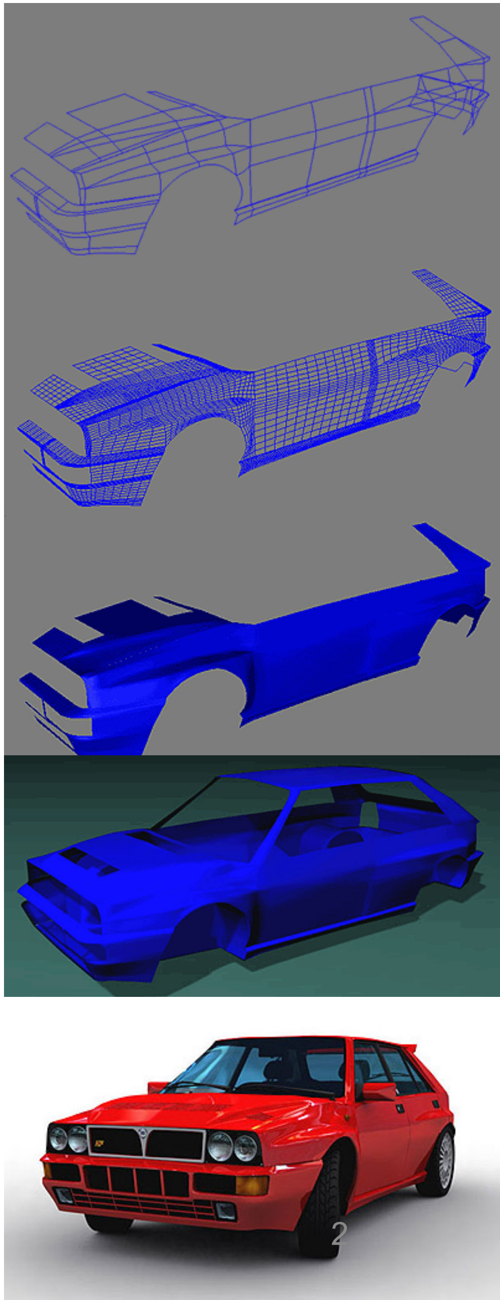
CMSC427
Parametric curves: Hermite, Catmull-Rom, Bezier

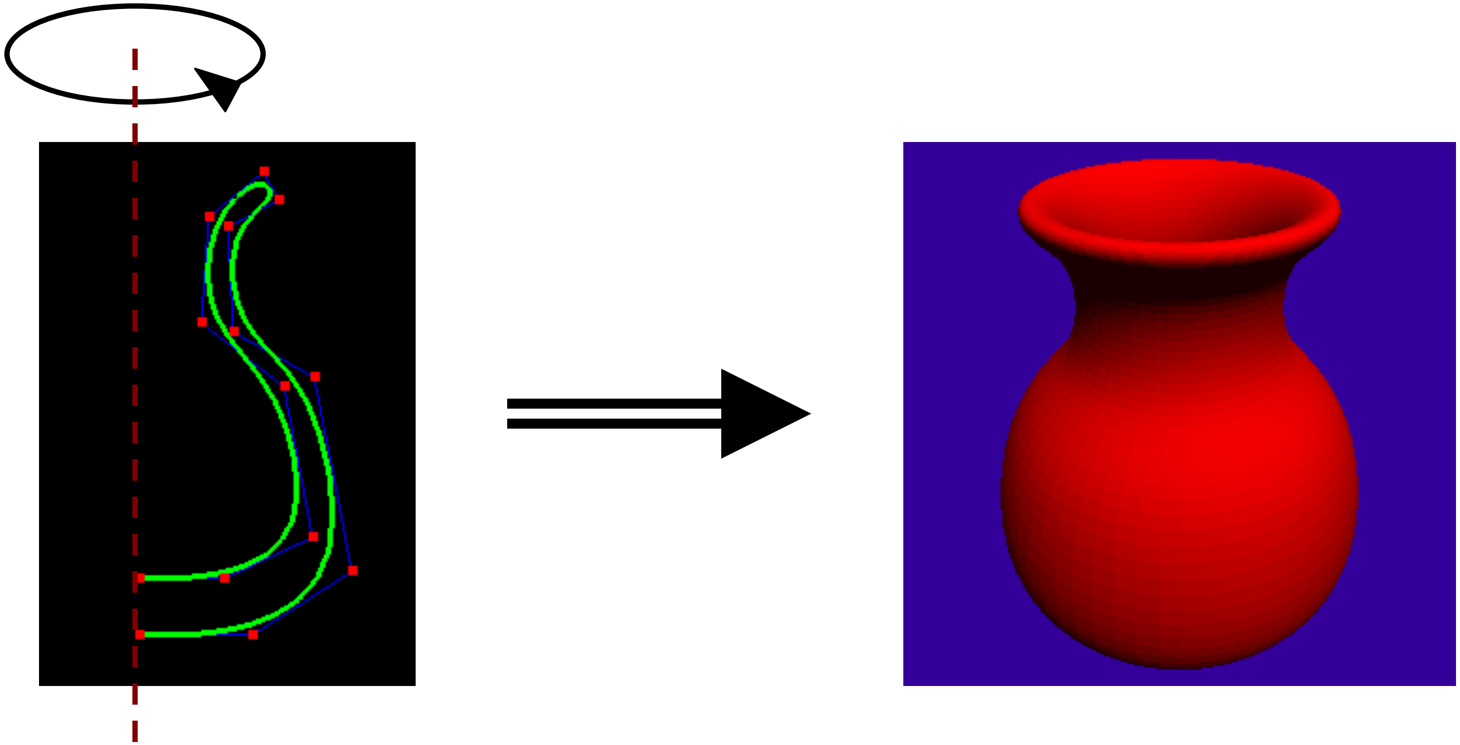
Modeling

- Creating 3D objects
- How to construct complicated surfaces?
- Goal
  - Specify objects with few control points
  - Resulting object should be visually pleasing (smooth)
- Start with curves, then generalize to surfaces



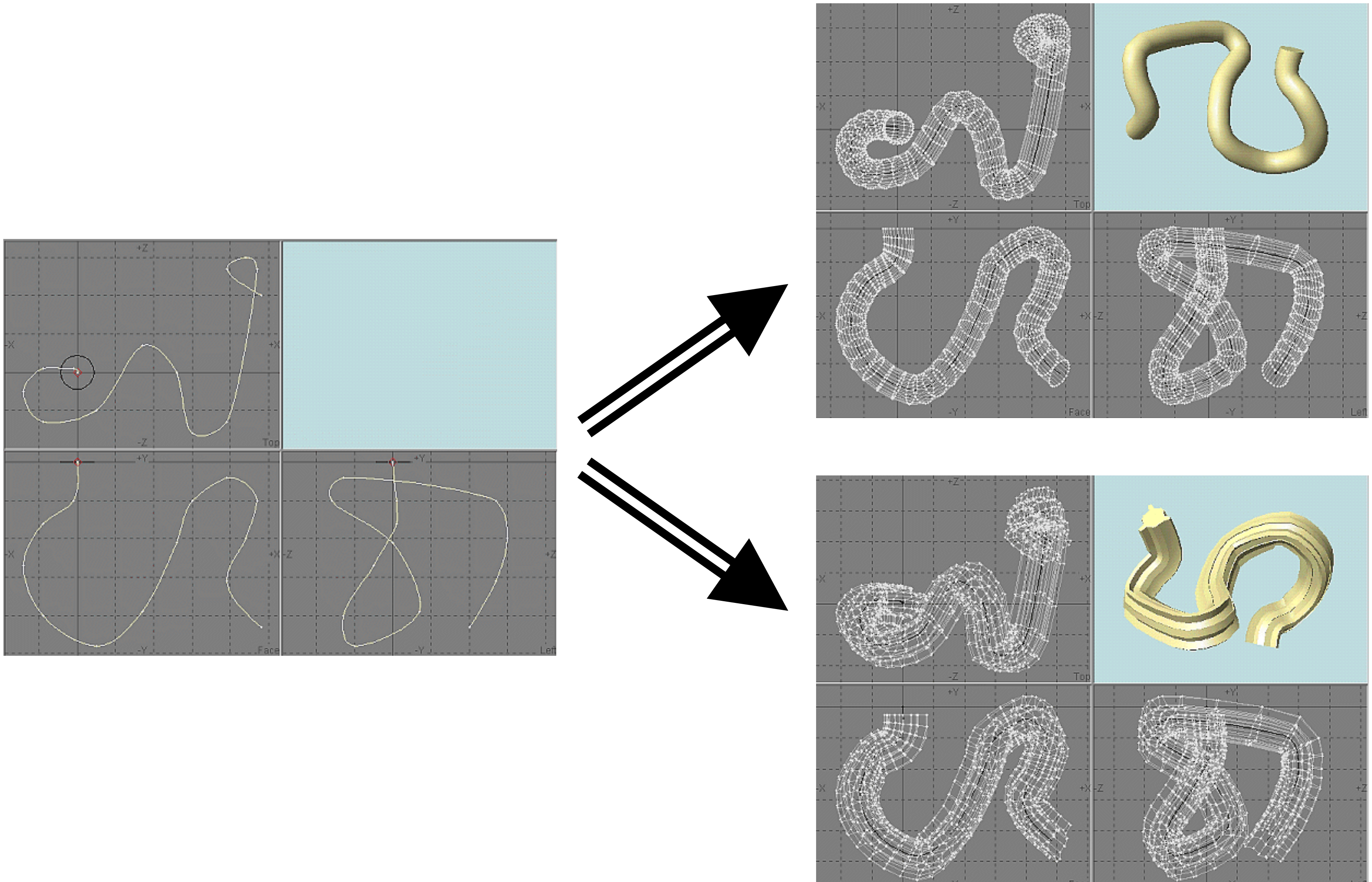
Usefulness of curves

- Surface of revolution



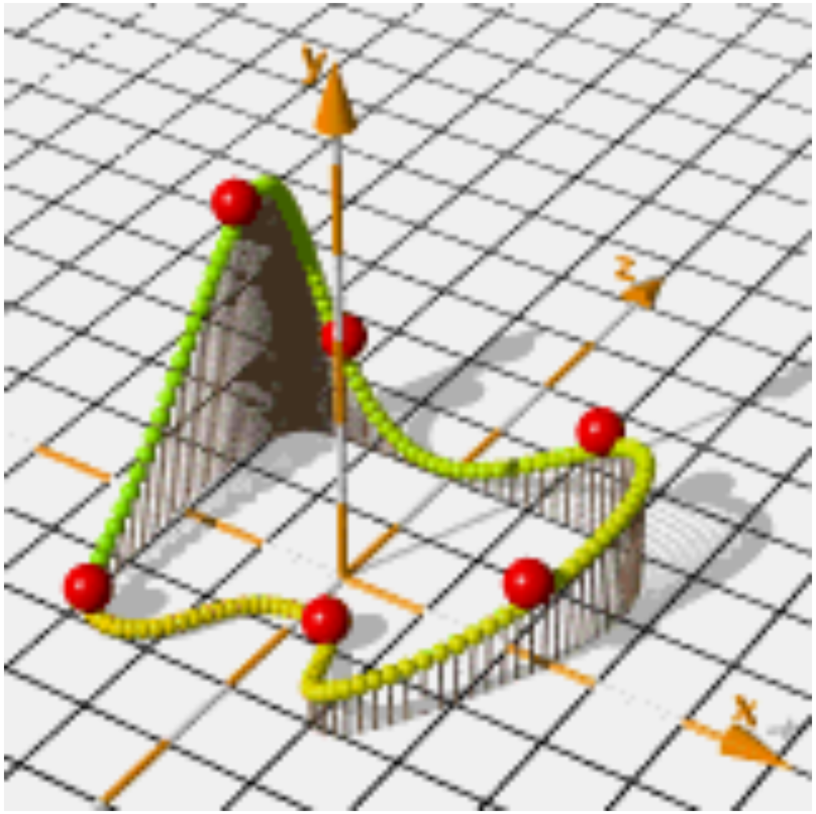
Usefulness of curves

- Extruded/swept surfaces



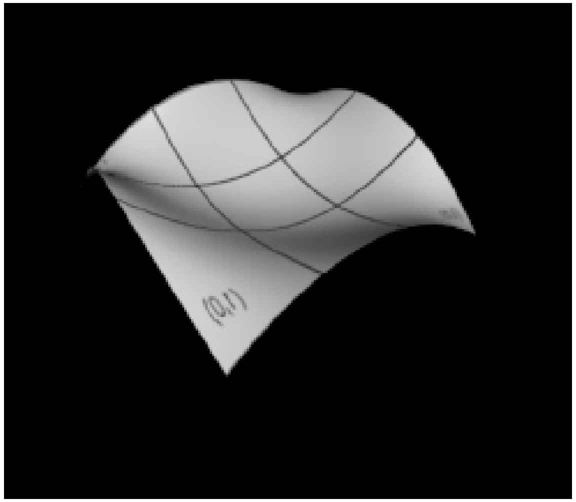
Usefulness of curves

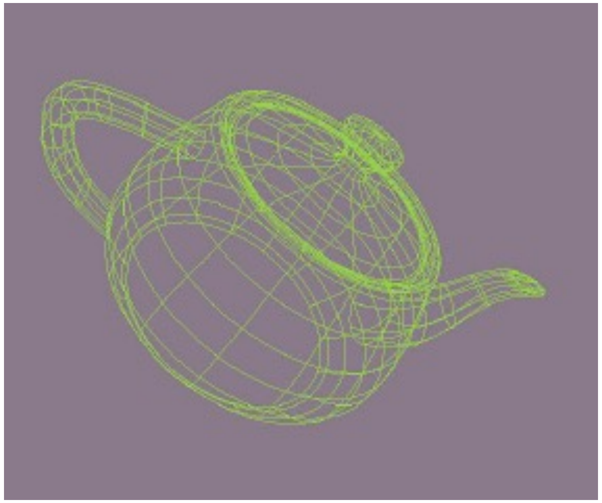
- Animation
  - Provide a “track” for objects
  - Use as camera path



Usefulness of curves

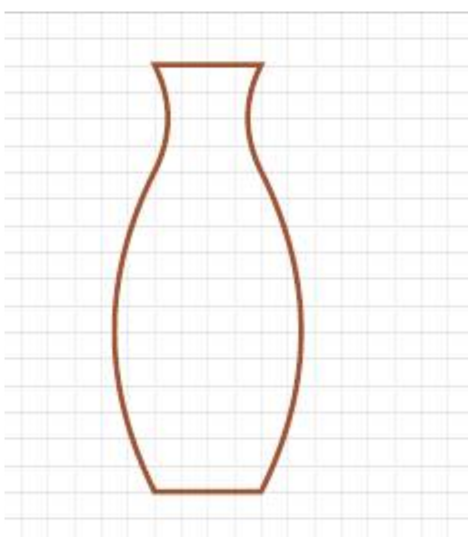
- Generalize to surface patches using “grids of curves”, next class

