On Random Sampling Auctions for Digital Goods

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Abstract

In the context of auctions for digital goods, an interesting Random Sampling Optimal Price auction (RSOP) has been proposed by Goldberg, Hartline and Wright; this leads to a truthful mechanism. Since random sampling is a popular approach for auctions that aims to maximize the seller's revenue, this method has been analyzed further by Feige, Flaxman, Hartline and Kleinberg, who have shown that it is 15-competitive in the worst case – which is substantially better than the previously proved bounds but still far from the conjectured competitive ratio of 4. In this paper, we prove that RSOP is indeed 4-competitive for a large class of instances in which there are at least 6 people receiving the item at the optimal uniform price. We also show that it is 4.68 competitive for the small class of remaining instances thus leaving a negligible gap between the lower and upper bound. We employ a mix of probabilistic techniques and dynamic programming to compute these bounds.

1 Introduction

In recent years, there has been a considerable amount of work in algorithmic mechanism design. One of the primary constraints that much of this work tries to enforce is *incentive compatibility*, which means that being truthful is the best for each agent. In this work, we study a popular random-sampling-based incentive-compatible mechanism ("RSOP") for auctions of digital goods where we aim to maximize the auctioneer's expected revenue; we prove by a mix of analytical methods and computing-based approaches (the latter based on rigorous mathematical arguments) that this mechanism has a much better competitive ratio than was known before, and place limits on how good this mechanism can be in the worst case.

Our basic problem is as follows. A seller (also referred to as auctioneer) has a good that she/he can make an unlimited number of copies of – such as a digital good. We also have N bidders with unknown valuations $v_1, v_2, \ldots, v_N$ for the good; this means that bidder i

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will buy the good iff it is offered at a price of at most v_i to him/her. We aim to design a (randomized) incentive-compatible mechanism that will maximize the seller's expected total revenue. (We assume that the seller can make up to N copies if necessary at negligible cost, so that the seller's revenue equals her/his profit.) A classical work of Myerson has studied this problem under the Bayesian setting, where we assume a distribution on the bids v_i; knowledge of the prior information about the bid distribution is essential to his work [11]. Here, we will work throughout with the classical "computer science" approach to this problem, which is to assume the worst case: this is the "prior-free" variant of our problem where we allow an arbitrary (unknown and worst-case) distribution of the bids. In the spirit of the competitive analysis of online algorithms, this naturally leads to the following notion of competitive ratio. Note that if the bids $v_1 \geq v_2 \geq \cdots \geq v_N$ are known in an instance I, then profit-maximization is trivial: letting $\lambda = \text{argmax}_{i \geq 2}\, i \cdot v_i$, we sell the good at price v_λ, to get an optimal revenue $OPT(I) = \lambda \cdot v_\lambda$.[1] The *competitive ratio* of an incentive-compatible mechanism is defined to be the largest possible value, taken over all possible instances I, of $OPT(I)$ divided by the expected profit obtained by our mechanism on I. Note that this ratio is at least 1.

The prior-free variant of our problem has been first investigated in [6, 5]. Random sampling is one of the most natural methods that is used in prior-free settings when the objective is to maximize the auctioneer's revenue. The work of [6] develops a natural random-sampling-based approach for our problem, *Random Sampling Optimal Price* (RSOP). In RSOP, the bidders are partitioned into two groups uniformly at random and the optimal price of each set is offered to the other set. It has been shown that RSOP returns a profit very close to optimal for many classes of interesting inputs ([12], [1]). There has also been a fair amount of work analyzing the competitive ratio of RSOP. In [5], Goldberg et al. showed that the competitive ratio of RSOP is 7600, and conjectured that the competitive ratio should be 4; note that this value of 4 cannot be lowered further since RSOP attains a value of 4 when we have $N = 2$ and $v_1 = 2v_2$. Later, Feige et al. improved the analysis and showed that this ratio is at most 15 [3]. There are at least two reasons for trying to prove that RSOP's competitive ratio is 4. First, RSOP is very natural and giving a tight analysis appears to be of inherent interest. Second, RSOP is very easily implementable and hence easily adaptible to different settings (e.g., double auctions [2], online limited-supply auctions [8], combinatorial auctions [1], [5], and for the "money burning" problem [9]).

Summary of our results: To describe our results, we will need the notion of "winners" (w.r.t. the optimal single-price auction). In our definition of $OPT(I)$ where we set $\lambda = \text{argmax}_{i \geq 2}\, i \cdot v_i$, let λ be the **largest** index that satisfies this definition. Recall that in the "offline" case where we know all the v_i and compute λ as this maximizing index, we

[1]There is a subtlety here that requires $\lambda \geq 2$ in the definition of $OPT(I)$, an issue that we will discuss later.

sell at the single price v_λ, which is then bought by bidders $1, 2, \ldots, \lambda$ to give an optimal revenue $OPT(I) = \lambda \cdot v_\lambda$ to the auctioneer. Since the number of bidders who get the good in this case is λ, we refer to λ as the number of "winners" (w.r.t. the optimal single-price auction). Note that λ is determined uniquely by the values $v_1 \geq v_2 \geq \cdots \geq v_N$.

Many of our results are motivated by the following question: the instance seen above where $n = \lambda = 2$ and the competitive ratio of RSOP is 4, seems quite unique. In particular, when selling a digital good, one expects the typical number of buyers to be "large". Does RSOP do much better than known before, when λ is large? Our main results are four-fold as follows, and are obtained by an improved probabilistic analysis aided by a dynamic programming computation and correlation inequalities:

I. **Improved upper bounds:** We prove that the competitive ratio of RSOP is:

- less than 4.68, improving upon the upper-bound 15 of Feige et al. [3];
- less than 4 if the number of winners λ is at least 6;
- upper-bounded by a quantity that approaches 3.3 as $\lambda \to \infty$.

These results indicate that RSOP does much better than known in the practically-interesting case where λ is "large", and that perhaps the only case where the competitive ratio of 4 is attained is the case where $N = 2$ and $v_1 = 2v_2$.

II. **Lower bounds:** We prove that even if λ gets arbitrarily large, one can construct instances I with such λ, for which the competitive ratio is at least 2.65.

III. **Combinatorial approach:** We also present a combinatorial approach for the case where the bid values are either 1 or h and show that the competitive ratio of RSOP is at most 4 in this case.

2 Problem Definition

We consider auctioning digital goods to N bidders with bid values $v_1, v_2, \ldots, v_N$. Without loss of generality, we assume $v_1 \geq v_2 \geq \cdots \geq v_N$. The Random Sampling Optimal Price auction partitions the bids into two sets A and B such that each bid v_i independently goes to either of A or B with probability 1/2. We then compute the optimal price of each set (among the two sets A and B) and offer it to the other set: note that the optimal price of a sequence $G = \langle u_1 \geq u_2 \geq \cdots \geq u_k \rangle$ of bids in nondecreasing order, is u_{λ_G} where $\lambda_G = \text{argmax}_{i \geq 1}\, i u_i$. (Thus, we will use this definition once with $G = A$ when we compute the optimal price for A and offer that price to B, and will use this definition again with $G = B$ when we compute the optimal price for B and offer that price to A.) For our input instance $I = v_1, v_2, \ldots, v_N$ of bids, we define the optimal profit of I as $OPT(I) = \lambda v_\lambda$ where $\lambda = \text{argmax}_{i \geq 2}\, i v_i$. Note that we force $\lambda \geq 2$ here: without this, it can be shown that no incentive compatible mechanism can achieve a constant fraction of the optimal profit in the case where $v_1 \gg v_2$ [7]. (Note that λ_G above is allowed to be one; it is only the λ that we use in the definition of $OPT(I)$ that is required to be at least two, in order to disallow negative results [7].)

3 Assumptions

To simplify the proofs we make the following assumptions throughout the rest of this paper.

- WLOG, we assume we have an infinite number of bids $v_1, v_2, \cdots$ in which all the bids after v_N are zero so our analysis will be independent of N.
- WLOG, to simplify the analysis, we assume that $OPT(I) = 1$ since we can always scale all the bids by a constant factor without affecting the mechanism.
- For the sake of notation we use $E[\text{RSOP}]$ to denote the expected profit of RSOP on an input instance where the expectation is taken over random partitions of the bids. Note that by our previous assumption that $OPT = 1$ we have $E[\text{RSOP}] \leq 1$ and the competitive ratio of RSOP can be defined as $\max_I \frac{1}{E[\text{RSOP}]}$.
- WLOG, we assume that v_1 is always in set B since the mechanism is symmetric for both A and B and so we can relabel the sets.
- WLOG, we only consider the profit obtained from B by offering the optimal price of A and we assume the obtained profit from A when offered the optimal price of B is 0. The justification for this assumption is that we are computing the $E[RSOP]$ for the worst case input. Note that for any given input instance we can replace v_1 with a very large bid such that the optimal price of set B is v_1 in which case by offering price of v_1 to set A we don't obtain any profit.

4 The Basic Lower Bound

In this section, we give a basic lower bound that shows RSOP is indeed 4-competitive for a large class of input instances. In the next section, we improve this result using a more sophisticated lower bound, but based on the same idea. We start by stating the main theorem of this section:

Theorem 4.1. *For any input instance $I = \{v_1, v_2, \cdots\}$ where there are more than 10 bids above the optimal uniform price (i.e. $\lambda > 10$), the expected profit of RSOP is at least $\frac{1}{4}$ (i.e., $E[\text{RSOP}] \geq \frac{1}{4}$). The actual computed lower bound values can be found in Table 1.*

We prove the theorem throughout the rest of this section. The outline of the proof is as follows. First, we define a lower bounding function (LBF) which, for each partition of bids to two sets (A, B), returns a value which is less than or equal to the profit of RSOP.

