P - P_0) \cdot v}{v \cdot v} v \right|$$

If v is of unit length, then

$$|a_2| = |(P - P_0) - ((P - P_0) \cdot v) \cdot v|$$

Example: What is distance of the point $(3,4)$ from the line defined by $P_0=(0,0)$ and $v=\langle 1,1 \rangle$?

$$d = \left| (P - P_0) - \frac{(P - P_0) \cdot v}{v \cdot v} v \right|$$

$$d = \left| \langle 3,4 \rangle - \frac{\langle 3,4 \rangle \cdot \langle 1,1 \rangle}{\langle 1,1 \rangle \cdot \langle 1,1 \rangle} \langle 1,1 \rangle \right|$$

$$d = \left| \langle 3,4 \rangle - \frac{7}{2} \langle 1,1 \rangle \right| = |\langle 3,4 \rangle - \langle 3.5,3.5 \rangle| = | \langle -0.5,0.5 \rangle | = 0.70711$$

Exercises:

1) What's the vector a_1 and scalar projection of $a=\langle 3,2,1 \rangle$ onto $b=\langle 0,2,2 \rangle$?

$$a_1 = \frac{a \cdot b}{b \cdot b} b = \frac{\langle 3,2,1 \rangle \cdot \langle 0,2,2 \rangle}{\langle 0,2,2 \rangle \cdot \langle 0,2,2 \rangle} \langle 0,2,2 \rangle = 0.75 \langle 0,2,2 \rangle = \langle 0,1.5,1.5 \rangle$$

$$|a_1| = |\langle 0,1.5,1.5 \rangle| = 2.1213$$

2) What the vector rejection a_2 for this example?

$$a_2 = a - a_1 = \langle 3,2,1 \rangle - \langle 0,1.5,1.5 \rangle = \langle 3,0.5,-0.5 \rangle \quad \text{Note that } a_1 \cdot a_2 = 0$$

3) And, putting (1) and (2) together, what's the normalized coordinate frame based on a_1 and a_2 for this case?

Normalize the two vectors a_1 and a_2 (call them A_1 and A_2 for ease of typing.)

$$\begin{aligned} A_1 &= \langle 0,1.5,1.5 \rangle / 2.1213 &= \langle 0, & 0.70711, & 0.70711 \rangle \\ A_2 &= \langle 3,0.5,-0.5 \rangle / 3.0822 &= \langle 0.97333, & 0.16222, & -0.16222 \rangle \end{aligned}$$

4) What's the distance from the point (7,3) to the line given by P0=(1,2) and v=<2,1>?

$$d = \left| (P - P_0) - \frac{(P - P_0) \cdot v}{v \cdot v} v \right|$$

$$d = \left| ((7, 3) - (1, 2)) - \frac{((7, 3) - (1, 2)) \cdot \langle 2, 1 \rangle}{\langle 2, 1 \rangle \cdot \langle 2, 1 \rangle} \langle 2, 1 \rangle \right|$$

$$= 1.7889$$

5) There's another equation for the distance from a point to a line based on the normal vector (hint: consider the perp vector if you have a line in point-vector form.) Develop an equation for the distance from a point P to a line defined by P0 and v, and then apply it to the data in (4).

If the line is defined by $P(t) = P_0 + tv$, then the normal vector is v_{perp} .

Let n be a normalized version of v_{perp} : $n = v_{\text{perp}}/|v_{\text{perp}}|$

The point-normal form for a line is $n \cdot (P - P_0) = 0$.

Here the projection of P-P0 is zero, so the distance from the point to the line is zero.

This same equation gives the distance of P from the line, so is what we want (but for distance, we need the absolute value $|n \cdot (P - P_0)|$)

For this case, with $P=(7,3)$ and $P_0=(1,2)$ and $v=\langle 2,1 \rangle$, we have

$$n = \langle -1, 2 \rangle / |\langle -1, 2 \rangle| = \langle -0.44721, 0.89443 \rangle$$

$$d = |n \cdot (P - P_0)| = |\langle -0.44721, 0.89443 \rangle \cdot ((7,3) - (1,2))| = 1.7889$$

Some of these exercises came from the file below. It has exercises with solutions, so is good for self-study. <https://dnichols30582.edublogs.org/files/2015/12/11.6-Solutions-1eupr84.pdf>