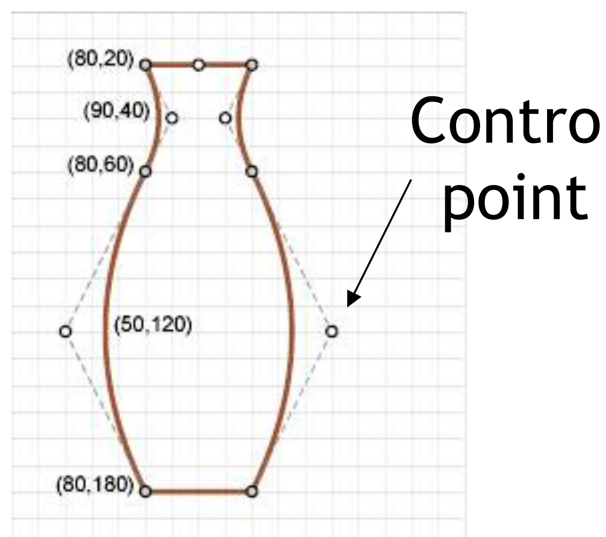




How to represent curves

- Specify every point along curve?
  - Hard to get precise, smooth results
  - Too much data, too hard to work with
- Idea: specify curves using small numbers of control points
- Mathematics: use polynomials to represent curves

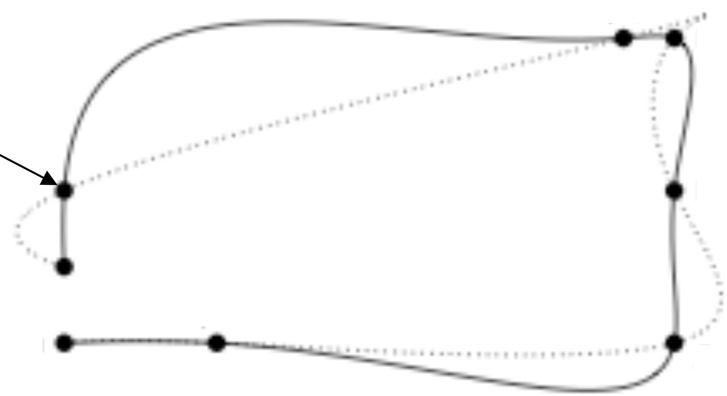




Interpolating polynomial curves

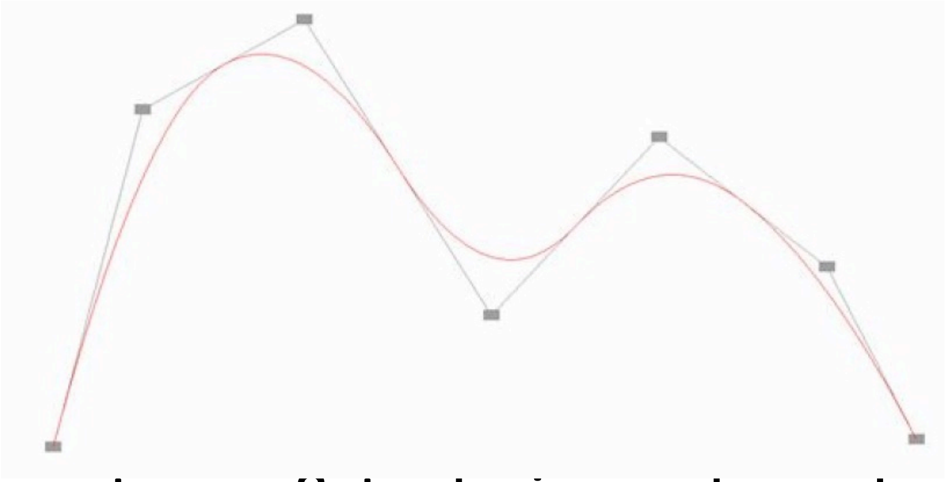
http://en.wikipedia.org/wiki/Polynomial_interpolation

- Curve goes through all control points
- Seems most intuitive
- Surprisingly, not usually the best choice
  - Hard to predict behavior
  - Overshoots, wiggles
  - Hard to get “nice-looking” curves



Approximating polynomial curves

- Curve is "influenced" by control points

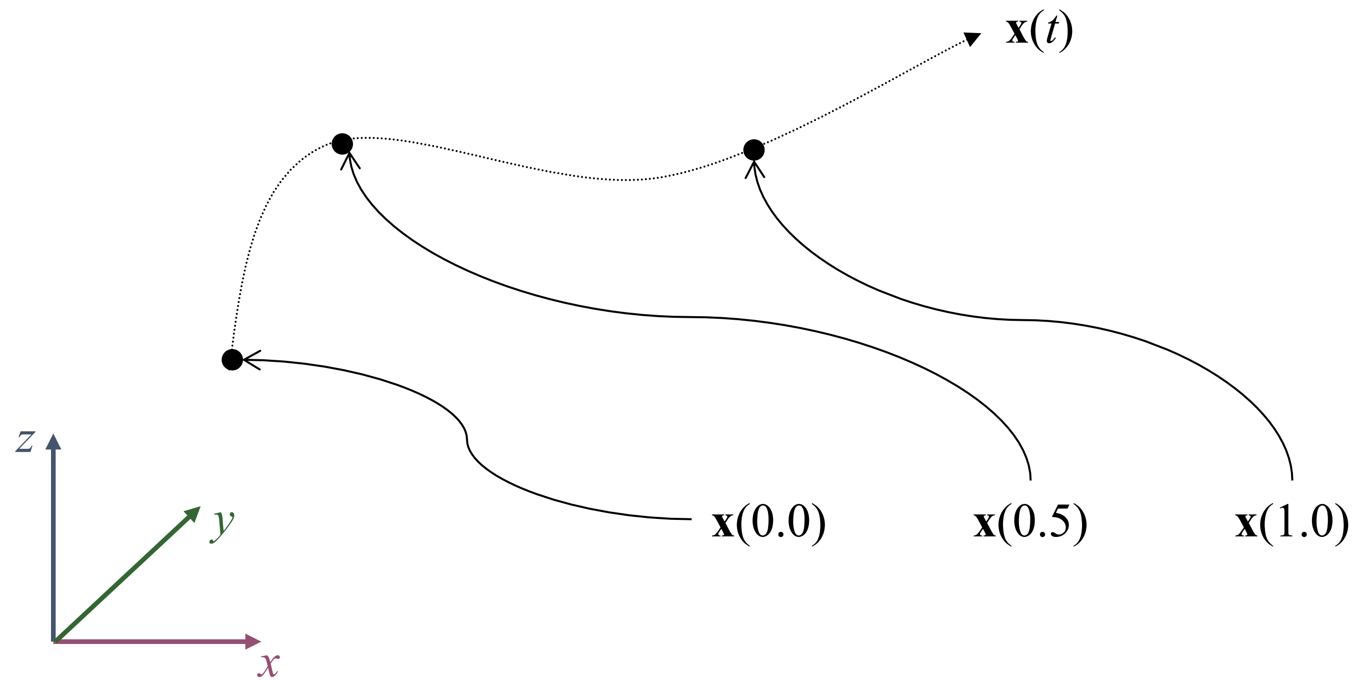


Control point

- Various types & techniques based on polynomial functions
  - Bézier curves, B-splines, NURBS
- Focus on Bézier curves

Mathematical definition

- A vector valued function of one variable $\mathbf{x}(t)$
  - Given t, compute a 3D point $\mathbf{x}=(x,y,z)$
  - May interpret as three functions $x(t)$, $y(t)$, $z(t)$
  - "Moving a point along the curve"

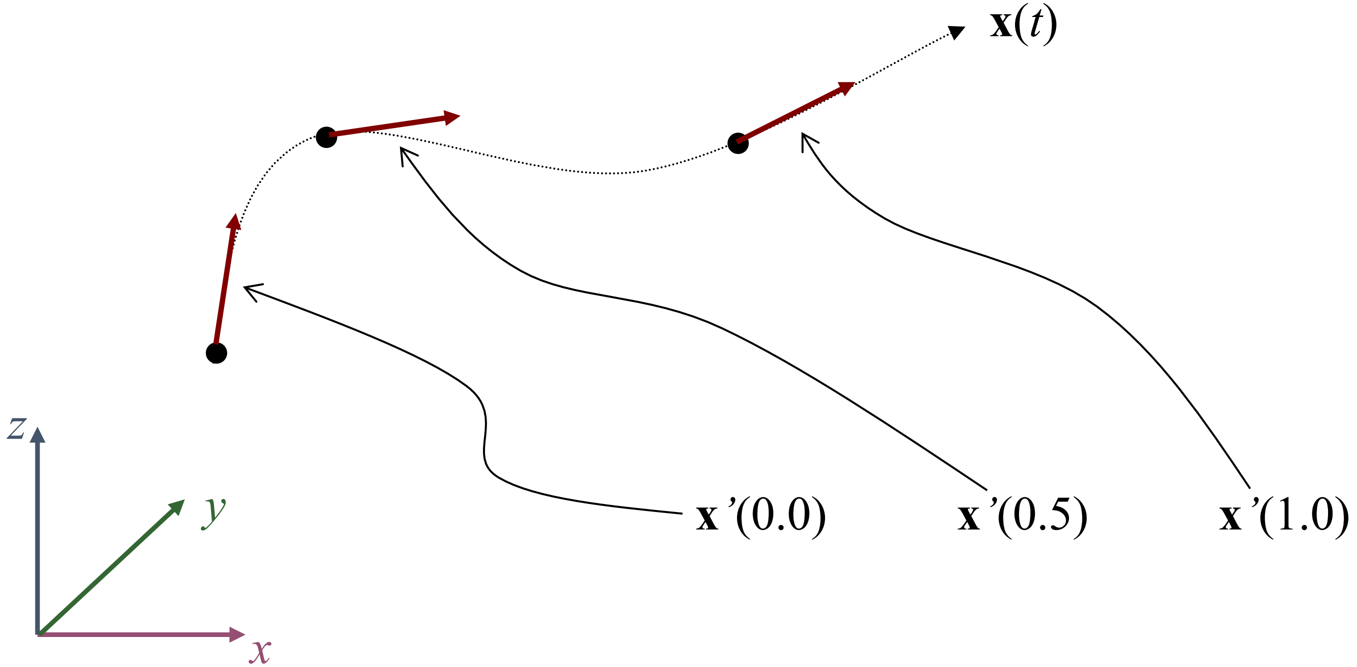


Tangent vector

- Derivative

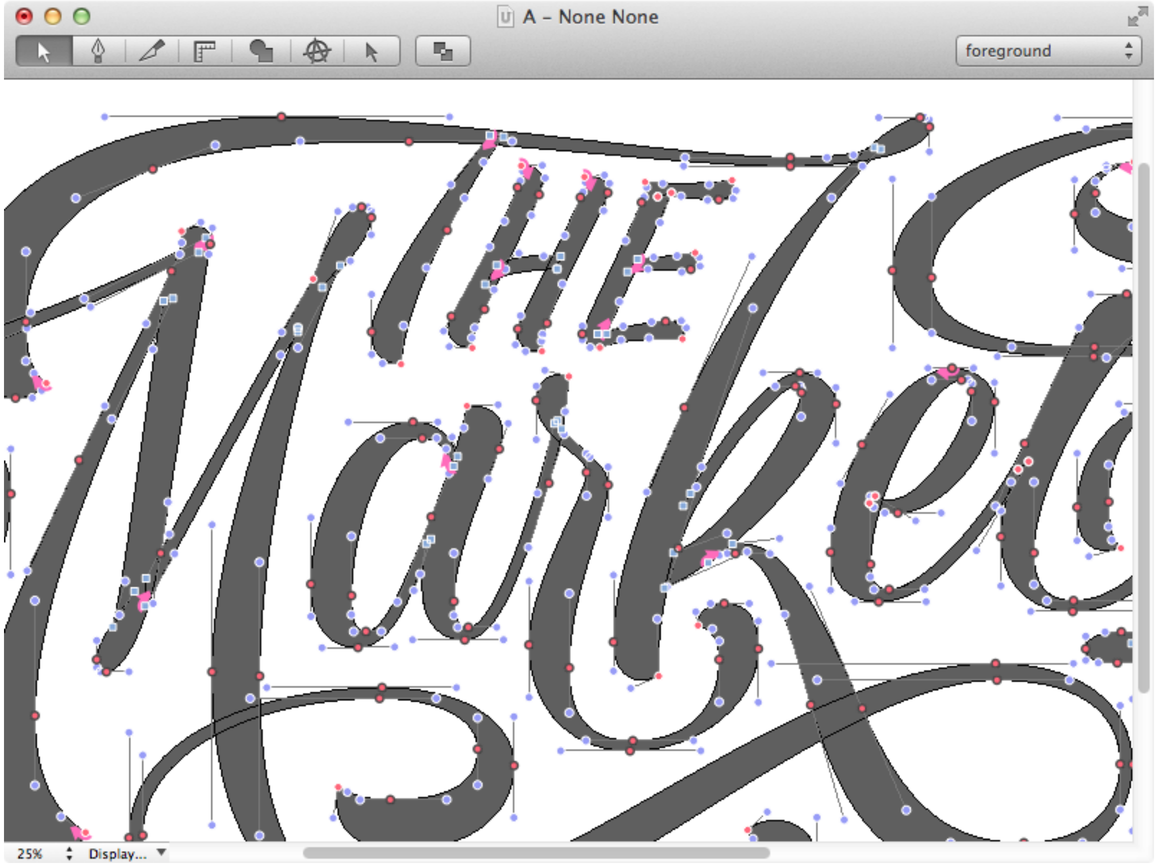
$$\mathbf{x}'(t) = \frac{d\mathbf{x}}{dt} = (x'(t), y'(t), z'(t))$$