Most importantly, our LBF only depends on λ and on how the bids are partitioned but is independent of the actual value of the bids $v_1, v_2, \cdots$. The expected value of the LBF is clearly a lower bound for $E[\text{RSOP}]$. After defining the LBF function, in subsection 4.1, we explain how we can compute the expected value of the LBF for any given λ. We then compute the LBF for all values of λ from 10 up to $\bar{\lambda} = 5000$ and show that the expected value of LBF is indeed greater than $\frac{1}{4}$ and so is $E[\text{RSOP}]$ for $10 \leq \lambda \leq \bar{\lambda}$. The computation of the lower bound involves a combination of probabilistic techniques and dynamic programming. Later, in subsection 4.2, we compute a lower bound on the expected value of the LBF assuming that $\lambda > \bar{\lambda} = 5000$ and show that it is indeed greater than $\frac{1}{4}$ and that completes the proof of Theorem 4.1.

Before we start with the proof, let us make the following observations which gives an intuition to our proof:

Observation 4.2. *For a given i, roughly, we expect about half of $v_1, \cdots, v_i$ to fall in set A and the other half to fall in set B. In other words, let $\mathbf{s}_i = \#\{j | j \leq i, v_j \in A\}$, we expect $\mathbf{s}_i \approx \frac{i}{2}$.*

Observation 4.3. *The optimal profit of set A is at least as much as the profit that we get if we offer v_λ to A. Let λ_A be the index of optimal price in A. The optimal profit of set A is at least $\mathbf{s}_\lambda v_\lambda$. Since we assumed $\lambda v_\lambda = OPT = 1$, essentially $v_\lambda = \frac{1}{\lambda}$ and therefore we can use $\frac{\mathbf{s}_\lambda}{\lambda}$ as a lower bound on the optimal profit of set A. Formally, assuming $\text{Prof}(A, v_{\lambda_A})$ denotes the profit that we get from a set A by offering the price v_{λ_A} to it:*

$$\text{Prof}(A, v_{\lambda_A}) \geq \frac{\mathbf{s}_\lambda}{\lambda} \tag{4.1}$$

Note that based on Observation 4.2 we expect this quantity to be about $\frac{1}{2}$.

Observation 4.4. *Define $\mathbf{z}_i = \frac{i - \mathbf{s}_i}{\mathbf{s}_i}$ which is the ratio of the number of bids from $v_1, \cdots, v_i$ that fall in B to the number of those that fall in A. It is easy to see that the ratio of profit of set B when offered v_{λ_A} to profit of set A when offered the same v_{λ_A} is the same as $\mathbf{z}_{\lambda_A}$. Formally:*

$$\frac{\text{Prof}(B, v_{\lambda_A})}{\text{Prof}(A, v_{\lambda_A})} = \mathbf{z}_{\lambda_A} \tag{4.2}$$

Notice that λ_A depends on the actual value of the bids and thus (4.2) is hard to work with. To work around that, we use $\mathbf{z} = \min_i \mathbf{z}_i$ as a lower bound for $\mathbf{z}_{\lambda_A}$. Therefore:

$$\frac{\text{Prof}(B, v_{\lambda_A})}{\text{Prof}(A, v_{\lambda_A})} \geq \mathbf{z} \tag{4.3}$$

The outline of the proof of our basic lower bound for $E[\text{RSOP}]$ is as follows. We combine Observation 4.3 and Observation 4.4 to get the following:

$$E[\text{RSOP}] \geq E[\text{Prof}(B, v_{\lambda_A})] \tag{4.4}$$

$$\geq E[\text{Prof}(A, v_{\lambda_A})\frac{\text{Prof}(B, v_{\lambda_A})}{\text{Prof}(A, v_{\lambda_A})}] \tag{4.5}$$

$$\geq E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}] \tag{4.6}$$

Note that (4.6) allows us to compute the lower bound regardless of the actual values of v_i because the right hand side of (4.6) is totally independent of the v_i values except for λ. Also note that for any given input instance I, λ depends only on I and not on how we partition the bids so in computing $E[\text{RSOP}]$, λ is a constant (for a fixed I) and not a random variable.

Ideally, we would like to separate $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ to $E[\frac{\mathbf{s}_\lambda}{\lambda}]E[\mathbf{z}]$, but since $\frac{\mathbf{s}_\lambda}{\lambda}$ and $\mathbf{z}$ are correlated we cannot do that. Nevertheless, the correlation decrease as λ increases which suggests that for sufficiently large λ we can separate the two terms. In subsection 4.1, we present a dynamic programming method for computing $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ for any fixed λ. We then use the dynamic program to compute the lower bound on $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ for values of $\lambda \leq \bar{\lambda} = 5000$. In subsection 4.2, we give a lower bound on $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ for all values of $\lambda > \bar{\lambda} = 5000$ by separating the $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ to $E[\frac{\mathbf{s}_\lambda}{\lambda}]E[\mathbf{z}]$ and subtracting the maximum possible difference caused by that.

4.1 When there are a few bids above the optimal uniform price

In this subsection we show the following:

- We show how we can compute a lower bound on $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ and therefore for $E[\text{RSOP}]$ for any fixed λ.
- We compute the above lower bound for all values of λ up to $\bar{\lambda} = 5000$ and verify that for $10 \leq \lambda \leq \bar{\lambda}$ it is indeed better than $\frac{1}{4}$. The computed lower bounds for various values of λ can be found in Table 1.

We can compute a lower bound for $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ and therefore for $E[\text{RSOP}]$ by defining a set of events and then breaking $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ over those events using the law of total expectation. As we showed before, $E[\text{RSOP}] \geq E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ so we only need to compute a lower bound on $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$. Since $\frac{\mathbf{s}_\lambda}{\lambda}$ and $\mathbf{z}$ are correlated random variables we cannot separate them in $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$. The idea is that when we condition $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ on any of these events we can derive lower bounds for both $\frac{\mathbf{s}_\lambda}{\lambda}$ and $\mathbf{z}$. We then use the above method to compute a lower bound on $E[\text{RSOP}]$ for all the values of $\lambda \leq \bar{\lambda} = 5000$ to show that for $10 \leq \lambda \leq \bar{\lambda}$ the lower is better than $\frac{1}{4}$. In the next subsection, we prove a lower bound of better than $\frac{1}{4}$ for all values of $\lambda > \bar{\lambda}$.

First we define the following notation:

$\mathcal{E}_R^T$ **:** If $T \subset \mathbb{N}$ is a subset of indices and R is an interval which is in $[0,\infty)$ and $\sup(R)$ is the supremum of R then $\mathcal{E}_R^T$ is the event in which for all indices $i \in T$, we have $\frac{\mathbf{s}_i}{i} \le \sup(R)$ and at least for one i in set T we have $\frac{\mathbf{s}_i}{i} \in R$. Formally, $\mathcal{E}_R^T = \{\forall i \in T : \frac{\mathbf{s}_i}{i} < \sup(R) \quad \wedge \quad \exists i \in T : \frac{\mathbf{s}_i}{i} \in R\}$.

For example, we might use $\mathcal{E}_{[0.4,0.5]}^{[4,10]}$ to denote the event in which for $4 \le i \le 10$ the $\frac{\mathbf{s}_i}{i}$ is at most 0.5 and there is some $4 \le j \le 10$ such that $\frac{\mathbf{s}_j}{j} \in [0.4, 0.5]$. As a shorthand we might sometimes use a single number instead of an interval to denote the interval from 0 up to and including that number. We may also omit the subset of indices altogether in which case we assume $[0,\infty)$. So we can derive the following alternate notations: $\mathcal{E}_\alpha^k$, $\mathcal{E}_\alpha$. We may also use one special notation $\mathcal{E}_\alpha^{k,j} = \{\forall i \le k : \frac{\mathbf{s}_i}{i} \le \alpha \wedge \mathbf{s}_k = j\}$.

$Pr[\mathcal{E}]$ **:** The probability of event $\mathcal{E}$ happening.

$\widehat{E}[X|\mathcal{E}]$ **:** The normalized conditional expected value of a random variable X which is:

$$\widehat{E}[X|\mathcal{E}] = E[X|\mathcal{E}]Pr[\mathcal{E}] \tag{4.7}$$

We first show the following:

Lemma 4.5. *For any sequence of* $\alpha_0, \cdots, \alpha_m$ *such that* $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$*, the following is a lower bound on* $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$*:*

$$E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}] \ge \sum_{i=1}^{m} (\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_{\alpha_i}] - \widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_{\alpha_{i-1}}])\frac{1-\alpha_i}{\alpha_i} \tag{4.8}$$

in which by definition $\mathcal{E}_{\alpha_i}$ *is the event in which for any index* j*, the fraction of the* $v_1, \cdots, v_j$ *that fall in set* A *is less than* α_i*.*

We actually prove a more general statement:

Lemma 4.6. *For any given positive random variable* $\mathbf{x}$ *and any sequence of* $\alpha_0, \cdots, \alpha_m$ *such that* $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$*, the following inequality always holds in which the random variable* $\mathbf{z}^1$ *is defined as* $\mathbf{z}^1 = \min(\mathbf{z}, 1)$*:*

$$E[\mathbf{x}\mathbf{z}] \ge E[\mathbf{x}\mathbf{z}^1] \ge \sum_{i=1}^{m} (\widehat{E}[\mathbf{x}|\mathcal{E}_{\alpha_i}] - \widehat{E}[\mathbf{x}|\mathcal{E}_{\alpha_{i-1}}])\frac{1-\alpha_i}{\alpha_i} \tag{4.9}$$

In which by definition $\mathcal{E}_{\alpha_i}$ *is the event in which for any index* j*, the fraction of the* $v_1, \cdots, v_j$ *that fall in set* A *is less than* α_i*.*

Proof. See section B. □

The intuition behind Lemma 4.6 is the following: We want to find lower bounds on $\mathbf{z}$ so we break the expected value over a set of small events. Under each event $\mathcal{E}_{\alpha_i}$ we have $\mathbf{z} \geq \frac{1-\alpha_i}{\alpha_i}$ based on the definition of $\mathcal{E}_{\alpha_i}$. Roughly, $\widehat{E}[\mathbf{x}|\mathcal{E}_{\alpha_i}] - \widehat{E}[\mathbf{x}|\mathcal{E}_{\alpha_{i-1}}]$ is the portion of the expected value for which the best lower bound for $\mathbf{z}$ that we can guarantee is $\frac{1-\alpha_i}{\alpha_i}$.

