CMSC 474, Introduction to Game Theory

Coalition Game Theory

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Coalitional Games with Transferable Utility

- Given a set of agents, a coalitional game defines how well each group (or coalition) of agents can do for itself—its payoff
  - Not concerned with
    - how the agents make individual choices within a coalition,
    - how they coordinate, or
    - any other such detail
- Transferable utility assumption: the payoffs to a coalition may be freely redistributed among its members
  - Satisfied whenever there is a universal currency that is used for exchange in the system
  - Implies that each coalition can be assigned a single value as its payoff
Coalitional Games with Transferable Utility

- A coalitional game with transferable utility is a pair $G = (N, v)$, where
  - $N = \{1, 2, \ldots, n\}$ is a finite set of players
  - $(v: 2^N \rightarrow \mathbb{R})$ associates with each coalition $S \subseteq N$ a real-valued payoff $v(S)$, that the coalition members can distribute among themselves
- $v$ is the characteristic function
  - We assume $v(\emptyset) = 0$
- A coalition’s payoff is also called its worth
- Coalitional game theory is normally used to answer two questions:
  1. Which coalition will form?
  2. How should that coalition divide its payoff among its members?
- The answer to (1) is often “the grand coalition” (all of the agents)
  - But this answer can depend on making the right choice about (2)
Example: A Voting Game

- Consider a parliament that contains 100 representatives from four political parties:
  - $A$ (45 reps.), $B$ (25 reps.), $C$ (15 reps.), $D$ (15 reps.)
- They’re going to vote on whether to pass a $100 million spending bill (and how much of it should be controlled by each party)
- Need a majority ($\geq 51$ votes) to pass legislation
  - If the bill doesn’t pass, then every party gets 0
- More generally, a voting game would include
  - a set of agents $N$
  - a set of winning coalitions $W \subseteq 2^N$
    - In the example, all coalitions that have enough votes to pass the bill
      - $v(S) = 1$ for each coalition $S \in W$
    - Or equivalently, we could use $v(S) = $100 million
      - $v(S) = 0$ for each coalition $S \notin W$
Superadditive Games

- A coalitional game $G = (N, v)$ is **superadditive** if the union of two disjoint coalitions is worth at least the sum of its members’ worths
  
  - for all $S, T \subseteq N$, if $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$

- The voting-game example is superadditive
  
  - If $S \cap T = \emptyset$, $v(S) = 0$, and $v(T) = 0$, then $v(S \cup T) \geq 0$
  
  - If $S \cap T = \emptyset$ and $v(S) = 1$, then $v(T) = 0$ and $v(S \cup T) = 1$
  
  - Hence $v(S \cup T) \geq v(S) + v(T)$

- If $G$ is superadditive, the grand coalition always has the highest possible payoff
  
  - For any $S \neq N$, $v(N) \geq v(S) + v(N-S) \geq v(S)$

- $G = (N, v)$ is **additive** (or **inessential**) if
  
  - For $S, T \subseteq N$ and $S \cap T = \emptyset$, then $v(S \cup T) = v(S) + v(T)$
Interference

- The book says that for superadditive games, coalitions can always work together without interfering with one another
  - In the spending-bill example, I think this ignores the question of how much of the bill should be controlled by each party
  - So what does “interfering” mean?

- $G = (N,v)$ is additive (or inessential) if there is no interference (either positive or negative) among disjoint coalitions
  - if $S, T \subseteq N$ and $S \cap T = \emptyset$, then $v(S \cup T) = v(S) + v(T)$
Constant-Sum Games

- $G$ is **constant-sum** if the worth of the grand coalition equals the sum of the worths of any two coalitions that partition $N$
  - $v(S) + v(N - S) = v(N)$, for every $S \subseteq N$

- Every additive game is constant-sum
  - additive $\Rightarrow v(S) + v(N - S) = v(S \cup (N - S)) = v(N)$

- But not every constant-sum game is additive
  - Example is a good exercise
**Convex Games**

- **G** is **convex (supermodular)** if for all $S, T \subseteq N$,
  - $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$

- It can be shown the above definition is equivalent to for all $i$ in $N$ and for all $S \subseteq T \subseteq N-\{i\}$,
  - $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$
  - Prove it as an exercise

- Recall the definition of a superadditive game:
  - for all $S, T \subseteq N$, if $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$

- It follows immediately that every super-additive game is a convex game
Simple Coalitional Games

- A game $G = (N, v)$ is **simple** for every coalition $S$,
  - either $v(S) = 1$ (i.e., $S$ wins) or $v(S) = 0$ (i.e., $S$ loses)
    - Used to model voting situations (e.g., the example earlier)