- A vector that … ent
- Length of $\mathbf{x}'(t)$ corresponds to speed



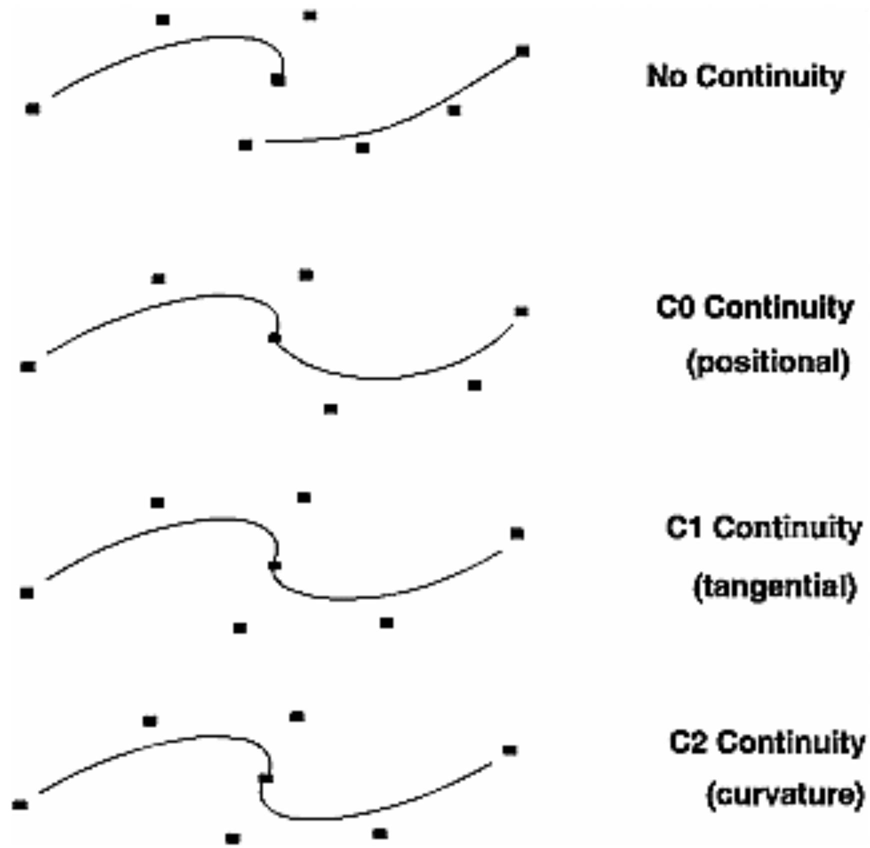
Piecewise polynomial curves

- Model complex shapes by sequence
- Use polyline to store control points



Continuity

- How piecewise curves join
- Ck continuity – kth derivatives match
- Gk continuity – kth derivatives are proportional

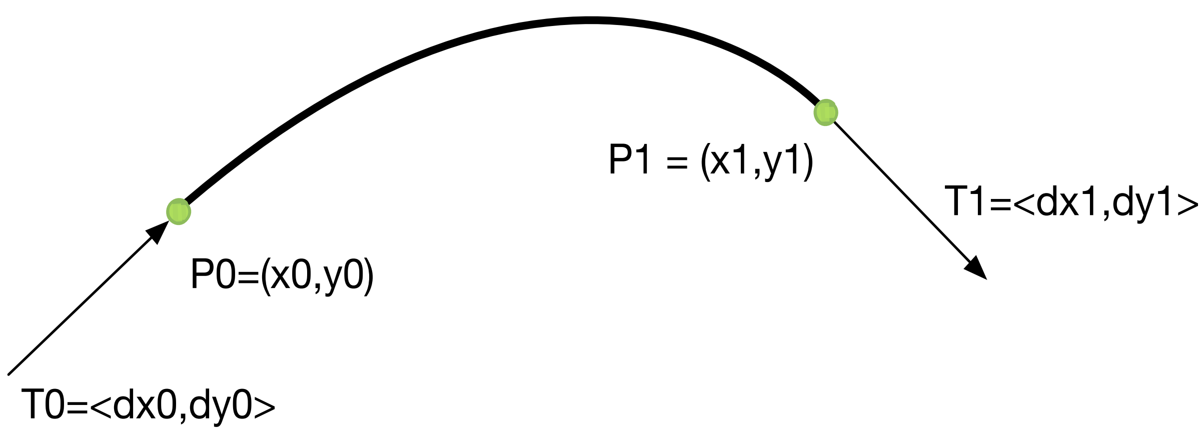


Hermite curves

- Cubic curve (here 2D)

$$x(t) = at^3 + bt^2 + ct + d$$
$$y(t) = et^3 + ft^2 + gt + h$$

- Interpolates end points P0 and P1
- Matches tangent at endpoints T0 and T1
  - (also dP0 and dP1 in these notes).



Computing coefficients a, b, c and d

- Derivative of x(t)

$$x'(t) = 3at^2 + 2bt + c$$

- Set t = 0 and 1 for endpoints
- Four constraints

$$x(0) = d$$
$$x'(0) = c$$
$$x(1) = a + b + c + d$$
$$x'(1) = 3a + 2b + c$$

Solve for a, b, c and d

- Solve for a, b, c and d

$$d = x0$$

$$c = dx0$$

$$b = -3x0 + 3x1 - 2dx0 - dx1$$

$$a = 2x0 - 2x1 + dx0 + dx1$$

Matrix version

- Constraints

$$x(0) = d$$
$$x'(0) = c$$
$$x(1) = a + b + c + d$$
$$x'(1) = 3a + 2b + c$$

- Give

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x0 \\ x1 \\ dx0 \\ dx1 \end{bmatrix}$$

Solve matrix version: basis matrix

- Since we have $MA = G$
- We can solve with $A = M^{-1}G$
- And get Hermite basis matrix M^{-1}

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x0 \\ x1 \\ dx0 \\ dx1 \end{bmatrix}$$

Vector version

- To include x, y and z, rewrite with vectors P0, P1 and tangents T0 and T1

$$\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{P0} \\ \boldsymbol{P1} \\ \boldsymbol{T0} \\ \boldsymbol{T1} \end{bmatrix}$$

- Coefficients a, b, c and d are now vectors

Full polynomial version

- Rewrite polynomial as dot product

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix}$$

- $$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{P0} \\ \boldsymbol{P1} \\ \boldsymbol{T0} \\ \boldsymbol{T1} \end{bmatrix}$$

Blending functions

- Instead of polynomial in t, look at curve as weighted sum of P0, P1, T0 and T1
- $x(t) = (2x0 - 2x1 + dx0 + dx1)t^3$
- $$+(-3x0 + 3x1 - 2dx0 - dx1)t^2$$
- $$+(dx0)t$$
- $$+x0$$

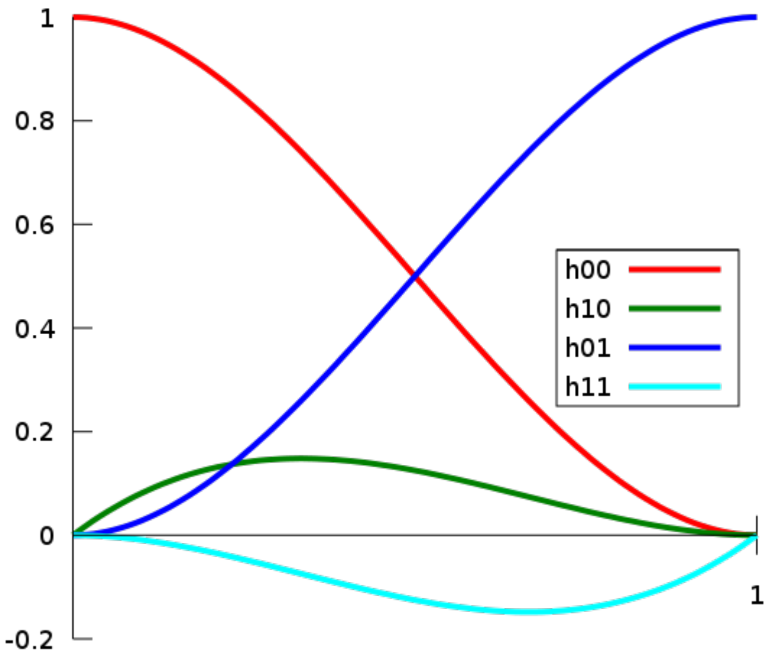
Blending functions

- Instead of polynomial in t, look at curve as weighted sum of P0, P1, T0 and T1

- $x(t) =$
- $(2t^3 - 3t^2 + 1)x0$
- $+(-2t^3 + 3t^2)x1$
- $+(t^3 - 2t^2 + t)dx0$
- $+(t^3 - t^2)dx1$

Blending functions

- $$h00(t) = (2t^3 - 3t^2 + 1)$$
  $$h01(t) = (-2t^3 + 3t^2)$$
  $$h10(t) = (t^3 - 2t^2 + t)$$
  $$h11(t) = (t^3 - t^2)$$



Computing Hermite tangents

- Have P(-1), P0, P1 and P2 as input
- Compute tangent with H matrix

$$\begin{bmatrix} x0 \\ x1 \\ dx0 \\ dx1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_{-1} \\ x_2 \end{bmatrix}$$

Combine with Hermite basis

- Unify notation

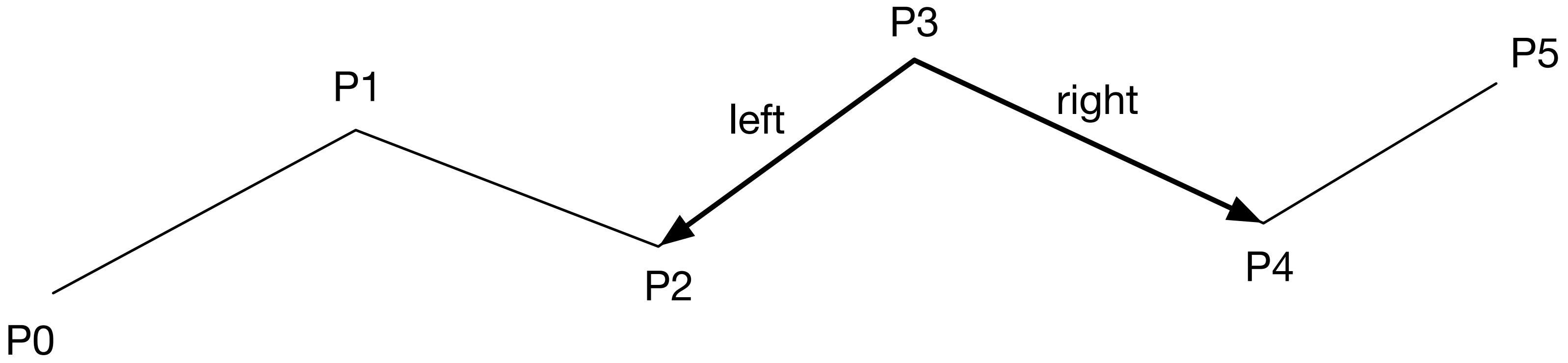
$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_{-1} \\ x_2 \end{bmatrix}$$

- Final matrix

- $$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 & -3 & -1 & 1 \\ -5 & 4 & 2 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_{-1} \\ x_2 \end{bmatrix}$$

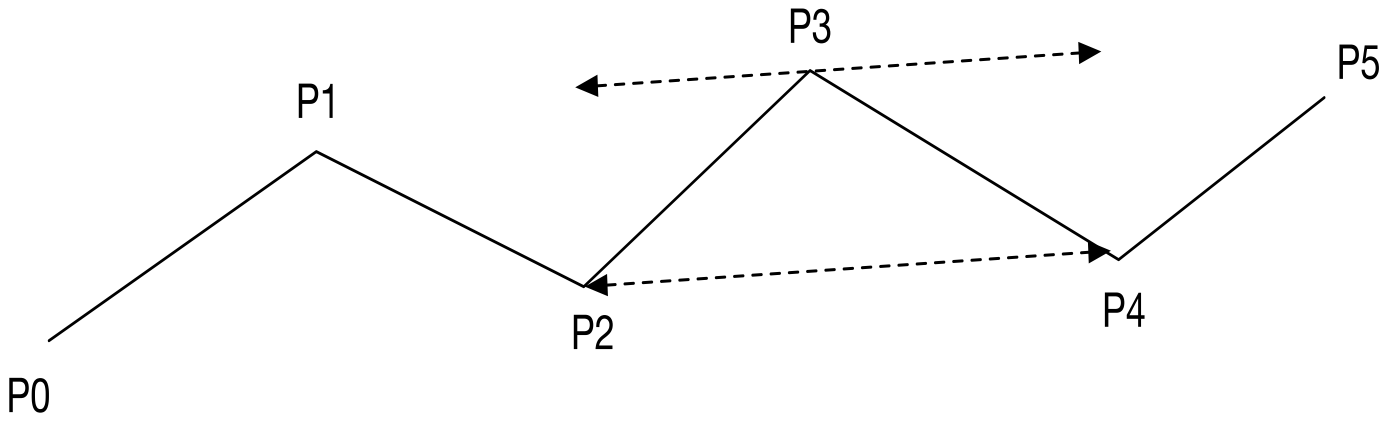
Catmull-Rom curves

- Hermite – problem with C1 continuity



Catmull-Rom curves

- Catmull-Rom – make tangent symmetric
- Define by two adjacent points
- Here T3 = P4-P2



Catmull-Rom curves

- Need to change H matrix
- ½ traditional for C-R curves

$$\begin{bmatrix} x0 \\ x1 \\ dx0' \\ dx1' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 \\ -1/2 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_{-1} \\ x_2 \end{bmatrix}$$

Catmull-Rom curves

- Which gives

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 \\ -1/2 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_{-1} \\ x_2 \end{bmatrix}$$

- Or

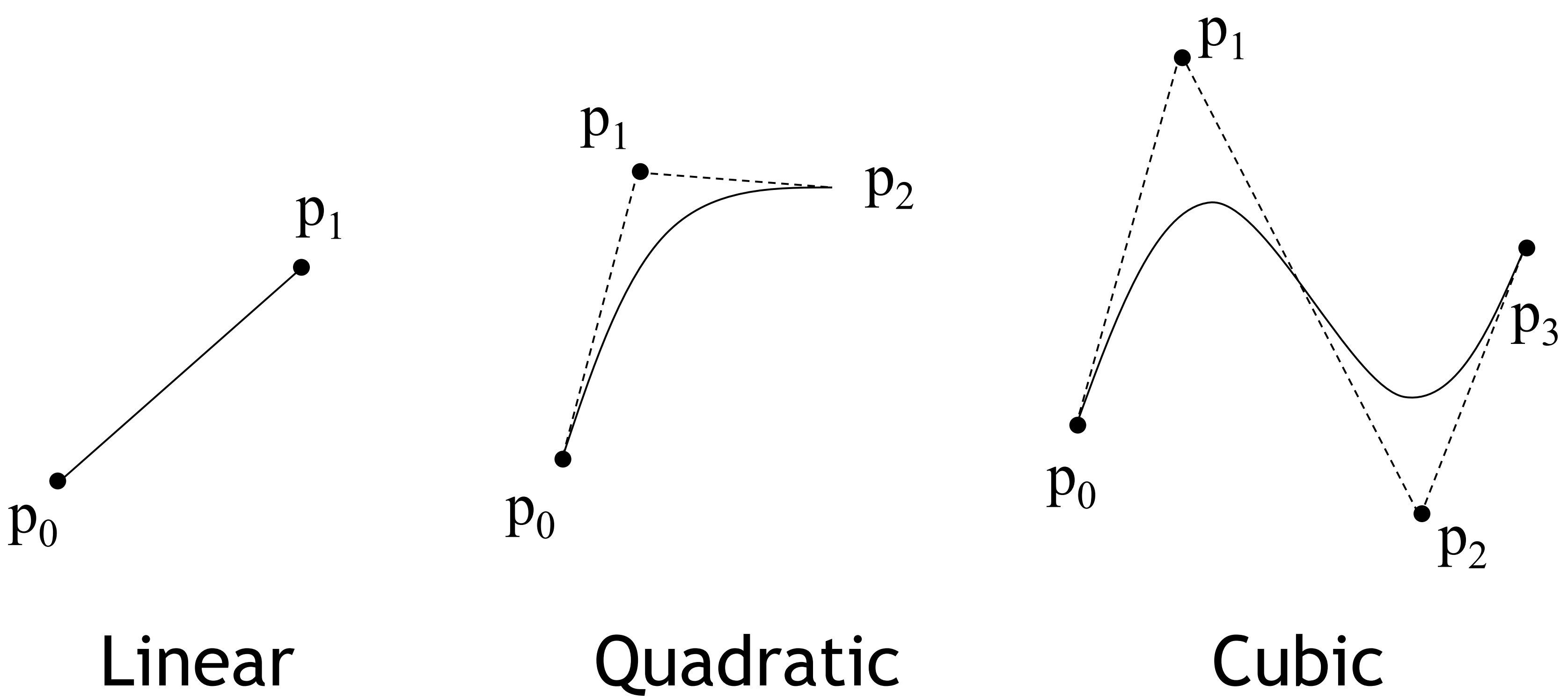
$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & -0.5 & 0.5 \\ -3.5 & 3 & 1 & -0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_{-1} \\ x_2 \end{bmatrix}$$