The choice of m and $\alpha_0, \cdots, \alpha_m$ in Lemma 4.6 greatly affects the value of the lower bound. Generally, increasing m improves the lower bound but at the cost of more computation. We will provide the values of α_i and m that we used to get our desired lower bound later.

We claim that the coefficient of each term $\widehat{E}[\mathbf{x}|\mathcal{E}_{\alpha_i}]$ on the right hand side of (4.6) is positive and therefore we can use a lower bound for each $\widehat{E}[\mathbf{x}|\mathcal{E}_{\alpha_i}]$ instead of its exact value and the inequality still holds. We prove our claim as follows. If we expand the sum on the right hand side of (4.9), each $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_{\alpha_i}]$ appears exactly twice except for $i = 0$ and $i = m$. Since $\alpha_0 = 0$ and $\alpha_m = 1$, the value of $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_{\alpha_0}]$ is 0 and also the coefficient of $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_{\alpha_m}]$ is 0. Except for those two, every other $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_{\alpha_i}]$ has a coefficient of $\frac{1-\alpha_i}{\alpha_i} - \frac{1-\alpha_{i+1}}{\alpha_{i+1}}$ which is positive and proves our claim. Therefore, we can relax the inequality by substituting each $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_{\alpha_i}]$ with its lower bound. Sofar, the problem has been reduced to computing a lower bound on $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_{\alpha_i}]$ which we explain next.

Lemma 4.7. *For any random variable* $\mathbf{x}$ *such that* $\mathbf{x} \in [0, 1]$ *and any* $\alpha \in [0, 1]$ *and any* $n \in \mathbb{N}$ *the following always holds:*

$$\widehat{E}[\mathbf{x}|\mathcal{E}_\alpha] \geq \widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^n] - Pr[\mathcal{E}_\alpha^n](1 - Pr[\mathcal{E}_\alpha^{(n,\infty)}]) \tag{4.10}$$

Proof. See section B. □

Intuitively, Lemma 4.7 is saying that if instead of computing $\widehat{E}[\mathbf{x}|\mathcal{E}_\alpha]$ we can approximate it by $\widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^n]$, the maximum that we may over-approximate is at most $Pr[\mathcal{E}_\alpha^n](1 - Pr[\mathcal{E}_\alpha^{(n,\infty)}])$ which is the probability of the event in which for any $j < n$, $\mathbf{s}_j < \alpha j$ and then there is some $j' > n$ such that $\mathbf{s}_{j'} \geq \alpha j'$. Note that since $\mathbf{x} \leq 1$, its normalized expected value conditioned on any event is less than the probability of that event. By choosing a large enough n we can make sure that the over approximation upper bound gets close enough to 0.

Again, in Lemma 4.7, increasing the n improves the lower bound, but the computation cost of $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^n]$ and $Pr[\mathcal{E}_\alpha^n]$ will increase.

To use Lemma 4.7 for $\mathbf{x} = \frac{\mathbf{s}_\lambda}{\lambda}$, effectively we need to be able to compute $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^n]$, $Pr[\mathcal{E}_\alpha^n]$ and $Pr[\mathcal{E}_\alpha^{(n,\infty)}]$. Next we show how to compute the first two exactly by using dynamic programming. Later in Lemma 4.9 we show how to get a lower bound on the third one.

Lemma 4.8. *The exact value of $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^n]$ and $Pr[\mathcal{E}_\alpha^n]$ can be computed using the following dynamic programming. Remember that $\mathcal{E}_\alpha^{k,j}$ is the event in which for all $r \leq k$, the fraction of $v_1, \cdots, v_r$ that fall in A is less than α and exactly j of $v_1, \cdots, v_k$ fall in A:*

$$Pr[\mathcal{E}_\alpha^{k,j}] = \begin{cases} \frac{1}{2} Pr[\mathcal{E}_\alpha^{k-1,j}] & \substack{j=0 \\ k>0} \\ \frac{1}{2} Pr[\mathcal{E}_\alpha^{k-1,j}] + \frac{1}{2} Pr[\mathcal{E}_\alpha^{k-1,j-1}] & 0 < j \leq \alpha k \\ 0 & j > \alpha k \\ 1 & j = k = 0 \end{cases} \tag{4.11}$$

$$\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k,j}] = \begin{cases} 0 & j = 0 \\ \frac{1}{2}\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k-1,j}] + \frac{1}{2}\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k-1,j-1}] & \substack{0<j\leq \alpha k \\ k>\lambda} \\ \frac{j}{\lambda} Pr[\mathcal{E}_\alpha^{k,j}] & \substack{0\leq j\leq \alpha k \\ k=\lambda} \end{cases} \tag{4.12}$$

$$Pr[\mathcal{E}_\alpha^k] = \sum_{j=0}^{k} Pr[\mathcal{E}_\alpha^{k,j}] \tag{4.13}$$

$$\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^k] = \sum_{j=0}^{k} \widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k,j}] \tag{4.14}$$

Proof. See section B. □

Intuitively, (4.11) means the event $\mathcal{E}_\alpha^{k,j}$ happens if either $\mathcal{E}_\alpha^{k-1,j}$ happens and v_j falls in set A (which happens with probability $\frac{1}{2}$) or $\mathcal{E}_\alpha^{k-1,j-1}$ happens and v_j falls in set B (again, with probability $\frac{1}{2}$). The intuition behind (4.12) is very similar to (4.11) when $k > \lambda$. When $k = \lambda$, under the event $\mathcal{E}_\alpha^{k,j}$ we know that exactly j of $v_1, \cdots, v_\lambda$ are in set A and so $\frac{\mathbf{s}_\lambda}{\lambda} = \frac{j}{k}$.

Computing $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^n]$ and $Pr[\mathcal{E}_\alpha^n]$ using the above recurrence relation and dynamic programming takes $O(n^2)$ time and $O(n)$ memory.

Finally, in order to complete our lower bounding method is $Pr[\mathcal{E}_\alpha^{(n,\infty)}]$. Next we show how we can find a lower bound for $Pr[\mathcal{E}_\alpha^{(n,\infty)}]$.

Lemma 4.9. *For any $\alpha \in [0.5, 1]$ and any $n, n' \in \mathbb{N}$ such that $n < n'$, the following two always hold:*

$$Pr[\mathcal{E}_\alpha^{(n,\infty)}] \geq (1 - \frac{{C_\alpha}^{n'+1}}{1 - C_\alpha}) \prod_{k=n+1}^{n'} (1 - {C_\alpha}^k) \tag{4.15}$$

in which :

$$C_\alpha = \frac{(\frac{1}{\alpha} - 1)^\alpha}{2(1-\alpha)} \tag{4.16}$$

Proof. See section B. □

(4.15) is based on a variant of Chernoff bound and gives a very good lower bound when n and n' are sufficiently large.

To get the desired lower bound for RSOP we set the parameters as the following. In using Lemma 4.6 we set $m = 100$, $\alpha_1 = 0.5$, $\alpha_m = 1.0$ and distributed the $\alpha_2, \cdots, \alpha_{m-1}$ evenly on $[0.5, 1.0]$ (that is $\alpha_i - \alpha_{i-1} = \frac{0.5}{m-1}$). We then used Lemma 4.7 to compute $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_{\alpha_i}]$ for each i together with Lemma 4.8 by setting $n = 5000$ and also used Lemma 4.9 to compute $Pr[\mathcal{E}_\alpha^{(n,\infty)}]$ by setting $n' = 100000$.

The results of our computation for various choices of λ is listed in Table 1. Notice that for $\lambda > 10$ we get a lower bound better than 0.25 and thus a competitive ratio better than 4.

4.2 When there are many bids above the optimal uniform price

In this subsection we show the following:

- We show how to compute a lower bound on $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ that holds for all values of $\lambda > \bar{\lambda}$.
- We compute the above lower bound for $\bar{\lambda} = 5000$ to get a lower bound of $\frac{1}{3.52}$, thus showing that for all $\lambda > \bar{\lambda}$, $E[\text{RSOP}] \geq E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}] > \frac{1}{3.52}$.

In the previous subsection, we showed how to compute a lower bound for $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ for any fixed value of λ and we used that to compute the $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ for all values of λ up to $\bar{\lambda}$.

The idea is that when λ is large (i.e., $\lambda > \bar{\lambda}$), the two random variables $\frac{\mathbf{s}_\lambda}{\lambda}$ and $\mathbf{z}$ are almost independent and so the expected value of their product is very close to the product of their expected values. Also for a large λ the value of $\frac{\mathbf{s}_\lambda}{\lambda}$ is very close to $\frac{1}{2}$ so $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ would be roughly $\frac{1}{2}E[\mathbf{z}]$.

Lemma 4.10. *For any $\alpha \in [0, 1]$ the following always holds:*

$$E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}] \geq \alpha(E[\mathbf{z}^1] - Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}]) \tag{4.17}$$

Proof. See section B. □

Intuitively, when λ is large, in (4.17) the $Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}]$ is very close to 0 even when $\alpha = \frac{1}{2} - \epsilon$ it roughly gives a lower bound of about $\frac{1}{2}E[\mathbf{z}^1]$. Next we show how to compute an upper bound on $Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}]$ to support our claim.