- Often add a requirement that if $S$ wins, all supersets of $S$ would also win:
  - if $v(S) = 1$, then for all $T \supseteq S$, $v(T) = 1$

- This doesn’t quite imply superadditivity
  - Consider a voting game $G$ in which 50% of the votes is sufficient to pass a bill
  - Two coalitions $S$ and $T$, each is exactly 50% $N$
  - $v(S) = 1$ and $v(T) = 1$
  - But $v(S \cup T) \neq 2$
Proper-Simple Games

- $G$ is a **proper simple game** if it is both simple and constant-sum
  - If $S$ is a winning coalition, then $N - S$ is a losing coalition
    - $v(S) + v(N - S) = 1$, so if $v(S) = 1$ then $v(N - S) = 0$

- Relations among the classes of games:
  - \{Additive games\} $\subseteq$ \{Super-additive games\} $\subseteq$ \{Convex games\}
  - \{Additive games\} $\subseteq$ \{Constant-sum game\}
  - \{Proper-simple games\} $\subseteq$ \{Constant-sum games\}
  - \{Proper-simple games\} $\subseteq$ \{Simple game\}
Analyzing Coalitional Games

- Main question in coalitional game theory
  - How to divide the payoff to the grand coalition?

- Why focus on the grand coalition?
  - Many widely studied games are super-additive
    - Expect the grand coalition to form because it has the highest payoff
  - Agents may be required to join
    - E.g., public projects often legally bound to include all participants

- Given a coalitional game $G = (N, v)$, where $N = \{1, \ldots, n\}$
  - We’ll want to look at the agents’ shares in the grand coalition’s payoff
    - The book writes this as $(\Psi)$ $\psi(N, v) = x = (x_1, \ldots, x_n)$, where $\psi_i(N, v) = x_i$ is the agent’s payoff
  - We won’t use the $\psi$ notation much
    - Can be useful for talking about several different coalitional games at once, but we usually won’t be doing that
Terminology

- **Feasible payoff set**
  
  \[ \text{Feasible payoff set} = \{ \text{all payoff profiles that don’t distribute more than the worth of the grand coalition} \} = \{(x_1, \ldots, x_n) \mid x_1 + x_2 + \ldots + x_n \leq v(N) \} \]

- **Pre-imputation set**
  
  \[ \text{Pre-imputation set} = \{ \text{feasible payoff profiles that are efficient, i.e., distribute the entire worth of the grand coalition} \} = \{(x_1, \ldots, x_n) \mid x_1 + x_2 + \ldots + x_n = v(N) \} \]

- **Imputation set**
  
  \[ \text{Imputation set} = \{ \text{payoffs in P in which each agent gets at least what he/she would get by going alone (i.e., forming a singleton coalition)} \} = \{(x_1, \ldots, x_n) \in P : \forall i \in N, x_i \geq v(\{i\}) \} \]
Fairness, Symmetry

- What is a **fair** division of the payoffs?
  - Three axioms describing fairness
    - *Symmetry, dummy player*, and *additivity* axioms

- Definition: agents *i* and *j* are **interchangeable** if they always contribute the same amount to every coalition of the other agents
  - i.e., for every *S* that contains neither *i* nor *j*, \( v(S \cup \{i\}) = v(S \cup \{j\}) \)

- **Symmetry axiom**: in a fair division of the payoffs, interchangeable agents should receive the same payments, i.e.,
  - if *i* and *j* are interchangeable and \((x_1, \ldots, x_n)\) is the payoff profile, then \(x_i = x_j\)
Dummy Players

- Agent $i$ is a **dummy player** if $i$’s contributes to any coalition is exactly the amount $i$ can achieve alone
  - i.e., for all $S$ s.t. $i \notin S$, $v(S \cup \{i\}) = v(S) + v(\{i\})$

- **Dummy player axiom**: in a fair distribution of payoffs, dummy players should receive payment equal to the amount they achieve on their own
  - i.e., if $i$ is a dummy player and $(x_1, \ldots, x_n)$ is the payoff profile, then $x_i = v(\{i\})$
Additivity

- Let $G_1 = (N, v_1)$ and $G_2 = (N, v_2)$ be two coalitional games with the same agents.

- Consider the combined game $G = (N, v_1 + v_2)$, where
  \[(v_1 + v_2)(S) = v_1(S) + v_2(S)\]

- **Additivity axiom**: in a fair distribution of payoffs for $G$, the agents should get the sum of what they would get in the two separate games.
  \[\psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2)\]