Bézier curves

http://en.wikipedia.org/wiki/B%C3%A9zier_curve

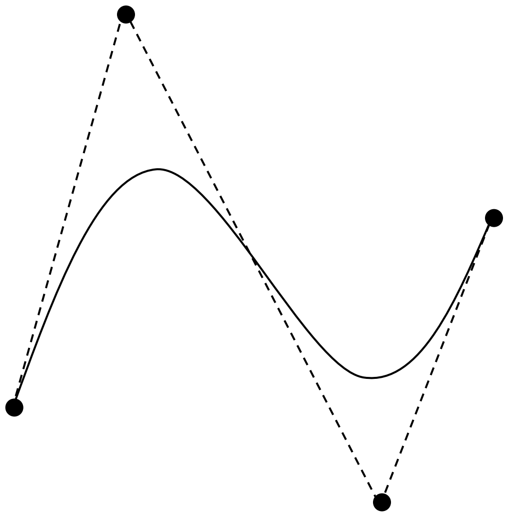
- A particularly intuitive way to define control points for polynomial curves
- Developed for CAD (computer aided design) and manufacturing
  - Before games, before movies, CAD was the big application for CG
- Pierre Bézier (1962), design of auto bodies for Peugeot, http://en.wikipedia.org/wiki/Pierre_B%C3%A9zier
- Paul de Casteljau (1959), for Citroen

- Higher order extension of linear interpolation
- Control points p_0, p_1, …



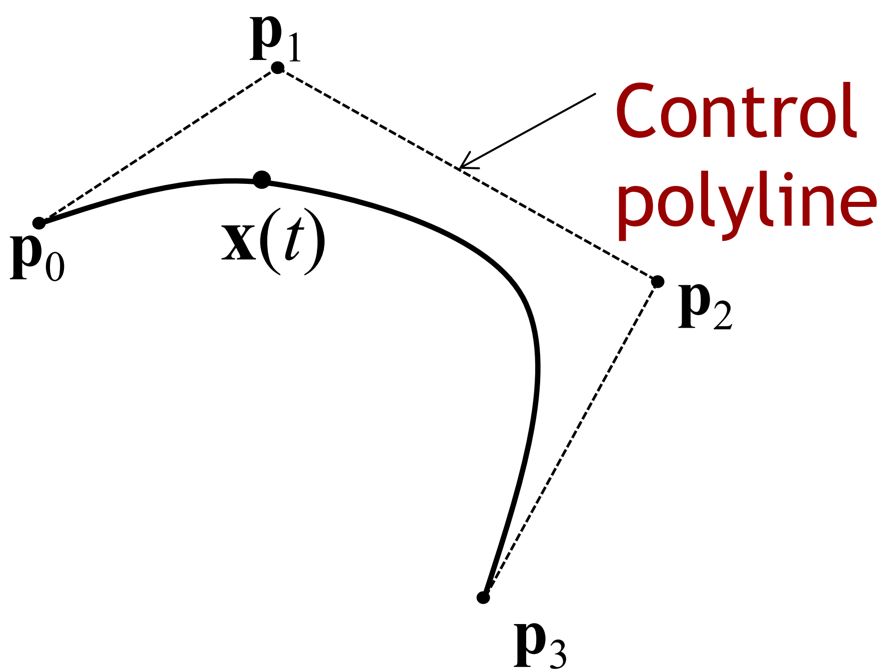
Bézier curves

- Intuitive control over curve given control points
  - Endpoints are interpolated, intermediate points are approximated
  - Convex Hull property
  - Variation-diminishing property



Cubic Bézier curve

- Cubic polynomials, most common case
- Defined by 4 control points
- Two interpolated endpoints
- Two midpoints control the tangent at the endpoints



Bézier Curve formulation

- Three alternatives, analogous to linear case

1. Weighted average of control points
2. Cubic polynomial function of t
3. Matrix form

- Algorithmic construction
  - de Casteljau algorithm

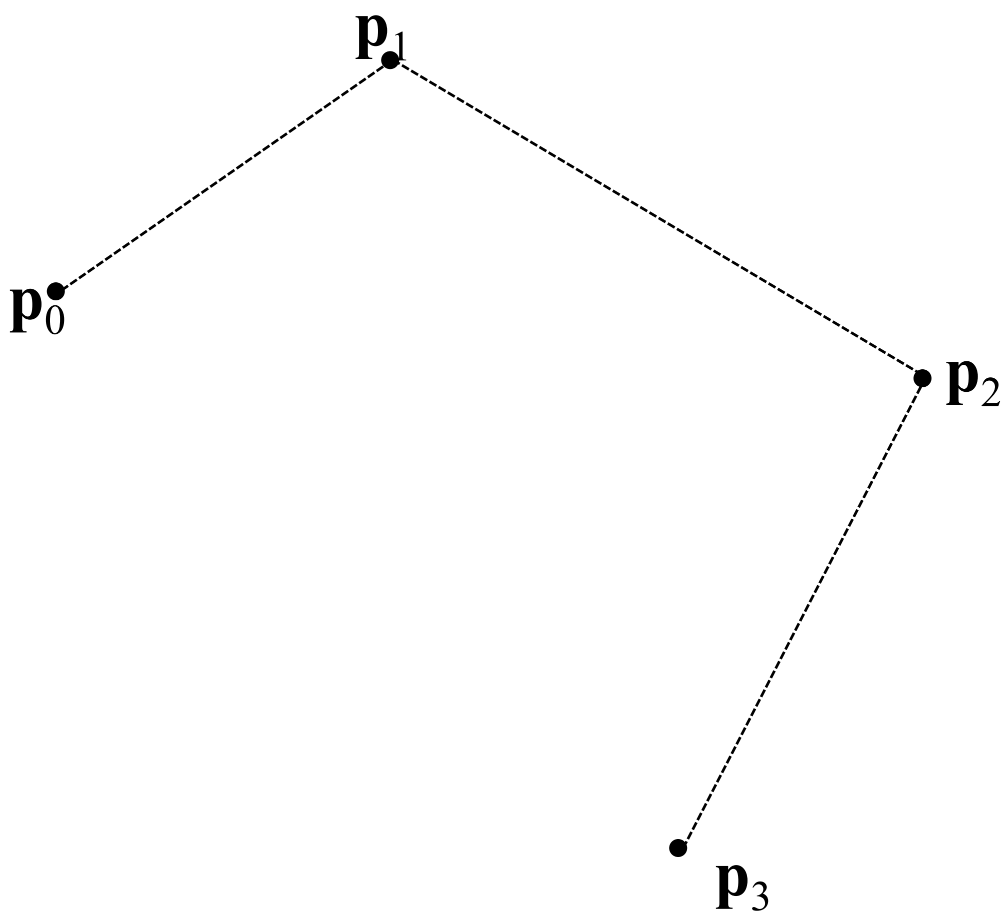
de Casteljau Algorithm

http://en.wikipedia.org/wiki/De_Casteljau's_algorithm

- A recursive series of linear interpolations
  - Works for any order, not only cubic
- Not terribly efficient to evaluate
  - Other forms more commonly used
- Why study it?
  - Intuition about the geometry
  - Useful for subdivision (later today)

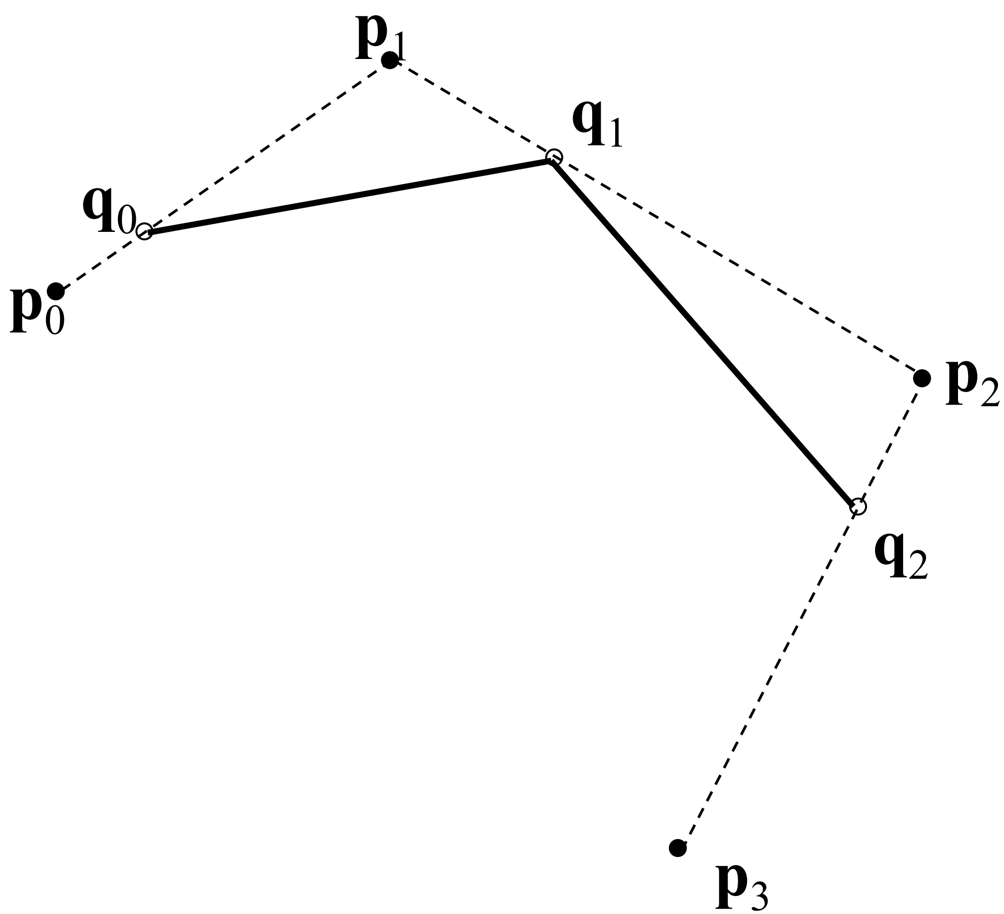
de Casteljau Algorithm

- Given the control points
- A value of t
- Here $t \approx 0.25$



de Casteljau Algorithm

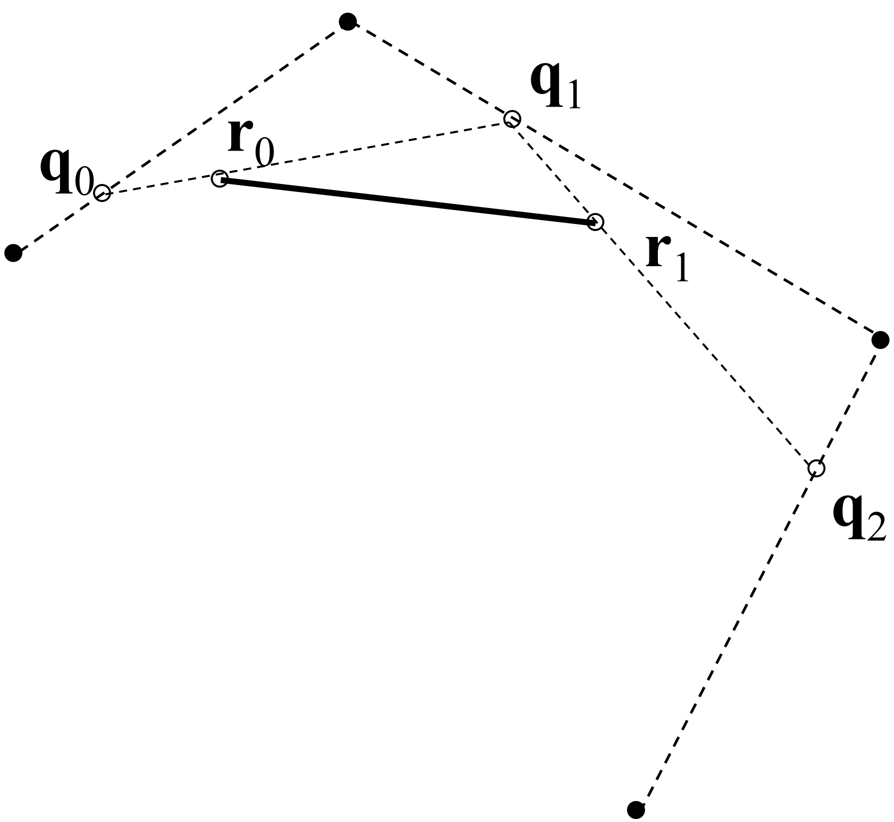
$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right)$$
$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right)$$
$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right)$$