Lemma 4.11. *For any $\alpha \in [0, 0.5]$, the following always holds:*

$$Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}] \leq 1 - {C_{\alpha'}}^{\lambda-1} \tag{4.18}$$

in which :

$$C_{\alpha'} = \frac{(\frac{1}{\alpha'} - 1)^{\alpha'}}{2(1-\alpha')}, \alpha' = 1 - \alpha - \frac{1}{\lambda - 1} \tag{4.19}$$

Proof. See section B. □

The only thing that remains is to compute a good lower bound on $E[\mathbf{z}^1]$.

Theorem 4.12. $E[\mathbf{z}] \geq E[\mathbf{z}^1] \geq 0.61$. *Intuitively,* $\mathbf{z}$ *is a measure of the least ratio of the number of bids in B to the number of bids in A among any prefix of the bids. A larger* $\mathbf{z}$ *indicates a more balanced partition. This is an important statistic for any random sampling method in general (note that* $\mathbf{z}$ *only depends on how we partition the bids and not the value of the bids).*

Proof. We can apply the Lemma 4.6 by plugging $\mathbf{x} = 1$ to compute $E[\mathbf{z}^1] = E[\mathbf{x}\mathbf{z}^1]$ to get the following:

$$E[\mathbf{z}^1] \geq \sum_{i=1}^{m} (\widehat{E}[1|\mathcal{E}_{\alpha_i}] - \widehat{E}[1|\mathcal{E}_{\alpha_{i-1}}])\frac{1-\alpha_i}{\alpha_i} \tag{4.20}$$

$$E[\mathbf{z}^1] \geq \sum_{i=1}^{m} (Pr[\mathcal{E}_{\alpha_i}] - Pr[\mathcal{E}_{\alpha_{i-1}}])\frac{1-\alpha_i}{\alpha_i} \tag{4.21}$$

To get (4.21) from (4.20) we have used the definition of of $\widehat{E}[]$ from (4.7). Also, we have that $Pr[\mathcal{E}_\alpha] \geq Pr[\mathcal{E}_\alpha^n]Pr[\mathcal{E}_\alpha^{(n,\infty)}]$ by the FKG inequality [4]. We can apply the FKG inequality because the two events $\mathcal{E}_\alpha^n$ and $\mathcal{E}_\alpha^{(n,\infty)}$ are positively correlated on the distributive lattice formed by partially ordering the instances of the partitioning by a subset relation on set A therefore their probability of their intersection is greater than or equal to the product of their probabilities. Again, if we substitute each $Pr[\mathcal{E}_{\alpha_i}]$ with its lower bound the inequality still holds because of the following. The coefficient of each $Pr[\mathcal{E}_{\alpha_i}]$ term after rearranging the sum on the right hand side of (4.21) is positive except for $Pr[\mathcal{E}_{\alpha_0}]$ which is itself 0 because $\alpha_0 = 0$. By tuning the parameters as we will explain at the end of this section we get a lower bound of $E[\mathbf{z}] \geq E[\mathbf{z}^1] \geq 0.61$. It is worth mentioning that by using a similar method, we computed an upper bound of $E[\mathbf{z}] \leq 0.63$ which indicates that our analysis of $E[\mathbf{z}]$ is very tight. □

That completes our method for computing a lower bound on $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ which is independent of λ for sufficiently large λ.

To compute $E[\mathbf{z}^1]$ we used (4.21) which we derived from Lemma 4.6 by setting $\mathbf{x} = 1$, $m = 100$, $\alpha_1 = 0.5$, $\alpha_m = 1.0$ and distributing the $\alpha_2, \cdots, \alpha_{m-1}$ evenly on $[0.5, 1.0]$ (that is $\alpha_i - \alpha_{i-1} = \frac{0.5}{m-1}$). Together with that we also used Lemma 4.9 by setting $\mathbf{x} = \frac{\mathbf{s}_\lambda}{\lambda}$, $n = 60000$ and $n' = 100000$ and Lemma 4.8 by setting $n = 60000$ to compute $Pr[\mathcal{E}_{\alpha_i}]$ for each i.

To get our desired lower bound on $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ when $\lambda \geq \bar{\lambda} = 5000$, we used Lemma 4.10 to separate the $\mathbf{z}$ and $\frac{\mathbf{s}_\lambda}{\lambda}$ as in (4.17). Using $E[\mathbf{z}] \geq 0.61$ together with Lemma 4.11 and setting $\alpha = 0.52$ we get that for any $\lambda > 5000$, $Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}] \leq 0.0183$ and so $E[\text{RSOP}] \geq 0.284$ which is equivalent to a competitive ratio of 3.52 which is better than 4.

5 The Exhaustive Search Lower-Bound

In the previous section, we showed that for $\lambda > 10$, $E[\text{RSOP}] \geq \frac{1}{4}$. In this section, we show the following:

- We show how to compute an improved lower bound on $E[\text{RSOP}]$ for any fixed $2 \leq \lambda \leq 10$.
- We compute the above lower bound on $E[\text{RSOP}]$ for all $2 \leq \lambda \leq 10$ to get a lower bound of $\frac{1}{4}$ when $6 \leq \lambda \leq 10$ and a lower bound of $\frac{1}{4.68}$ when $2 \leq \lambda \leq 6$. The computed values of our lower bound for all values of $2 \leq \lambda \leq 10$ can be found in Table 2.

In the rest of this section we explain an *Exhaustive-Search* approach for improving the lower-bound of RSOP for the cases where λ is small (i.e., $\lambda \leq 10$). The basic lower bound of $E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}]$ in section 4 does not work well enough in these cases mainly because $\frac{\mathbf{s}_\lambda}{\lambda}$ and $\mathbf{z}$ are negatively correlated and their correlation is much stronger when λ is small. Also because v_1 is always in B and so $\mathbf{s}_1$ is always 0, the expected value of $\frac{\mathbf{s}_\lambda}{\lambda}$ decreases as λ decreases such that for $\lambda = 2$ we have $\frac{\mathbf{s}_\lambda}{\lambda} = \frac{1}{4}$ which is far from $\frac{1}{2}$. The idea is to try all possible values for the first few v_i but instead of using an exact value for each v_i we use an interval for each v_i and we try all the possible combination of these intervals to cover all the possible input instances. We then report the lowest $E[\text{RSOP}]$ of all the different combinations as the lower bound.

Theorem 5.1. *For any input instance $I = \{v_1, v_2, \cdots\}$ where there are between 6 to 10 bids above the optimal uniform price (i.e. $6 \leq \lambda \leq 10$), the expected profit of RSOP is at least $\frac{1}{4}$ (i.e., $E[\text{RSOP}] \geq \frac{1}{4}$). Also, if there are between 2 to 5 bids above the optimal price, the expected profit of RSOP is at least $\frac{1}{4.68}$. The actual computed lower bound values can be found in Table 2.*

Due to the complexity of the proofs and lack of space we only give an outline of our method [2].

First we define λ' as the index of the winning price after λ in the optimal single price auction (i.e., we are choosing the winning price from the bids whose index are greater than λ). Again we don't take λ' as a random variable. Instead we provide a lower bound for RSOP for any fixed λ and λ' and another lower bound for sufficiently large λ'. Note that λ' depends on the set of bids as a whole and does not depend on how the bids are partitioned by RSOP. Formally $\lambda' = \max\ \text{argmax}_{i>\lambda} iv_i$.

Algorithm 5.2. ***Exhaustive-Search***$(m, \lambda, \lambda', r, r')$
For some given $m \geq \lambda$ we consider the first m highest bids, that is $v_1, \cdots, v_m$ and also $v_{\lambda'}$. We then restrict each bid v_i where $i \in \mathbb{S} = \{1, \cdots, m, \lambda'\}$ to some interval $[l_i, h_i]$ as we explain later and find a lower-bound for the utility of RSOP assuming those restrictions. We try all the possible combination of these intervals for the first m bids and for $v_{\lambda'}$ so as to cover all possible cases (remember that $v_\lambda = \frac{1}{\lambda}$ since we assumed that $OPT = 1$). Then we take the lowest lower bound among all those combination and report it as the lower bound of $E[\text{RSOP}]$ for that specific choice of λ and λ'. We will also provide a way of computing a lower bound which is independent of the actual λ' when λ' is greater than a certain value. We then take the minimum of that for all choices of λ' and use it as a lower bound for $E[\text{RSOP}]$ for the specific choice of λ (remember that we are only interested in $\lambda \leq 10$ since for $\lambda > 10$ the basic lower bound of section 4 is already better than 0.25).

In order to try all the combination of intervals we do the following. Since $OPT = 1$, each bid v_i is always in the interval $[0, \frac{1}{i}]$. For some given parameter r, we divide this interval to r smaller intervals $[\frac{0}{r}\frac{1}{i}, \frac{1}{r}\frac{1}{i}], \cdots, [\frac{r-1}{r}\frac{1}{i}, \frac{r}{r}\frac{1}{i}]$. For each $i \in \mathbb{S}$, we set $[l_i, u_i]$ to one of the mentioned r intervals. We will do the same thing for $v_{\lambda'}$ except that we divide it to r' different intervals for some given r'. As a result we can have either $r'(m-2)^r$ or $r'(m-1)^r$ possible combinations depending on whether $\lambda' \leq m$ or $\lambda' > m$. Note that v_λ is always restricted to be exactly $\frac{1}{\lambda}$ because $OPT = 1$. Also note that some of these combinations might be partially or even entirely impossible because they should satisfy the constraint of $v_{i-1} \geq v_i$ and $\lambda' v_{\lambda'} > iv_i$ for all $i > \lambda$. So we discard or refine some combinations (for example by setting $u_i \leftarrow \min(u_i, u_{i-1})$).

Next we show how we compute the lower bound based on the range restrictions of Algorithm 5.2.

