CMCS427 Notes
Finding a local coordinate frame for a parametric curve

Given parametric form for curve, such as this twisted cubic, $p(t) = < t, t^2, t^3 >$, it's possible to compute an orthogonal set of vectors (T, N and B) at each point of the curve which define a local coordinate frame.

Phase 1: Compute unnormalized versions of T, N and B.

This phase is a useful exercise in computing partial derivatives of a parametric curve, and using the cross product to compute the binormal vector.

The first derivative of the curve is the tangent vector T: $T(t) = p'(t) = < 1,2t, 3t^2 >$

The second derivative is the normal vector N: $N(t) = p''(t) = < 0,2,6t >$

However – since the tangent vector is not of unit length, and normalized with respect to arc length, N is not properly a normal vector. You can see that T and N are not orthogonal to each other. For the moment we will ignore this and work on the mechanics of computing the binormal vector.

The binormal vector B is then given by T x N so we have

$$B(t) = \det\left(\begin{bmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{bmatrix}\right)$$

Which gives

$$B(t) = i\begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} - j\begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} + k\begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}$$

$$B(t) =< 6t^2, -6t, 2 >$$

We can observe now that B is orthogonal to T and N:

$$T(t) \bullet B(t) = < 1,2t, 3t^2 > \bullet < 6t^2, -6t, 2 > = 6t^2 - 12t^2 + 6t^2 = 0$$

$$N(t) \bullet B(t) = < 0,2,6t > \bullet < 6t^2, -6t, 2 > = 0 - 12t + 12t = 0$$

At this point we've computed initial values of T, N and B. We don't have normalized, orthogonal frame yet. But, we have that T is the tangent vector, the two vectors T and N define the plane in which the curve is turning, and B is the normal to that plane.

Phase 2: Compute normalized versions of T, N and B

Getting to properly normalized versions of T, N and B adds considerable complexity. The normalized version of T from above is

$$\hat{T}(t) = \frac{< 1,2t, 3t^2 >}{\sqrt{1 + 4t^2 + 9t^4}}$$

And now N, the derivative of T, is (from Wolfram Alpha). This version of N is normal to T.

$$\frac{d}{dt}\left(\frac{\{1, 2t, 3t^2\}}{\sqrt{9|t|^4 + 4|t|^2 + 1}}\right) =$$

$$\left\{-\frac{36t^3 + 8t}{2(9t^4 + 4t^2 + 1)^{3/2}}, \frac{2}{\sqrt{9t^4 + 4t^2 + 1}} - \frac{t(36t^3 + 8t)}{(9t^4 + 4t^2 + 1)^{3/2}}, \frac{6t}{\sqrt{9t^4 + 4t^2 + 1}} - \frac{3t^2(36t^3 + 8t)}{2(9t^4 + 4t^2 + 1)^{3/2}}\right\}$$

You can see that the work to normalize and take derivatives can be tedious and involved.

There are three ways to get a fully orthogonal set of vectors.

A) As above, compute with normalized versions of T and N.

B) Using the versions of T, N and B, computed in Phase 1, now take N2 = T x B. This new vector N2 is orthogonal to T and B, and in the plane spanned by T and the original N.

C) Compute a different version of N2 by Gram Schmit normalization. Subtract from N the component in the same direction as T, and what remains is orthogonal to T.