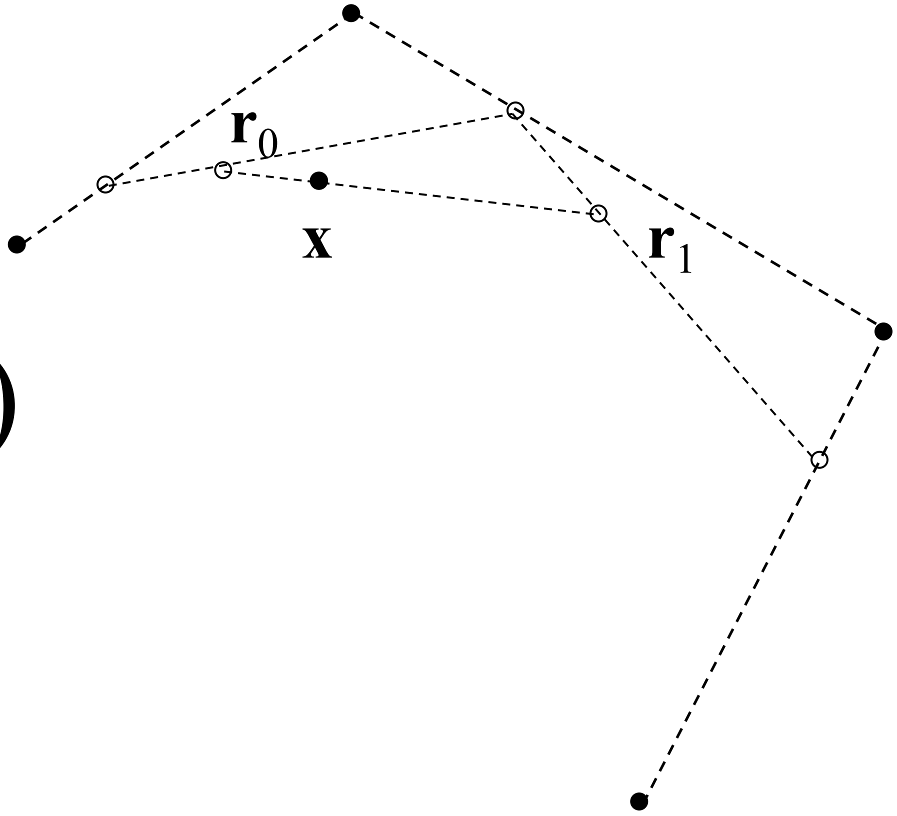
de Casteljau Algorithm

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right)$$

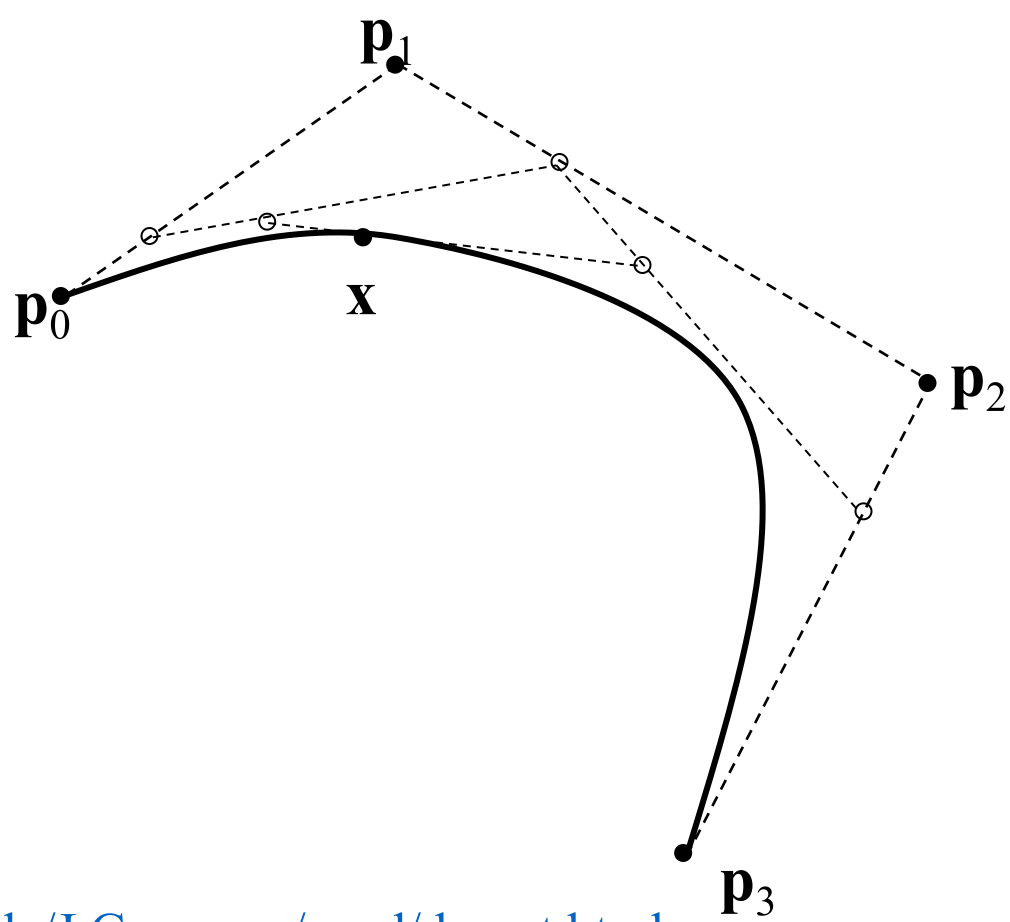


de Casteljau Algorithm



$$\mathbf{x}(t) = Lerp\left(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\right)$$

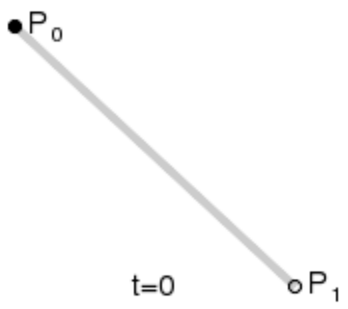
de Casteljau algorithm



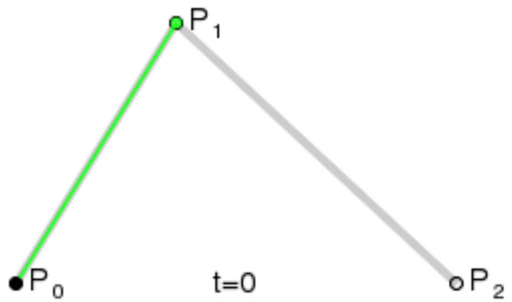
- Applets
  - http://www2.mat.dtu.dk/people/J.Gravesen/cagd/decast.html
  - http://www.caffeineowl.com/graphics/2d/vectorial/bezierintro.html

de Casteljau Algorithm

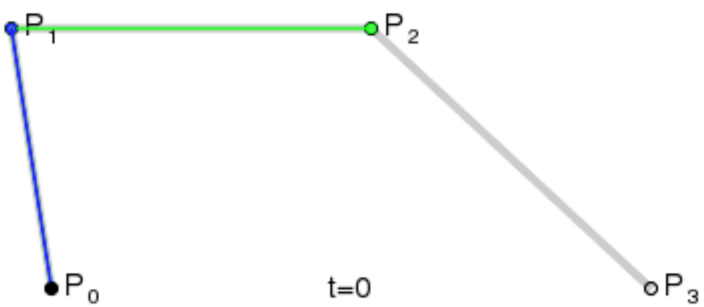
http://en.wikipedia.org/wiki/De_Casteljau's_algorithm



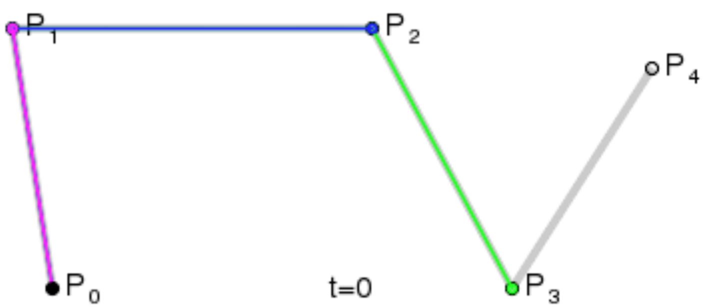
Linear



Quadratic



Cubic



Quartic

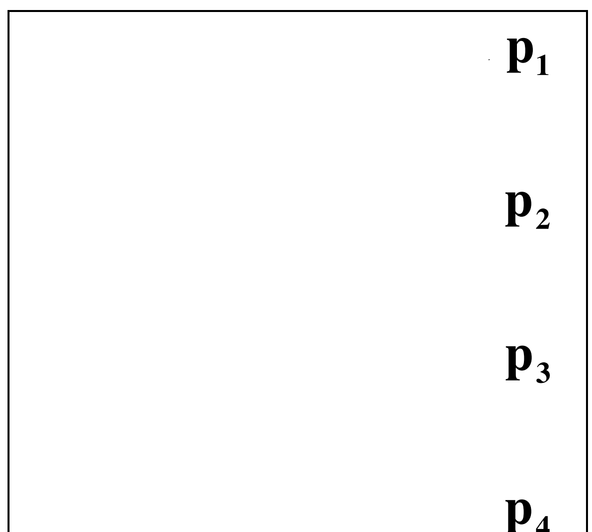
Recursive linear interpolation

$\mathbf{p}_0$

$\mathbf{p}_1$

$\mathbf{p}_2$

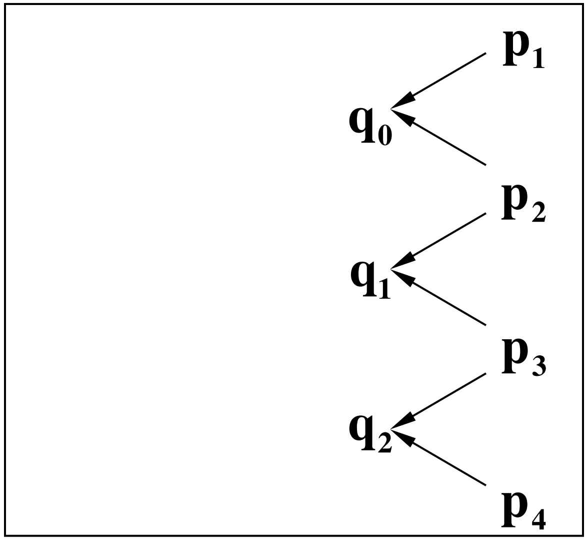
$\mathbf{p}_3$



Recursive linear interpolation

$$\mathbf{q}_0 = Lerp(t, \mathbf{p}_0, \mathbf{p}_1)$$
$$\mathbf{q}_1 = Lerp(t, \mathbf{p}_1, \mathbf{p}_2)$$
$$\mathbf{q}_2 = Lerp(t, \mathbf{p}_2, \mathbf{p}_3)$$

$\mathbf{p}_0$
$\mathbf{p}_1$
$\mathbf{p}_2$
$\mathbf{p}_3$



Recursive linear interpolation

$$\mathbf{p}_0$$

$$\mathbf{q}_0 = Lerp(t, \mathbf{p}_0, \mathbf{p}_1)$$

$$\mathbf{r}_0 = Lerp(t, \mathbf{q}_0, \mathbf{q}_1)$$

$$\mathbf{p}_1$$

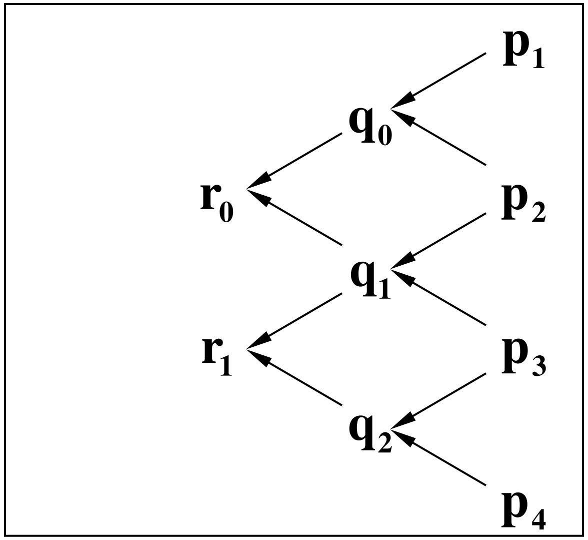
$$\mathbf{q}_1 = Lerp(t, \mathbf{p}_1, \mathbf{p}_2)$$

$$\mathbf{r}_1 = Lerp(t, \mathbf{q}_1, \mathbf{q}_2)$$

$$\mathbf{p}_2$$

$$\mathbf{q}_2 = Lerp(t, \mathbf{p}_2, \mathbf{p}_3)$$

$$\mathbf{p}_3$$



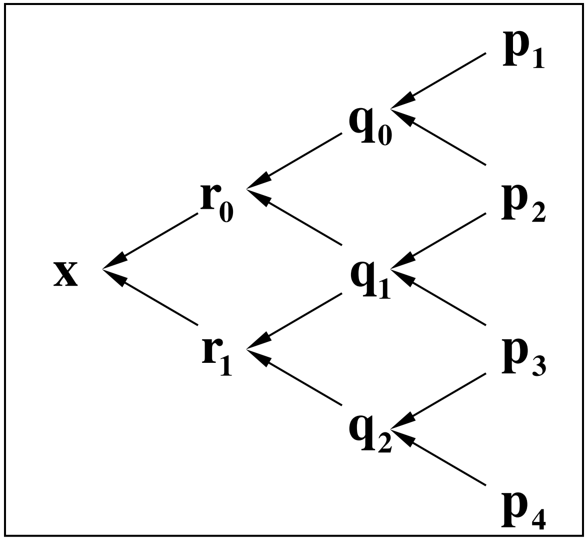
Recursive linear interpolation

$$\mathbf{x} = Lerp(t, \mathbf{r}_0, \mathbf{r}_1)$$

$$\mathbf{r}_0 = Lerp(t, \mathbf{q}_0, \mathbf{q}_1) \qquad \mathbf{r}_1 = Lerp(t, \mathbf{q}_1, \mathbf{q}_2)$$

$$\mathbf{q}_0 = Lerp(t, \mathbf{p}_0, \mathbf{p}_1) \qquad \mathbf{q}_1 = Lerp(t, \mathbf{p}_1, \mathbf{p}_2) \qquad \mathbf{q}_2 = Lerp(t, \mathbf{p}_2, \mathbf{p}_3)$$

$$\mathbf{p}_0 \qquad \mathbf{p}_1 \qquad \mathbf{p}_2 \qquad \mathbf{p}_3$$



Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = \left(1 - t\right)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = \left(1 - t\right)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = \left(1 - t\right)\mathbf{p}_2 + t\mathbf{p}_3$$

Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = \left(1 - t\right)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = \left(1 - t\right)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = \left(1 - t\right)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right)$$

Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = (1-t)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right) = (1-t)\left((1-t)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right) = (1-t)\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left((1-t)\mathbf{p}_2 + t\mathbf{p}_3\right)$$

Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = \left(1 - t\right)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = \left(1 - t\right)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = \left(1 - t\right)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right) = \left(1 - t\right)\left(\left(1 - t\right)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left(\left(1 - t\right)\mathbf{p}_1 + t\mathbf{p}_2\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right) = \left(1 - t\right)\left(\left(1 - t\right)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left(\left(1 - t\right)\mathbf{p}_2 + t\mathbf{p}_3\right)$$

$$\mathbf{x}(t) = Lerp\left(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\right)$$

Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = \left(1-t\right)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = \left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = \left(1-t\right)\mathbf{p}_2 + t\mathbf{p}_3$$

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$$\begin{aligned}\mathbf{x}(t) &= Lerp\left(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\right) \\ &= \left(1-t\right)\left(\left(1-t\right)\left(\left(1-t\right)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left(\left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2\right)\right) \\ &\quad + t\left(\left(1-t\right)\left(\left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left(\left(1-t\right)\mathbf{p}_2 + t\mathbf{p}_3\right)\right)\end{aligned}$$