Algorithm 5.3. ·
Restricted-RSOP-Lowerbound$(m, \lambda, \lambda', r, r', \{(l_i, u_i)\})$
Here we use $E[\mathbf{u}'_A \mathbf{z}']$ as a lower bound for $E[\text{RSOP}]$ in which again $\mathbf{u}'_A$ is a random variable indicating the lower bound on the utility of set A and $\mathbf{z}'$ a random variable indicating the restricted least prefix ration of B to A *which is slightly different from $\mathbf{z}$. In $\mathbf{z}'$ we are considering the range restrictions that we explain next. To compute the lower-bound, we*

[2] The complete proof would take up about 2 or 3 times the proofs of the basic lower bound of section 4

enumerate all 2^{m-1} possible ways of partitioning $v_1, \cdots, v_m$ and refer to them with events $\mathcal{D}_1, \cdots, \mathcal{D}_{2^{m-1}}$. Then based on the law of total expectation we can compute a lower-bound by $E[\text{RSOP}] \geq E[\mathbf{u}'_A \mathbf{z}'] = \sum_{i=1}^{2^{m-1}} \widehat{E}[\mathbf{u}'_A \mathbf{z}' | \mathcal{D}_i]$. Basically, under each event $\mathcal{D}_i$, we fix the partitioning of the first m bids and then apply all the previous techniques that we discussed in section 4 to the tail of the bids that is $v_{m+1}, v_{m+2}, \cdots$ with some modification which we explain next. First, instead of using $\frac{\mathbf{s}_\lambda}{\lambda}$ as a lower bound for the utility of set A we use $\mathbf{u}'_A = \max_{i \in \mathbb{S}} \mathbf{s}_i l_i$ as a lower bound on the profit of set A. We also modify the (4.11), (4.12), (4.13), (4.14) to condition them on event $\mathcal{D}_i$. Also we replace the term $\frac{j}{\lambda} Pr[\mathcal{E}_\alpha^{k,j}]$ in (4.12) with $\mathbf{u}'_A Pr[\mathcal{E}_\alpha^{k,j}]$. The most important change in the computations from section 4 is that whenever the value of $\mathbf{z}$ is conditioned on an event $\mathcal{E}_\alpha^T$ (as defined in subsection 4.1) if $\alpha \lambda' u_{\lambda'} < \max_{i \in \{1, \cdots, m\}} \mathbf{s}_i l_i$ we can argue that because by definition of λ', $\lambda' v_{\lambda'} \geq i v_i$ for all $i > \lambda$, then the winning price in set A should be among $v_2, \cdots, v_m$ (because for all $j > m$ we have $\alpha j v_j < \max_{i \in \{1, \cdots, m\}} \mathbf{s}_i l_i$ and $\alpha j v_j$ is the maximum utility one can possibly get in set A by choosing v_j as the winning price under event $\mathcal{E}_\alpha^T$).

By choosing $m = 11$, $r = 3$, $r' = 100$ and the rest of the parameters as in section 4 we get a lower bound of 0.213845 for $\lambda = 2$ over all values of λ' which is equivalent to a competitive ratio of 4.68 which is also the upper bound of competitive ratio of RSOP over all λ. Table 2 shows the exhaustive search lower-bounds for $2 \leq \lambda \leq 10$. In our computations, we noticed that $\lambda' = \lambda + 1$ was the worst case among all choices of λ'.

6 An Upper Bound For The Performance of RSOP For Any λ

In previous works, it has been shown that $E[\text{RSOP}]$ is $\frac{1}{4}$ for some instances (e.g. [3], [5]). However in all those instances, $\lambda = 2$.In this section, we show that the lower bound for $E[\text{RSOP}]$ cannot be improved further than 3/8 for any value of λ.

Theorem 6.1. *For any λ there exists an input instance I for which $E[\text{RSOP}] \leq \frac{3}{8}$.*

Before proving the theorem we define the following.

Definition 6.2 (Equal Revenue Instance)**.** *We refer to the input instance with N bidders in which $v_i = \frac{1}{i}$ as* Equal Revenue *with N bidders. Notice that choosing any of the v_i as the winning price yields a profit of* 1*.*

Observation 6.3. *For an equal revenue input instance,* RSOP *always offers the worst price to the other set. In other words, the optimal price of set A is the worst price that we could offer to set B and vice versa.*

The previous observation suggests that an equal revenue instance might actually be the worst case input instance for RSOP however that is not quite true at least for small values of

N. Furthermore, analyzing the performance of RSOP on equal revenue instances for general N is not easy. Therefore, we define a modified version of RSOP, call it RSOP$'$ which is very similar to RSOP and yields about the same profit. We then analyze the performance of RSOP$'$ on equal revenue instances and use that to upper bound the performance of RSOP. In RSOP$'$, as in RSOP, we partition the bidders into two sets at random and then offer the best single price of each set to the other set. The only difference is in the case that one of the sets is empty. In this case, in RSOP$'$, the offered price from the empty side to the other set will be $\frac{1}{N}$ instead of 0.

Lemma 6.4. $E[\text{RSOP}']$ *on an equal revenue instance with* N *bidders is decreasing function of* N.

Proof. The proof is by induction. Assume $\forall i, j : i < j \leq N - 1$, $E[\text{RSOP}']$ for an equal revenue instance with i elements is larger than $E[\text{RSOP}']$ for an equal revenue instance with j elements. Now, we need to show $\forall i, j : i < j \leq N$ this property holds as well. It is enough to show that $E[\text{RSOP}']$ for an equal revenue instance with N bidders is less than $E[\text{RSOP}']$ for an equal revenue instance with $N - 1$ bidders. Consider the random partitions of the instance with N bidders. As before, WLOG assume that $v_1 \in B$. Now, categorize partitions to two groups:

1. Partitions in which $v_N \in B$. These partitions can be built by considering all the partitions for $N - 1$ bidders and adding v_N to B in each partition. Call the original partitions for $N - 1$ bidders, A' and B'.

2. Partitions in which $v_N \in A$. Again we can build all these partitions by considering the partitions for $N - 1$ bidders and adding v_N to A. Call the original partitions without v_N, $A'$and B'.

Each of the above cases can happen with probability $\frac{1}{2}$. We compare the expected profit of each case with $E[\text{RSOP}']$ for equal revenue instance with $N - 1$ bidders. In fact, we will show that the expected profit of partitions belonging to case 1, is exactly the same as $E[\text{RSOP}']$ for equal revenue instance with $N - 1$ bidders. Also, we show that the expected revenue of cases of partitions belonging to case 2, is at most equal to $E[\text{RSOP}']$ of the equal revenue instance with $N - 1$ bidders.

There is a one-to-one correspondence between the partitions belonging to case 1 and partitions of the equal revenue instance with $N - 1$ bidders. We can see that the profit of each partition is exactly the same as the profit of its corresponding partition with $N - 1$ bidders. Consider the partition A and B and its corresponding partition A' and B'. If $A' \neq \emptyset$ (and correspondingly $A \neq \emptyset$) , the offered price to B' is the same as the offered price to B by A and it is always larger than $\frac{1}{N}$. It means that the profit obtained from the elements in B' that belongs to B is also the same and we don't obtain any profit from v_N since it is smaller than the the offered price. If $A = A' = \emptyset$, the offered price to the other set, for the equal revenue case with $N - 1$ bidders, is $\frac{1}{N-1}$ and the obtained profit from

B' is $(N-1).\frac{1}{N-1} = 1$. For the case with N bidders, the offered price to the other set is $\frac{1}{N}$ however we have also N bidders in B so the total profit obtained from B is $N.\frac{1}{N}$ which gives the same profit.

We have also a one-to-one correspondence between partitions in case 2 and the partitions of the equal revenue instance with $N-1$ bidders. If $A' \neq \emptyset$, then the obtained profit from B is at most equal to the obtained profit from B'. There are two possible cases here. Either the offered price to B and B' are the same, in which case the obtained profit from both sets are the same as well. In the other case, adding $\frac{1}{N}$ to A' (to obtain A) has changed the best price for A. In the latter case, the offered price by A to B should be $\frac{1}{N}$. Also note that, in the partition of an equal revenue instance, the best price for set A is the worst offered price for set B, which means that we are only reducing the profit obtained from B when we change the selected price in A to $\frac{1}{N}$ from the selected price for A'. Also if $A' = \emptyset$, the obtained profit in the equal revenue instance with $N-1$ bidders is 1. However in the corresponding instance, containing $v_N = \frac{1}{N}$ in A, the offered price to B is $\frac{1}{N}$ and we have only $N-1$ elements in B in this case. So the total obtained profit is $\frac{n-1}{n} < 1$. So the expected profit of all the partitions belonging to the second category is less than $E[\text{RSOP}']$ for equal revenue instances with $N-1$ bidders. Putting both cases together, we can conclude that the total expected profit is only decreased when the number of bidders is increased. □

Also it can be shown that for equal revenue instances $E[\text{RSOP}] = E[\text{RSOP}'] - \frac{1}{2^{N-1}}$. The profit obtained by both methods are always the same except for the case that $A = \emptyset$. This event happens with probability $\frac{1}{2^{N-1}}$ and the obtained profit is 1. (The obtained profit in RSOP' is 1 and the profit of RSOP is 0 in this case.)

It can be shown that for $N \leq 6$, for the equal revenue instances, $E[\text{RSOP}'] \leq \frac{1}{2.65}$. Using Lemma 6.4, we can conclude that $E[\text{RSOP}] \leq 1/2.65$ for the equal revenue instance for any N. Finally, for any given winner index j, we show how to find an instance for which we have $\lambda = j$ and also $E[\text{RSOP}]$ for that instance is equal to $E[\text{RSOP}]$ for the equal revenue instance with j bidders. For a given j, we define its corresponding instance as follows (and refer to it as *perturbed equal revenue*): Consider the equal revenue instance with j bidders. Construct the perturbed equal revenue instance by changing only v_j to $\frac{1}{j} + \epsilon$ instead of $\frac{1}{j}$. (The value of the rest of the bids are similar to the equal revenue instance.)