Weighted average of control points

- Regroup

$$\mathbf{x}(t) = (1-t)\Big((1-t)\big((1-t)\mathbf{p}_0 + t\mathbf{p}_1\big) + t\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big)\Big)$$
$$+t\Big((1-t)\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big) + t\big((1-t)\mathbf{p}_2 + t\mathbf{p}_3\big)\Big)$$

Weighted average of control points

- Regroup

$$\mathbf{x}(t) = (1-t)\Big((1-t)\big((1-t)\mathbf{p}_0 + t\mathbf{p}_1\big) + t\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big)\Big)$$
$$+t\Big((1-t)\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big) + t\big((1-t)\mathbf{p}_2 + t\mathbf{p}_3\big)\Big)$$

$$\mathbf{x}(t) = (1-t)^3\,\mathbf{p}_0 + 3(1-t)^2\,t\mathbf{p}_1 + 3(1-t)t^2\mathbf{p}_2 + t^3\mathbf{p}_3$$

Weighted average of control points

- Regroup

$$\mathbf{x}(t) = (1-t)\Big((1-t)\big((1-t)\mathbf{p}_0 + t\mathbf{p}_1\big) + t\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big)\Big)$$
$$+t\Big((1-t)\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big) + t\big((1-t)\mathbf{p}_2 + t\mathbf{p}_3\big)\Big)$$

$$\mathbf{x}(t) = (1-t)^3\mathbf{p}_0 + 3(1-t)^2 t\mathbf{p}_1 + 3(1-t)t^2\mathbf{p}_2 + t^3\mathbf{p}_3$$

$$\mathbf{x}(t) = \overbrace{\left(-t^3 + 3t^2 - 3t + 1\right)}^{B_0(t)}\mathbf{p}_0 + \overbrace{\left(3t^3 - 6t^2 + 3t\right)}^{B_1(t)}\mathbf{p}_1$$
$$+\underbrace{\left(-3t^3 + 3t^2\right)}_{B_2(t)}\mathbf{p}_2 + \underbrace{\left(t^3\right)}_{B_3(t)}\mathbf{p}_3$$

Bernstein polynomials

Cubic Bernstein polynomials

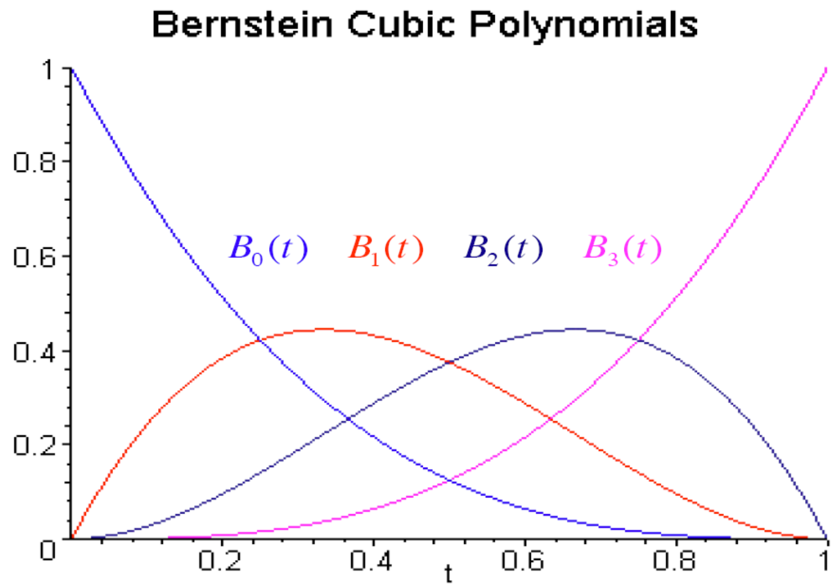
http://en.wikipedia.org/wiki/Bernstein_polynomial

$$\mathbf{x}(t) = B_0(t)\mathbf{p}_0 + B_1(t)\mathbf{p}_1 + B_2(t)\mathbf{p}_2 + B_3(t)\mathbf{p}_3$$

The cubic *Bernstein polynomials :*

$$B_0(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1(t) = 3t^3 - 6t^2 + 3t$$
$$B_2(t) = -3t^3 + 3t^2$$
$$B_3(t) = t^3$$

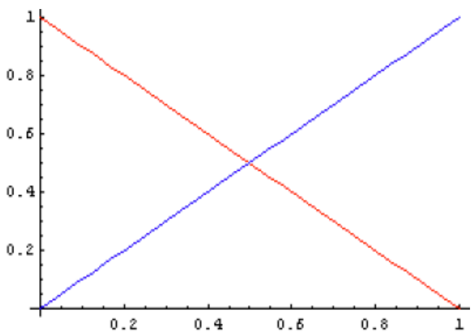
$$\sum B_i(t) = 1$$



- Partition of unity, at each t always add to 1
- Endpoint interpolation, B_0 and B_3 go to 1

General Bernstein polynomials

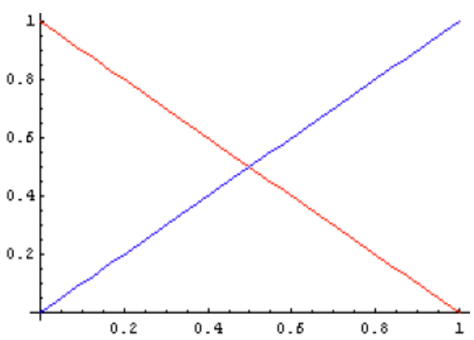
$$B_0^1(t) = -t + 1$$
$$B_1^1(t) = t$$

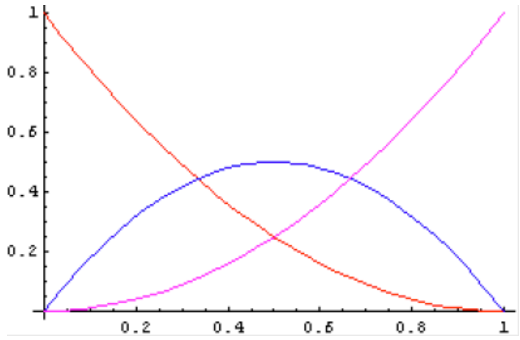


General Bernstein polynomials

$$B_0^1(t) = -t + 1$$
$$B_1^1(t) = t$$

$$B_0^2(t) = t^2 - 2t + 1$$
$$B_1^2(t) = -2t^2 + 2t$$
$$B_2^2(t) = t^2$$





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$$B_0^3(t) = -t^3 + 3t^2 - 3t + 1$$
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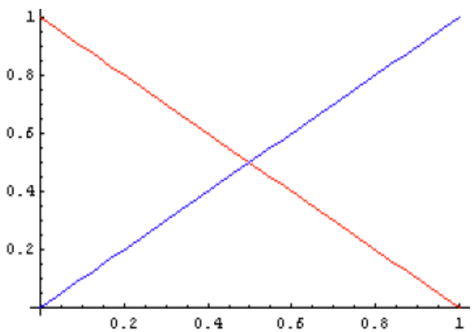
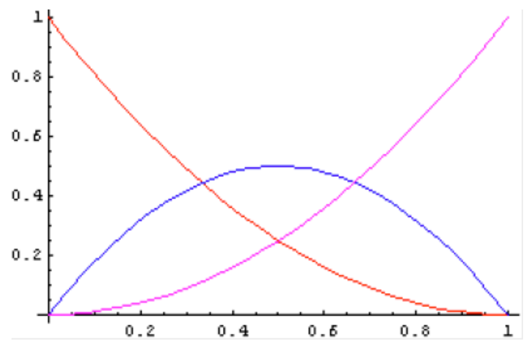
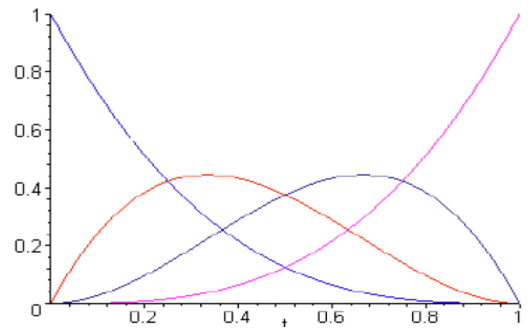
$$B_3^3(t) = t^3$$







General Bernstein polynomials

$$B_0^1(t) = -t + 1$$
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$$B_2^2(t) = t^2$$

$$B_0^3(t) = -t^3 + 3t^2 - 3t + 1$$
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$$B_2^3(t) = -3t^3 + 3t^2$$
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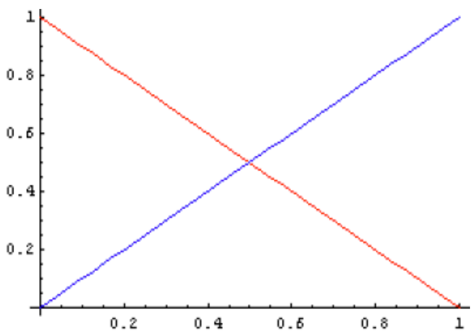
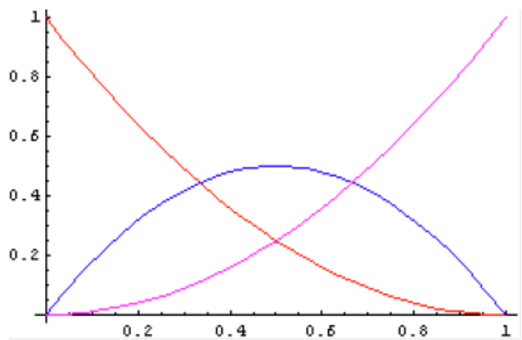
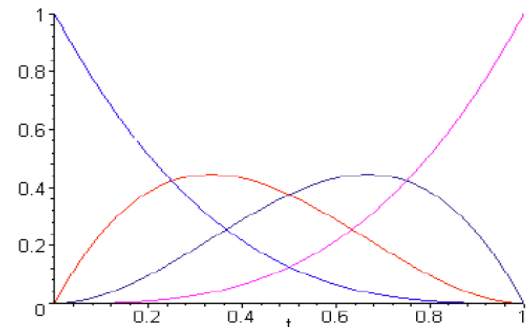






Order n: $$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}(t)^i \qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$\sum B_i^n(t) = 1$$

Partition of unity, endpoint interpolation

General Bézier curves

- nth-order Bernstein polynomials form nth-order Bézier curves
- Bézier curves are weighted sum of control points using nth-order Bernstein polynomials

Bernstein polynomials of order n:

$$B_i^n(t) = \binom{n}{i} (1-t)^{n-i} (t)^i$$

Bézier curve of order n:

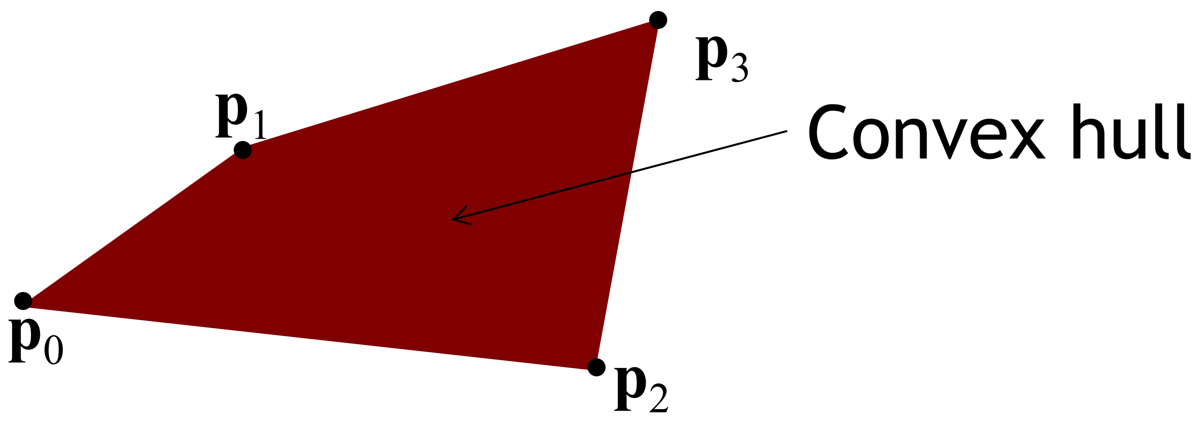
$$\mathbf{x}(t) = \sum_{i=0}^{n} B_i^n(t)\mathbf{p}_i$$

Bézier curve properties

- Convex hull property
- Variation diminishing property
- Affine invariance

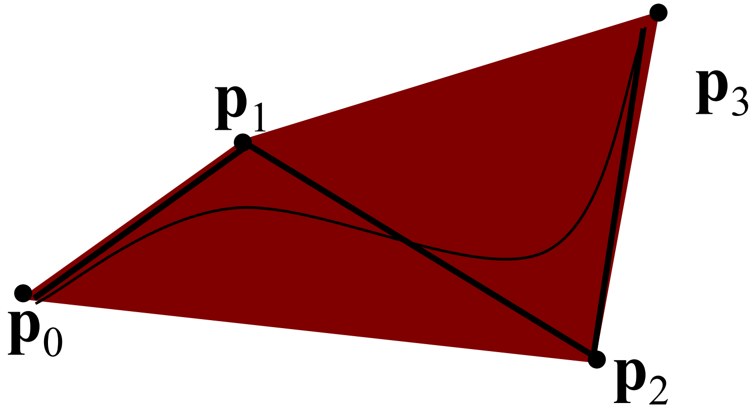
Convex hull, convex combination

- Convex hull of a set of points
  - Smallest polyhedral volume such that
    (i) all points are in it
    (ii) line connecting any two points in the volume lies completely inside it (or on its boundary)
- Convex combination of the points
  - Weighted average of the points, where weights all between 0 and 1, sum up to 1
- Any convex combination always lies within the convex hull

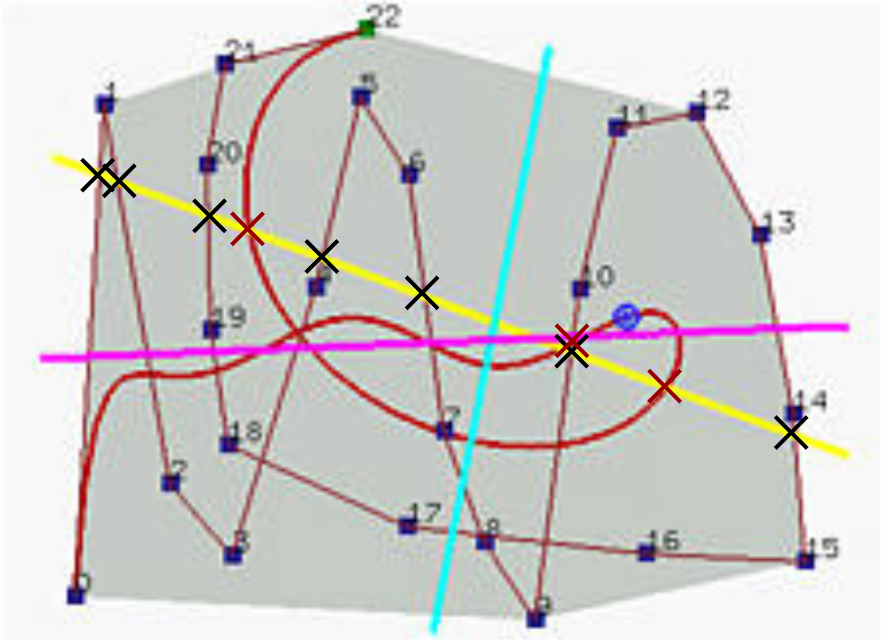


Convex hull property

- Bézier curve is a convex combination of the control points
  - Bernstein polynomials add to 1 at each value of t
- Curve is always inside the convex hull of control points
- Makes curve predictable
- Allows efficient culling, intersection testing, adaptive tessellation



- If the curve is in a plane, this means no straight line intersects a Bézier curve more times than it intersects the curve's control polyline
- "Curve is not more wiggly than control polyline"



Yellow line: 7 intersections with control polyline
3 intersections with curve

Affine invariance

- Two ways to transform Bézier curves
  1. Transform the control points, then compute resulting point on curve
  2. Compute point on curve, then transform it
- Either way, get the same transform point!
  - Curve is defined via affine combination of points (convex combination is special case of an affine combination)
  - Invariant under affine transformations
  - Convex hull property always remains

Cubic polynomial form

Start with Bernstein form:

$$\mathbf{x}(t) = \left(-t^3 + 3t^2 - 3t + 1\right)\mathbf{p}_0 + \left(3t^3 - 6t^2 + 3t\right)\mathbf{p}_1 + \left(-3t^3 + 3t^2\right)\mathbf{p}_2 + \left(t^3\right)\mathbf{p}_3$$

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Regroup into coefficients of t :

$$\mathbf{x}(t) = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)t^3 + \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)t^2 + \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)t + \left(\mathbf{p}_0\right)1$$

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$$\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

$$\mathbf{a} = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)$$

$$\mathbf{b} = \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)$$

$$\mathbf{c} = \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)$$

$$\mathbf{d} = \left(\mathbf{p}_0\right)$$

- Good for fast evaluation, precompute constant coefficients ($\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$)

Cubic polynomial form

Start with Bernstein form:

$$\mathbf{x}(t) = \left(-t^3 + 3t^2 - 3t + 1\right)\mathbf{p}_0 + \left(3t^3 - 6t^2 + 3t\right)\mathbf{p}_1 + \left(-3t^3 + 3t^2\right)\mathbf{p}_2 + \left(t^3\right)\mathbf{p}_3$$

Regroup into coefficients of t :

$$\mathbf{x}(t) = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)t^3 + \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)t^2 + \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)t + \left(\mathbf{p}_0\right)1$$

$$\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

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$$\mathbf{c} = \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)$$

$$\mathbf{d} = \left(\mathbf{p}_0\right)$$

- Good for fast evaluation, precompute constant coefficients ($\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$)
- Not much geometric intuition

Cubic matrix form

$$\mathbf{x}(t) = \begin{bmatrix} \vec{\mathbf{a}} & \vec{\mathbf{b}} & \vec{\mathbf{c}} & \mathbf{d} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\vec{\mathbf{a}} = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)$$

$$\vec{\mathbf{b}} = \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)$$

$$\vec{\mathbf{c}} = \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)$$

$$\mathbf{d} = \left(\mathbf{p}_0\right)$$

$$\mathbf{x}(t) = \underbrace{\begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}}_{\mathbf{G}_{Bez}} \underbrace{\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{Bez}} \underbrace{\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}}_{\mathbf{T}}$$

- Can construct other cubic curves by just using different basis matrix $\mathbf{B}$
- Hermite, Catmull-Rom, B-Spline, …

Cubic matrix form

- 3 parallel equations, in x, y and z:

$$\mathbf{x}_x(t) = \begin{bmatrix} p_{0x} & p_{1x} & p_{2x} & p_{3x} \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\mathbf{x}_y(t) = \begin{bmatrix} p_{0y} & p_{1y} & p_{2y} & p_{3y} \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\mathbf{x}_z(t) = \begin{bmatrix} p_{0z} & p_{1z} & p_{2z} & p_{3z} \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Matrix form

- Bundle into a single matrix

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} & p_{2x} & p_{3x} \\ p_{0y} & p_{1y} & p_{2y} & p_{3y} \\ p_{0z} & p_{1z} & p_{2z} & p_{3z} \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\mathbf{x}(t) = \mathbf{G}_{Bez} \mathbf{B}_{Bez} \mathbf{T}$$
$$\mathbf{x}(t) = \mathbf{C}\ \mathbf{T}$$

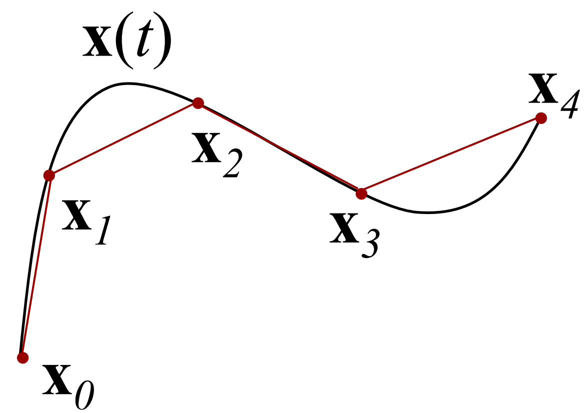
- Efficient evaluation
  - Precompute $\mathbf{C}$
  - Take advantage of existing 4x4 matrix hardware support

Drawing Bézier curves

- Generally no low-level support for drawing smooth curves
  - I.e., GPU draws only straight line segments
- Need to break curves into line segments or individual pixels
- Approximating curves as series of line segments called tessellation
- Tessellation algorithms
  - Uniform sampling
  - Adaptive sampling
  - Recursive subdivision

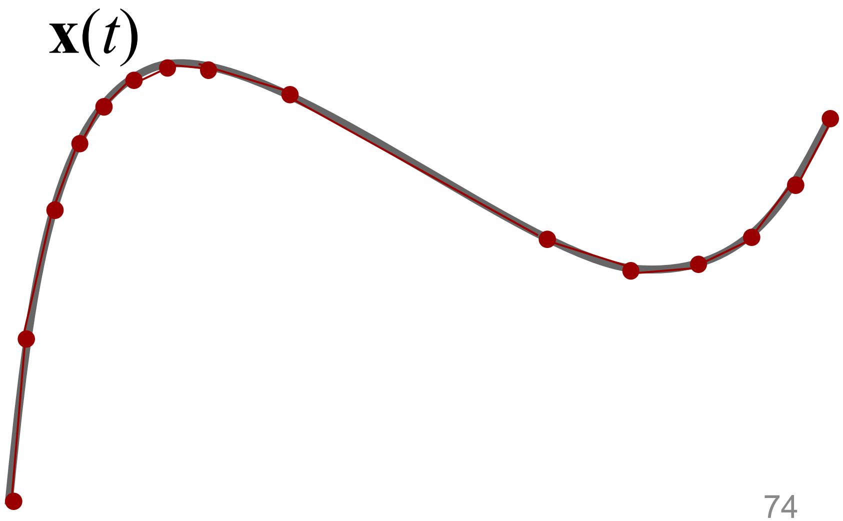
Uniform sampling

- Approximate curve with N-1 straight segments
  - N chosen in advance
  - Evaluate
    $$\mathbf{x}_i = \mathbf{x}(t_i) \text{ where } t_i = \frac{i}{N} \text{ for } i = 0, 1, \ldots, N$$
    $$\mathbf{x}_i = \mathbf{a}\frac{i^3}{N^3} + \mathbf{b}\frac{i^2}{N^2} + \mathbf{c}\frac{i}{N} + \mathbf{d}$$
  - Connect th
- Too few points?
  - Bad approximation
  - “Curve” is faceted
- Too many points?
  - Slow to draw too many line segments
  - Segments may draw on top of each other



Adaptive Sampling

- Use only as many line segments as you need
  - Fewer segments where curve is mostly flat
  - More segments where curve bends
  - Segments never smaller than a pixel
- Various schemes for sampling, checking results, deciding whether to sample more



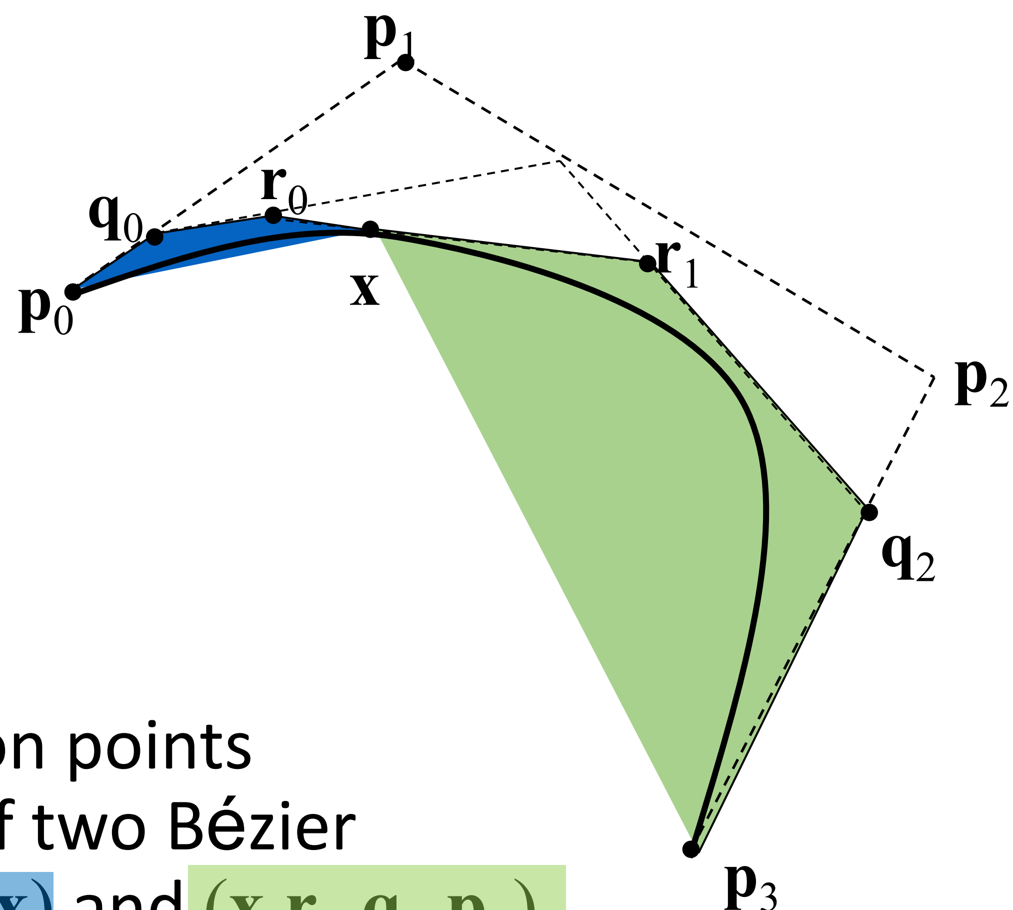
Recursive Subdivision

- Any cubic (or k-th order) curve segment can be expressed as a cubic (or k-th order) Bézier curve

  "Any piece of a cubic (or k-th order) curve is itself a cubic (or k-th order) curve"

- Therefore, any Bézier curve can be subdivided into smaller Bézier curves

de Casteljau subdivision



- de Casteljau construction points are the control points of two Bézier sub-segments $(\mathbf{p}_0,\mathbf{q}_0,\mathbf{r}_0,\mathbf{x})$ and $(\mathbf{x},\mathbf{r}_1,\mathbf{q}_2,\mathbf{p}_3)$

Adaptive subdivision algorithm

1. Use de Casteljau construction to split Bézier segment in middle (t=0.5)
2. For each half
   - If "flat enough": draw line segment
   - Else: recurse from 1. for each half

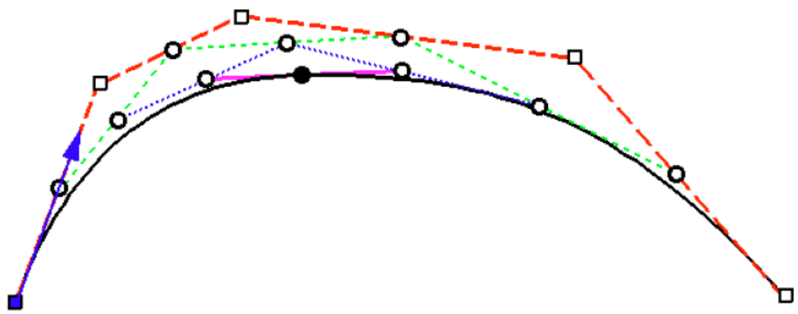
- Curve is flat enough if hull is flat enough
- Test how far away midpoints are from straight segment connecting start and end
  - If about a pixel, then hull is flat enough

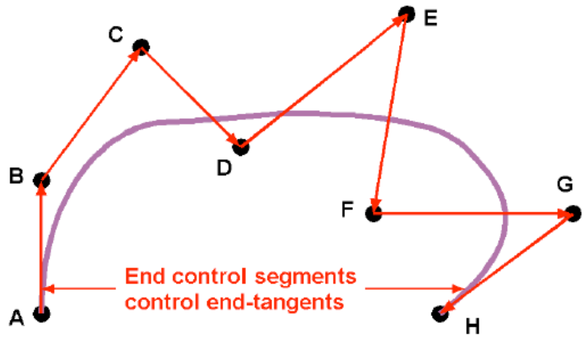
Curves

- Introduction
- Polynomial curves
- Bézier curves
- Drawing Bézier curves
- Piecewise curves

More control points

- Cubic Bézier curve limited to 4 control points
  - Cubic curve can only have one inflection
  - Need more control points for more complex curves
- k-1 order Bézier curve with k control points

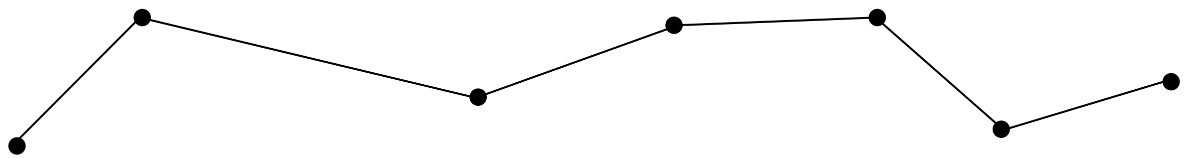




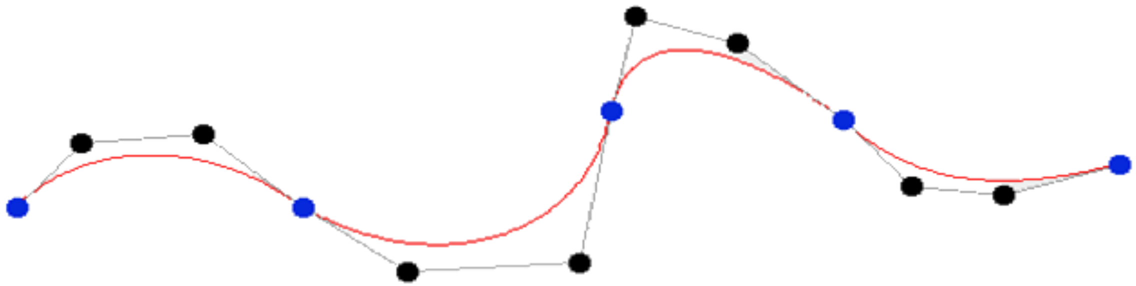
- Hard to control and hard to work with
  - Intermediate points don't have obvious effect on shape
  - Changing any control point changes the whole curve
- Want local support
  - Each control point only influences nearby portion of curve