It is easy to see that the benefit obtained by RSOP from the equal revenue instance with j bidders is converging to the benefit obtained from perturbed equal revenue instance when $\epsilon \to 0$ which completes the proof of the theorem.

7 The interesting case of h and 1

In this section, we describe a combinatorial approach which shows that $E[\text{RSOP}]$ is at least $\frac{1}{4}$ of the optimal profit for all the instances where bidders have only one of the two possible valuations, 1 and h. We call an instance, an *equal profit* instance, if selecting either 1 or h as the uniform price returns the same profit. In the rest of this section, for a given instance of input, we denote the number of h bids by N_h and the number of 1 bids by N_1. Also the profit obtained from a set S by offering price p, is represented by $\text{Prof}(S,p)$. We first show that:

Lemma 7.1. *For an* equal profit *instance,* $E[\text{RSOP}] \geq \frac{1}{4}OPT + \frac{h}{4}$.

Proof. The proof is based on induction on N_h. We first show that for the base case of $N_h = 1$, we have $E[\text{RSOP}] \geq \frac{h}{2} = \frac{h}{4} + \frac{h}{4}$.

Because this is an equal profit instance, when $N_h = 1$, it should be that $N_1 = h - 1$. Now consider the partitioning of the bidders into two groups A and B. WLOG, assume that $v_1 \in B$ which means the optimal price of set B which is offered to set A is h and $\text{Prof}(A,h) = 0$. On the other hand, since the valuations of all bidders in set A are 1 the optimal price of set A which is offered to set B is always 1. To compute $\text{Prof}(B,1)$ it is enough to compute $E[|B|]$. Since bidders are partitioned uniformly at random, we can conclude that $E[|B|] = \frac{h-1}{2} + 1 \geq h/2$ which completes the proof for $N_h = 1$.

To prove the induction step for N_h, we assume that for all values of $N_h \leq k$, $E[\text{RSOP}] \geq OPT/4 + h/4$. Now consider an *equal profit* instance I with $N_h = k + 1$. We can write all the possible ways of partitioning the bids in this new instance as the cartesian product of all the possible ways to partition the bids into two *equal profit* instances, one with $N_h = 1$ and the other with $N_h = k$. In other words, call the instance with $N_h = 1$, I_1 and the instance with $N_h = k$, I_2. Construct all the possible partitions of bidders into two groups (A and B) for the equal revenue instance with $N_h = k + 1$. We can see that any possible partition in I can be constructed by combining exactly one partition of I_1 and one partition of I_2(one-to-one mapping). For a given partition A and B of an instance I, call the corresponding partitions from I_1, A_1 and B_1 and the corresponding partition from I_2, A_2 and B_2, so $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. In the rest of this section, we use the simple observation that in any *equal profit* instance I, if the optimal price for set A is 1, then the optimal price for B has to be h and vice versa. In the rest of the proof, we use the notion of *price pair* to present the optimal prices of each side of a partition. (e.g. *price pair* $(1,h)$ means that the optimal price for set A is 1 and the optimal price for set B is h.)

We have 4 possible *price pair*'s for a combination of two partitions taken from I_1 and I_2. However, 2 of these 4 cases can be reduced to the other 2 by renaming A and B, so we only consider the first 2 cases:

- The *price pair* of both (A_1,B_1) and (A_2,B_2) are $(1,h)$. Call the combination of these partitions (A,B). We can see that the *price pair* for (A,B) would be $(1,h)$ as well. So the extracted profit from each side, is exactly equal to the sum of the profits obtained from (A_1,B_1) and (A_2,B_2).

- The *price pair* of (A_1,B_1) is $(1,h)$ but the *price pair* for (A_2,B_2) is $(h,1)$. Since, we are considering an *equal profit* instance, we know that *price pair* for (A,B) should be either $(1,h)$ or $(h,1)$ as well. WLOG assume the *price pair* of (A,B) is $(1,h)$. We can see that, the profit extracted from bidders in I_1 in (A,B) partition is exactly the same as the extracted profit in (A_1,B_1) instance since the offered prices to each side are the same. Now, for the bidders belonging to I_2, the extracted profit in (A,B) is at least as high as the extracted profit in (A_2,B_2) partition. The reason is that, in I_2 the offered price to B_2 is h however the best price for B_2 is 1.(Since the *price pair* for (A_2,B_2) was $(h,1)$) So by offering price 1 to B_2, the extracted profit from bidders on the B_2 side is only increased. Also by using the same argument, offering price h to elements in A_2 is only increasing the extracted profit from them. So we can conclude that, in this case, the extracted profit in (A,B), is at least as high as the sum of the extracted profit from (A_1,B_1) and (A_2,B_2).

We can rewrite $E[\text{RSOP}]$ as the sum of the expected profit obtained from bidders in I_1 and the expected profit obtained from bidders in I_2. Since every partition of bidders in I_1 appears in the same number of partitions of I and by using the above argument, we can conclude that the expected profit obtained from bidders in I_1, is at least as much $E[\text{RSOP}]$ for the *equal profit* instance I_1. Using similar argument for I_2, we can see that $E[\text{RSOP}]$ for the *equal profit* instance I, is at least as much as the sum of the $E[\text{RSOP}]$ for *equal profit* instances I_1 and I_2. Now, by using induction, we have $E_I[\text{RSOP}] \geq E_{I_1}[\text{RSOP}] + E_{I_2}[\text{RSOP}] \geq \frac{h-1}{4} + \frac{h}{4} + \frac{h}{4} + \frac{h}{4} > OPT/4 + h/4$. □

Next, we show how to use lemma 7.1 to prove that:

Lemma 7.2. *The competitive ratio of RSOP for any instance with only two kind of valuations is at most* 4*.*

In lemma 7.1, we proved that the competitive ratio of RSOP is at most 4 for *equal profit* instances. Here, we show that in fact, we can generalize the result to any instance consisting of 1 and h bids. We face two scenarios here:

1. Either $n_1 \geq n_h(h-1)$ which means that our instance is a combination of an *equal profit* instance and a extra set of bidders with value 1.

2. Or $n_1 < n_h(h-1)$. That means, we have an instance which is a combination of an *equal profit* instance and some extra (at least 1) bidder(s) with valuation h and less than $h-1$ extra bidder(s) with valuation 1.

We give the proof for each scenario separately. Again, we denote the original instance by I, the *equal profit* part of I, by I_1 and the rest by I_2. Also, for a partition (A,B) of I, we denote the part of A belonging to I_1 by A_1 and the part belonging to I_2 by A_2.(Similarly for B with B_1 and B_2)

In scenario 1, either the *price pair* of (A_1,B_1) is $(1,h)$ or it is $(h,1)$. In the first case , we can conclude that (A,B) is either $(1,1)$ or $(1,h)$ which means that the offered price from A to B is always 1. So the obtained profit from set B is equal to the sum of the profits of B_1 and B_2 in I_1 and I_2 instances. If the offered price from B to A is 1, with the similar argument given in Lemma 7.1, we can see that the profit obtained from B is at least as much as the total profit of B_1 and B_2 in I_1 and I_2 instances. However if the offered price is h, we get the same profit from the elements that were coming from A_1 and we loose all the profit that was obtained from A_2. However the amount of loss can be upper bounded by the number of 1's in I_2 which is at most h. The conclusion is that the obtained profit from (A, B) for instance I, is at least as much as the the profit that we could obtain from (A_1, B_1) for instance I_1. By using lemma 7.1 we know that the obtained profit by RSOP from (A_1, B_1) is at least $N_h/4 \cdot h + h/4$. Also the optimal profit that can be obtained from (A, B) is at most $N_h \cdot h + h$. That means that we already obtained $1/4$ of the optimal profit by RSOP.

In scenario 2, the best price for I is h. We call the number of h bids in I_1 by N_h^1 and the number of h bids belonging to I_2 by N_h^2. The optimal profit can be defined by $N_h \cdot h$. Here we are in one of the following cases:

- Either the *price pair* of (A_1, B_1) is $(1, h)$ and for (A_2, B_2) is $(1, h)$ (which means that the number of h bids in A_2 is 0). In this case, the *price pair* of (A, B) is $(1, h)$. This means that the benefit that we obtain from bidders in I_1 in (A, B) is the same as the profit we obtained in (A_1, B_1). However, we are loosing the profit from h bids in B_2.

- Or $(A_1, B_1) = (1, h)$ and $(A_2, B_2) = (h, h)$. There are two possibilities here: Either *price pair* of (A, B) is (h, h) or it is $(1, h)$. If the *price pair* is (h, h), the profit obtained from A_1 in (A, B) is the same as the obtained profit in I_1 with partition (A_1, B_1). However the benefit obtained from B_1 can only increase since we offer price h. Also, in this case, we extract all the profit from h bids in I_2.

  On the other hand, if $(A, B) = (1, h)$ we again extract the same profit from the instance I_1 and also we obtain all the profit from the h bids in A_2.

So in both cases, the profit extracted in I from the bidders belonging to I_1, is at least as much as the amount extracted in RSOP from those bidders in I_1 instance. Also we always extract all the profit from bidders with h value that are belonging to A_2. Assuming that we are partitioning the bidders always uniformly at random, we can conclude that the expected number of h bids belonging to A_2 is $N_h^2/2$. So the total profit obtained by RSOP from I is at least the profit obtained by RSOP from I_1 plus $h \cdot N_h^2/2$. In other words the profit that will be obtained in this scenario is at least $h \cdot N_h^1/4 + h/4 + N_h^2/2 > h \cdot N_h/4$. Thus, $E[\text{RSOP}] \geq OPT/4$ for all instances with only two different bid values.

8 Conclusion

In this work, we further improved upon the bounds on the competitiveness of RSOP through a mix of probabilistic techniques and computer-aided analysis methods. More specifically, we proved that the competitive ratio of RSOP is: (i) less than 4.68, (ii) less than 4 if the number of winners λ is at least 6; and (iii) upper-bounded by a quantity that approaches 3.3 as $\lambda \to \infty$, and (iv) quite robust as λ gets large. These indicate that RSOP does much better than known in the practically-interesting case where λ is "large", and that perhaps the only case where the competitive ratio of 4 is attained is the case where $n = 2$ and $v_1 = 2v_2$. We also proved that even if λ gets arbitrarily large, one can construct instances I with such λ, for which the competitive ratio is at least 2.65. Furthermore, we presented a combinatorial approach for the case that the bid values are chosen from $\{1, h\}$ and showed that the competitive ratio of RSOP is at most 4 in this case.

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A Results

λ	$E[RSOP]$	Competitive-Ratio
2	0.125148	7.99
3	0.166930	5.99
4	0.192439	5.20
5	0.209222	4.78
6	0.221407	4.52
7	0.230605	4.34
8	0.237862	4.20
9	0.243764	4.10
10	0.248647	4.02
11	0.252774	3.96
15	0.264398	3.78
20	0.273005	3.66
30	0.282297	3.54
50	0.290384	3.44
100	0.296993	.37
200	0.300549	.33
300	0.301784	.31
500	0.302792	.30
1000	0.303560	.29
1500	0.303818	.29
2000	0.303949	.29

Table 1: The result of using the basic lower-bound by choosing $n = 5000$

B Proofs

Proposition B.1. *For any random variable X and any event $\mathcal{E} = \cup_i \mathcal{E}_i$ such that for all $i \neq j : \mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ we may write $\widehat{E}[X|\mathcal{E}] = \sum_i \widehat{E}[X|\mathcal{E}_i]$.*

λ	$E[RSOP]$	Competitive-Ratio
2	0.2138	4.68
3	0.2178	4.59
4	0.238	4.20
5	0.243	4.11
6	0.2503	3.99
7	0.2545	3.93
8	0.2602	3.84
9	0.2627	3.81
10	0.2669	3.75

Table 2: The result of using the exhaustive-search lower-bound by choosing $m = 11$, $r = 3$, $r' = 100$

Proof. It follows immediately from the law of total expectation. □

Proof of Lemma 4.6. First we define a set of m events $\mathcal{E}_{[\alpha_0,\alpha_1)}, \cdots, \mathcal{E}_{[\alpha_{m-1},\alpha_m)}$. The first inequality in (4.9) is trivial. To prove the second one, we first break the $E[\mathbf{x}\mathbf{z}^1]$ to small pieces each one conditioned on one of those events. Notice that for $i \neq j$ we have $\mathcal{E}_{[\alpha_i,\alpha_{i+1})} \cap \mathcal{E}_{[\alpha_j,\alpha_{j+1})} = \emptyset$ so these events are disjoints which means based on Proposition B.1, we can write $E[\mathbf{x}\mathbf{z}^1]$ as the following:

$$E[\mathbf{x}\mathbf{z}^1] \geq \sum_{i=1}^{m} \widehat{E}[\mathbf{x}\mathbf{z}^1 | \mathcal{E}_{[\alpha_{i-1},\alpha_i)}] \tag{B.1}$$

$$E[\mathbf{x}\mathbf{z}^1] \geq \sum_{i=1}^{m} \widehat{E}[\mathbf{x}\frac{1-\alpha_i}{\alpha_i} | \mathcal{E}_{[\alpha_{i-1},\alpha_i)}] \tag{B.2}$$

$$E[\mathbf{x}\mathbf{z}^1] \geq \sum_{i=1}^{m} \widehat{E}[\mathbf{x} | \mathcal{E}_{[\alpha_{i-1},\alpha_i)}]\frac{1-\alpha_i}{\alpha_i} \tag{B.3}$$

Note that $\widehat{E}[\mathbf{x}|\mathcal{E}]$ denotes $E[\mathbf{x}|\mathcal{E}]Pr[\mathcal{E}]$ which is the normalized expected value. According to the definition of the event $\mathcal{E}_{[\alpha_{i-1},\alpha_i)}$, for each such event, we can use $\frac{1-\alpha_i}{\alpha_i}$ as a lower bound for $\mathbf{z}^1$ to get (B.2). Since $\frac{1-\alpha_i}{\alpha_i}$ is a constant we can take it out of the expected value to get (B.3).

Now consider the event $\mathcal{E}_{[0,\alpha_i)}$. We can decompose that to two disjoint events $\mathcal{E}_{[0,\alpha_{i-1})}$ and $\mathcal{E}_{[\alpha_{i-1},\alpha_i)}$ so we can write:

$$\widehat{E}[\mathbf{x}|\mathcal{E}_{[0,\alpha_i)}] = \widehat{E}[\mathbf{x}|\mathcal{E}_{[0,\alpha_{i-1})}] + \widehat{E}[\mathbf{x}|\mathcal{E}_{[\alpha_{i-1},\alpha_i)}] \tag{B.4}$$

We can then combine (B.4) and (B.3) to get the following which is the claim of the lemma:

$$E[\mathbf{x}\mathbf{z}^1] \geq \sum_{i=1}^{m} (\widehat{E}[\mathbf{x}|\mathcal{E}_{[0,\alpha_i)}] - \widehat{E}[\mathbf{x}|\mathcal{E}_{[0,\alpha_{i-1})}]) \frac{1-\alpha_i}{\alpha_i} \tag{B.5}$$

□

Proof of Lemma 4.7. We can decompose the event $\mathcal{E}_\alpha^n$ to two disjoint events $\mathcal{E}_\alpha^{[0,n]} \cap \mathcal{E}_\alpha^{(n,\infty)}$ and $\mathcal{E}_\alpha^{[0,n]} \cap \overline{\mathcal{E}}_\alpha^{(n,\infty)}$ so we can write:

$$\widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^n] = \widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^{[0,n]} \cap \mathcal{E}_\alpha^{(n,\infty)}] + \widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^{[0,n]} \cap \overline{\mathcal{E}}_\alpha^{(n,\infty)}] \tag{B.6}$$

$$\widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^n] = \widehat{E}[\mathbf{x}|\mathcal{E}_\alpha] + \widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^{[0,n]} \cap \overline{\mathcal{E}}_\alpha^{(n,\infty)}] \tag{B.7}$$

$$\widehat{E}[\mathbf{x}|\mathcal{E}_\alpha] = \widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^n] - \widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^{[0,n]} \cap \overline{\mathcal{E}}_\alpha^{(n,\infty)}] \tag{B.8}$$

In order to derive (B.6), we have used Proposition B.1. Also we have substituted $\mathcal{E}_\alpha^{[0,n]} \cap \mathcal{E}_\alpha^{(n,\infty)}$ with the equivalent single event $\mathcal{E}_\alpha$ to get (B.7). Rearranging the terms, we get (B.8). We can further relax the inequality by substituting an upper bound for $\widehat{E}[\mathbf{x}|\mathcal{E}_{\alpha_i}^{[0,n]} \cap \overline{\mathcal{E}}_{\alpha_i}^{(n,\infty)}]$ which we give next:

$$\widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^{[0,n]} \cap \overline{\mathcal{E}}_\alpha^{(n,\infty)}] \leq Pr[\mathcal{E}_\alpha^{[0,n]} \cap \overline{\mathcal{E}}_\alpha^{(n,\infty)}] \tag{B.9}$$

$$\widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^{[0,n]} \cap \overline{\mathcal{E}}_\alpha^{(n,\infty)}] \leq Pr[\mathcal{E}_\alpha^{[0,n]}] Pr[\overline{\mathcal{E}}_\alpha^{(n,\infty)}] \tag{B.10}$$

$$\widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^{[0,n]} \cap \overline{\mathcal{E}}_\alpha^{(n,\infty)}] \leq Pr[\mathcal{E}_\alpha^{[0,n]}](1 - Pr[\mathcal{E}_\alpha^{(n,\infty)}]) \tag{B.11}$$

(B.9) can be derived from the definition of $\widehat{E}[]$ in (4.7) and the fact that $\mathbf{x}$ is at most 1. Since the two events $\mathcal{E}_{\alpha_i}^{[0,n]}$ and $\overline{\mathcal{E}}_{\alpha_i}^{(n,\infty)}$ are negatively correlated (to prove that we can define a distributive lattice on the different instances of the partitioning such that for two partition instances (A, B) and (A', B') we define $(A, B) < (A', B')$ if and only if $A \subset A'$, then $\mathcal{E}_{\alpha_i}^{[0,n]}$ is a decreasing function on the lattice and $\overline{\mathcal{E}}_{\alpha_i}^{(n,\infty)}$ is an increasing function on the lattice), so by FKG inequality we can argue that the probability of the intersection of those two events is less than or equal to the product of their individual probabilities and so we get (B.10). By combining (B.10) and (B.8) we get the following which is the claim of the lemma:

$$\widehat{E}[\mathbf{x}|\mathcal{E}_\alpha] \geq \widehat{E}[\mathbf{x}|\mathcal{E}_\alpha^n] - Pr[\mathcal{E}_\alpha^{[0,n]}](1 - Pr[\mathcal{E}_\alpha^{(n,\infty)}]) \tag{B.12}$$

□

Proof of Lemma 4.8. The proof of (4.11) is trivial. Let $\mathcal{A}_i$ denote the event that v_i falls in set A. The event $\mathcal{E}_\alpha^{k,j}$ can be decomposed to two disjoint events $\mathcal{E}_\alpha^{k-1,j} \cap \overline{\mathcal{A}}_k$ and $\mathcal{E}_\alpha^{k-1,j-1} \cap \mathcal{A}_k$ and therefore its probability is the sum of the probabilities of those two. Each of those two events is also the intersection of two independent events and can be written as the product of the probabilities of those events. Also $Pr[\mathcal{A}_k] = \frac{1}{2}$. Therefore we can conclude the (4.11).