Piecewise curves (splines)

- Sequence of simple (low-order) curves, end-to-end
  - Piecewise polynomial curve, or splines
    http://en.wikipedia.org/wiki/Spline_(mathematics)
- Sequence of line segments
  - Piecewise linear curve (linear or first-order spline)



- Sequence of cubic curve segments
  - Piecewise cubic curve, here piecewise Bézier (cubic spline)



Piecewise cubic Bézier curve

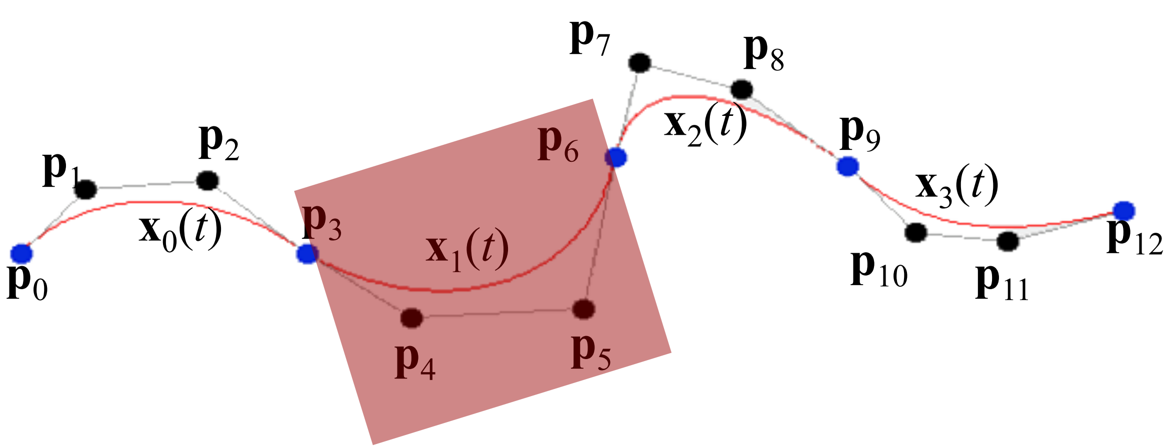
- Given $3N+1$ points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{3N}$
- Define N Bézier segments:

$$\mathbf{x}_0(t) = B_0(t)\mathbf{p}_0 + B_1(t)\mathbf{p}_1 + B_2(t)\mathbf{p}_2 + B_3(t)\mathbf{p}_3$$

$$\mathbf{x}_1(t) = B_0(t)\mathbf{p}_3 + B_1(t)\mathbf{p}_4 + B_2(t)\mathbf{p}_5 + B_3(t)\mathbf{p}_6$$

$$\vdots$$

$$\mathbf{x}_{N-1}(t) = B_0(t)\mathbf{p}_{3N-3} + B_1(t)\mathbf{p}_{3N-2} + B_2(t)\mathbf{p}_{3N-1} + B_3(t)\mathbf{p}_{3N}$$

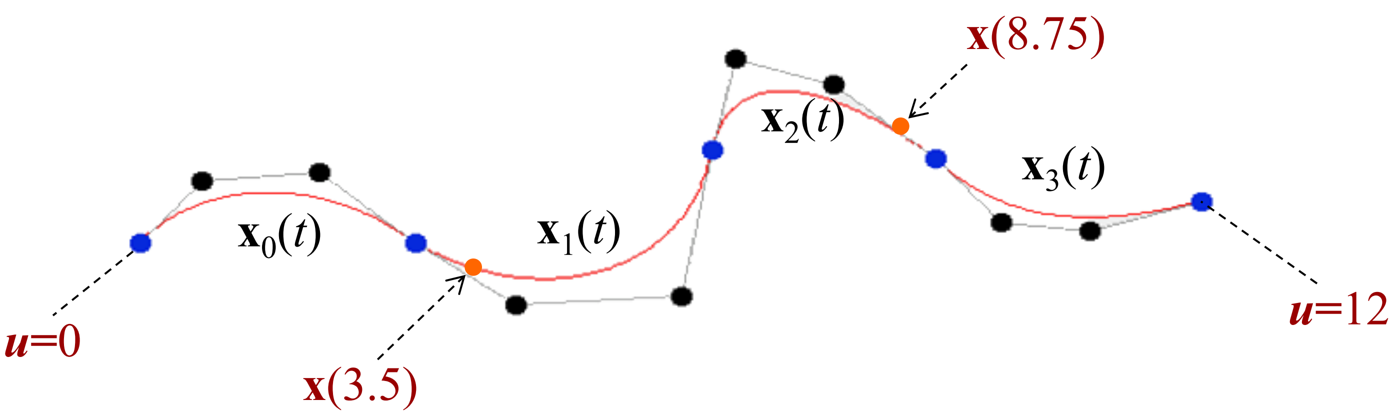


Piecewise cubic Bézier curve

- Global parameter u, $0 <= u <= 3N$

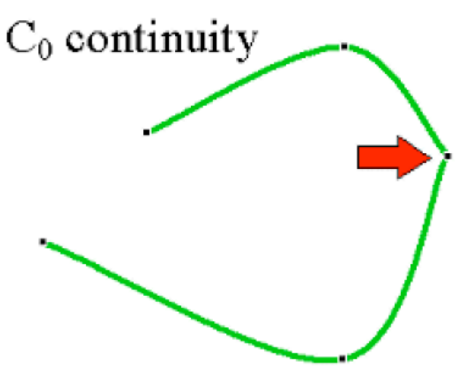
$$\mathbf{x}(u) = \begin{cases} \mathbf{x}_0(\frac{1}{3}u), & 0 \le u \le 3 \\ \mathbf{x}_1(\frac{1}{3}u - 1), & 3 \le u \le 6 \\ \vdots & \vdots \\ \mathbf{x}_{N-1}(\frac{1}{3}u - (N-1)), & 3N - 3 \le u \le 3N \end{cases}$$

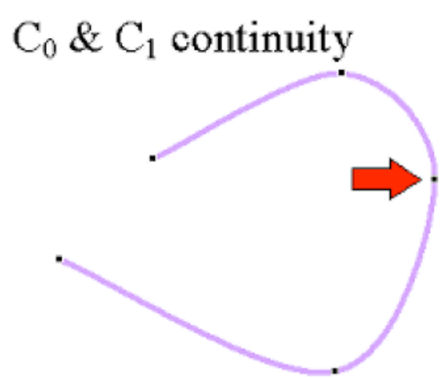
$$\mathbf{x}(u) = \mathbf{x}_i\left(\tfrac{1}{3}u - i\right), \text{ where } i = \left\lfloor \tfrac{1}{3}u \right\rfloor$$

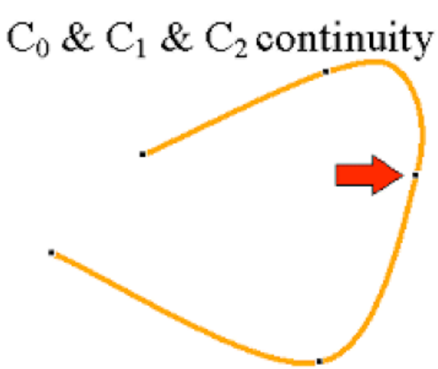


Continuity

- Want smooth curves
- C^0 continuity
  - No gaps
  - Segments match at the endpoints
- C^1 continuity: first derivative is well defined
  - No corners
  - Tangents/normals are C^0 continuous (no jumps)
- C^2 continuity: second derivative is well defined
  - Tangents/normals are C^1 continuous
  - Important for high quality reflections on surfaces

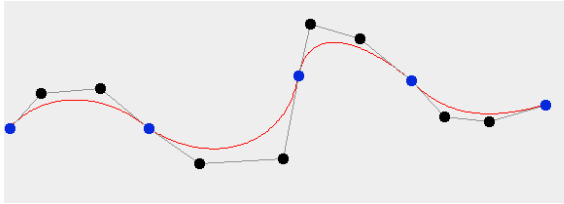


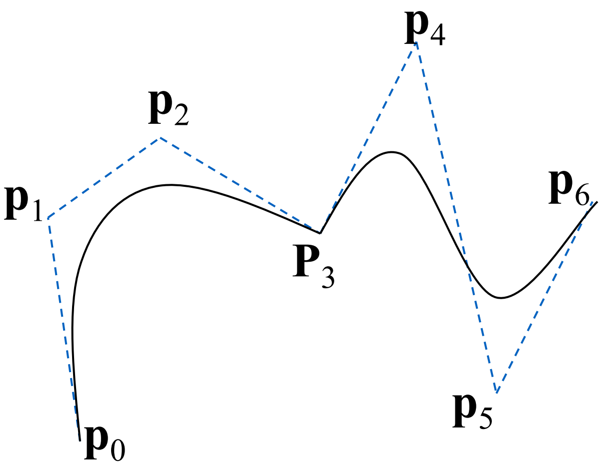




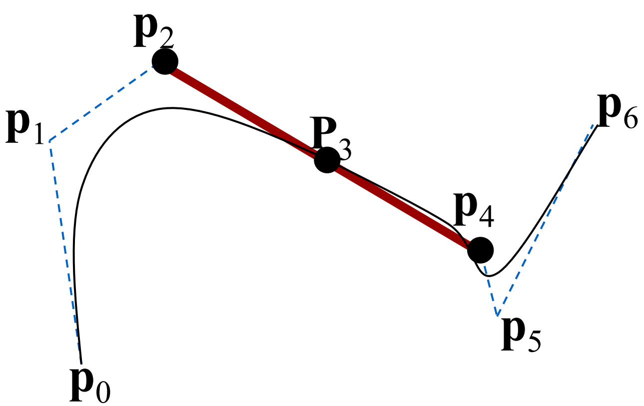
Piecewise cubic Bézier curve

- C^0 continuous by construction
- C^1 continuous at segment endpoints $\mathbf{p}_{3i}$ if $\mathbf{p}_{3i} - \mathbf{p}_{3i-1} = \mathbf{p}_{3i+1} - \mathbf{p}_{3i}$
- C^2 is harder to get





C^0 continuous



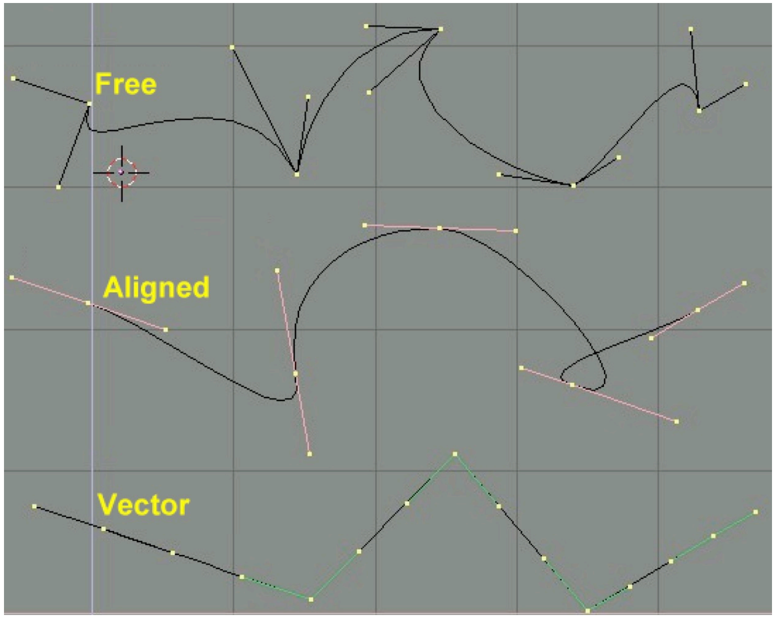
C^1 continuous

Piecewise cubic Bézier curves

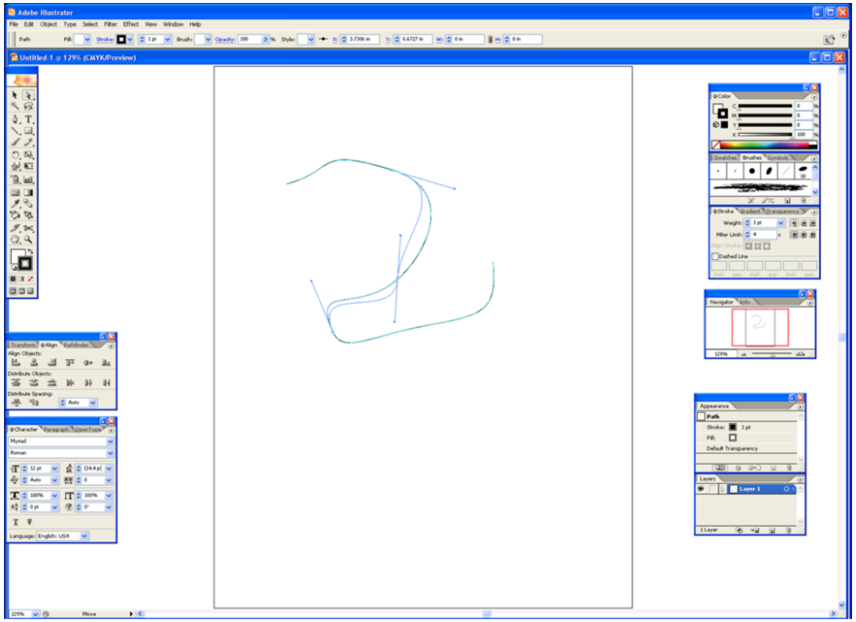
- Used often in 2D drawing programs
- Inconveniences
  - Must have 4 or 7 or 10 or 13 or … (1 plus a multiple of 3) control points
  - Some points interpolate (endpoints), others approximate (handles)
  - Need to impose constraints on control points to obtain C^1 continuity
  - C^2 continuity more difficult
- Solutions
  - User interface using “Bézier handles”
  - Generalization to B-splines, next time

Bézier handles

- Segment end points (interpolating) presented as curve control points
- Midpoints (approximating points) presented as "handles"
- Can have option to enforce C^1 continuity



[www.blender.org]



Adobe Illustrator