To prove (4.12) we consider the case of $k = \lambda$ and the case of $k > \lambda$ separately. When $k = \lambda$, from the definition of $\mathcal{E}_\alpha^{k,j}$ we can immediately conclude that $\mathbf{s}_\lambda = j$ and so $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k,j}] = \frac{j}{\lambda}$ and by definition of $\widehat{E}[]$ in (4.7) we get $\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k,j}] = \frac{j}{\lambda} Pr[\mathcal{E}_\alpha^{k,j}]$. For the case of $k > \lambda$, again we break the event $\mathcal{E}_\alpha^{k,j}$ to two disjoint events $\mathcal{E}_\alpha^{k-1,j} \cap \overline{\mathcal{A}}_k$ and $\mathcal{E}_\alpha^{k-1,j-1} \cap \mathcal{A}_k$ and then compute them separately and add them together. Next we give the proof for the first event (The proof for the second one is the same):

$$\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k-1,j} \cap \overline{\mathcal{A}}_k] = E[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k-1,j} \cap \overline{\mathcal{A}}_k] Pr[\mathcal{E}_\alpha^{k-1,j} \cap \overline{\mathcal{A}}_k] \tag{B.13}$$

$$\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k-1,j} \cap \overline{\mathcal{A}}_k] = E[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k-1,j}] Pr[\mathcal{E}_\alpha^{k-1,j}] Pr[\overline{\mathcal{A}}_k] \tag{B.14}$$

$$\widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k-1,j} \cap \overline{\mathcal{A}}_k] = \widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k-1,j}] \frac{1}{2} \tag{B.15}$$

To derive (B.14) we have used the fact that $E[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k-1,j} \cap \overline{\mathcal{A}}_k] = E[\frac{\mathbf{s}_\lambda}{\lambda}|\mathcal{E}_\alpha^{k-1,j}]$ since $\mathcal{E}_\alpha^{k-1,j}$ only depend on $\mathcal{A}_1, \cdots, \mathcal{A}_{k-1}$ and independent of $\mathcal{A}_k$ and $\mathbf{s}_\lambda$ does not change by the event $\overline{\mathcal{A}}_k$.

The proof of (4.13) and (4.14) follow immediately from Proposition B.1 since the $\mathcal{E}_\alpha^{k,j}$ events are disjoint for different values of j. □

The following theorem is a standard Chernoff-Hoeffding bound [10]:

Theorem B.2 (Chernoff-Hoeffding). *For (i.i.d) random variables $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ where $p = E[\mathbf{x}_i]$ and $\mathbf{x}_i \in \{0, 1\}$ and for some $\varepsilon > 0$ we have*

$$Pr\left[\frac{1}{n}\sum \mathbf{x}_i \geq p + \varepsilon\right] \leq \left(\left(\frac{p}{p+\varepsilon}\right)^{p+\varepsilon}\left(\frac{1-p}{1-p-\varepsilon}\right)^{1-p-\varepsilon}\right)^n \tag{B.16}$$

Proof of Lemma 4.9. Let $\mathbf{a}_i$ be an indicator random variable which is 1 when v_i falls in set A and 0 otherwise. We use Theorem B.2 by setting $\mathbf{x}_i = \mathbf{a}_i$ and $p = 0.5$ and $\varepsilon = \alpha - 0.5$ to get an upper bound on $Pr[\overline{\mathcal{E}}_\alpha^{[n,n]}]$ and thus a lower bound on $1 - Pr[\mathcal{E}_\alpha^{[n,n]}]$. Note that $\mathbf{a}_1$ is always 0 and consequently $\mathbf{x}_1 = 0$, however that only decreases the probability on the left hand side of (B.16) so it still holds. After simplifying we get the following:

$$Pr[\mathcal{E}_\alpha^{[n,n]}] \geq 1 - {C_\alpha}^n \quad \text{in which :} \quad C_\alpha = \frac{(\frac{1}{\alpha} - 1)^\alpha}{2(1-\alpha)} \tag{B.17}$$

For any $n' \in (n, \infty)$ we can write $\mathcal{E}_\alpha^{(n,\infty)} = \mathcal{E}_\alpha^{(n',\infty)} \cap (\bigcap_{k=n+1}^{n'} \mathcal{E}_\alpha^{[k,k]})$. Note that all the $\mathcal{E}_\alpha^{[k,k]}$ events for different values of k and $\mathcal{E}_\alpha^{(n',\infty)}$ are positively correlated. So we can use the FKG inequality to get (B.18):

$$Pr[\mathcal{E}_\alpha^{(n,\infty)}] \geq Pr[\mathcal{E}_\alpha^{(n',\infty)}] \prod_{k=n+1}^{n'} Pr[\mathcal{E}_\alpha^{[k,k]}] \tag{B.18}$$

$$Pr[\mathcal{E}_\alpha^{(n,\infty)}] \geq (1 - \sum_{k=n'+1}^{\infty} Pr[\overline{\mathcal{E}}_\alpha^{[k,k]}]) \prod_{k=n+1}^{n'} Pr[\mathcal{E}_\alpha^{[k,k]}] \tag{B.19}$$

$$Pr[\mathcal{E}_\alpha^{(n,\infty)}] \geq (1 - \sum_{k=n'+1}^{\infty} {C_\alpha}^k) \prod_{k=n+1}^{n'} (1 - {C_\alpha}^k) \tag{B.20}$$

$$Pr[\mathcal{E}_\alpha^{(n,\infty)}] \geq (1 - \frac{{C_\alpha}^{n'+1}}{1 - C_\alpha}) \prod_{k=n+1}^{n'} (1 - {C_\alpha}^k) \tag{B.21}$$

Computing (B.21) for any arbitrary $n' \in (n, \infty)$ will give us a lower bound. Choosing a larger n' gives a better bound.

□

Proof of Lemma 4.10. Note that we always have $\mathbf{z} \geq \mathbf{z}^1$. So we can write:

$$E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}] \geq E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}^1] \tag{B.22}$$

$$E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}] \geq \widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}^1|\overline{\mathcal{E}}_\alpha^{[\lambda,\lambda]}] + \widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}^1|\mathcal{E}_\alpha^{[\lambda,\lambda]}] \tag{B.23}$$

$$E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}] \geq \widehat{E}[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}^1|\overline{\mathcal{E}}_\alpha^{[\lambda,\lambda]}] \tag{B.24}$$

$$E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}] \geq \alpha\widehat{E}[\mathbf{z}^1|\overline{\mathcal{E}}_\alpha^{[\lambda,\lambda]}] \tag{B.25}$$

$$E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}] \geq \alpha(E[\mathbf{z}^1] - \widehat{E}[\mathbf{z}^1|\mathcal{E}_\alpha^{[\lambda,\lambda]}]) \tag{B.26}$$

$$E[\frac{\mathbf{s}_\lambda}{\lambda}\mathbf{z}] \geq \alpha(E[\mathbf{z}^1] - Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}]) \tag{B.27}$$

(B.23) and (B.26) use Proposition B.1. Also, based on the definition of event $\overline{\mathcal{E}}_\alpha^{[\lambda,\lambda]}$, we have $\frac{\mathbf{s}_\lambda}{\lambda} > \alpha$ and so we can replace $\frac{\mathbf{s}_\lambda}{\lambda}$ with α to get (B.25). (B.27) can be derived from (B.26) by substituting $\mathbf{z}^1$ with 1 and applying the definition of $\widehat{E}[]$ from (4.7). □

Proof of Lemma 4.11. Let $\mathbf{a}_i$ be an indicator random variable which is 1 when v_i falls in set A and 0 otherwise. We can rewrite $Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}]$ as the following:

$$Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}] = Pr[\frac{\mathbf{s}_\lambda}{\lambda} < \alpha] \tag{B.28}$$

$$Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}] = Pr[\frac{\lambda - \mathbf{s}_\lambda}{\lambda} > 1 - \alpha] \tag{B.29}$$

$$Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}] = Pr[\frac{\sum_{j=1}^{\lambda} \overline{\mathbf{a}}_j}{\lambda} > 1 - \alpha] \tag{B.30}$$

$$Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}] = Pr[\frac{1 + \sum_{j=2}^{\lambda} \overline{\mathbf{a}}_j}{\lambda} > 1 - \alpha] \tag{B.31}$$

$$Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}] = Pr[\frac{\sum_{j=2}^{\lambda} \overline{\mathbf{a}}_j}{\lambda} > 1 - \alpha - \frac{1}{\lambda}] \tag{B.32}$$

$$Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}] \leq Pr[\frac{\sum_{j=2}^{\lambda} \overline{\mathbf{a}}_j}{\lambda - 1} > 1 - \alpha - \frac{1}{\lambda}] \tag{B.33}$$

Now we can apply Theorem B.2 to (B.33) by setting $\mathbf{x}_j = \overline{\mathbf{a}}_{j-1}$ (note that $\overline{\mathbf{a}_1}$ is always 1), $n = \lambda - 1$, $p = 0.5$ and $\epsilon = 0.5 - \alpha - \frac{1}{\lambda}$ and after simplifying we get the following upper bound:

$$Pr[\mathcal{E}_\alpha^{[\lambda,\lambda]}] \leq 1 - {C_{\alpha'}}^{\lambda-1} \text{in which :} \quad C_{\alpha'} = \frac{(\frac{1}{\alpha'} - 1)^{\alpha'}}{2(1-\alpha')}, \alpha' = 1 - \alpha - \frac{1}{\lambda - 1} \tag{B.34}$$

□