CMSC 651 Advanced Algorithms	Lecture 6
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1 F-heap

1.1 Properties

F-heaps have the following properties:

- maintain the minimum key in the heap all the time,
- relax the condition that we have at most one B_k tree for any k,i.e., we can have any number of B_k trees co-existing.
- trees are not true binomial trees.

For example,

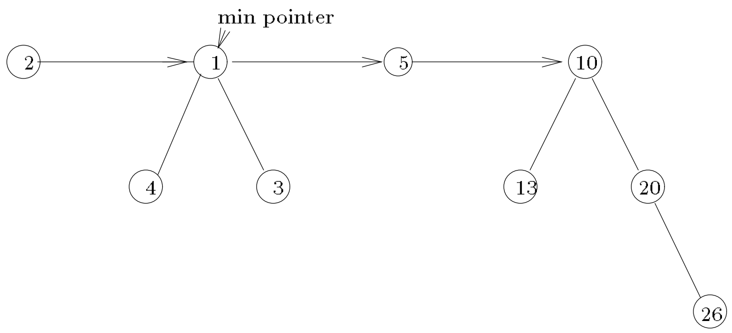


Figure 1: Example of F-heap

Property: size of a tree, rooted at x, is in exponential in the degree of x.

In Binomial trees we have the property that $size(x) \geq 2^k$, here we will not have as strong a property, but we will have the following:

$$size(x) \geq \phi^k$$

where k is degree of x and $\phi = \frac{1+\sqrt{5}}{2}$.

Note that since $\phi > 1$ this is sufficient to guarantee a $O(\log n)$ bound on the depth and the number of children of a node.

1.2 Decrease-Key operation

Mark strategy:

- when a node is cut off from its parent, tick one mark to its parent,
- when a node gets 2 marks, cut the edge to its parent and tick its parent as well,
- when a node becomes a root, erase all marks attached to that node,
- every time a node is cut, give \$1 to that node as a new root and \$2 to its parent.

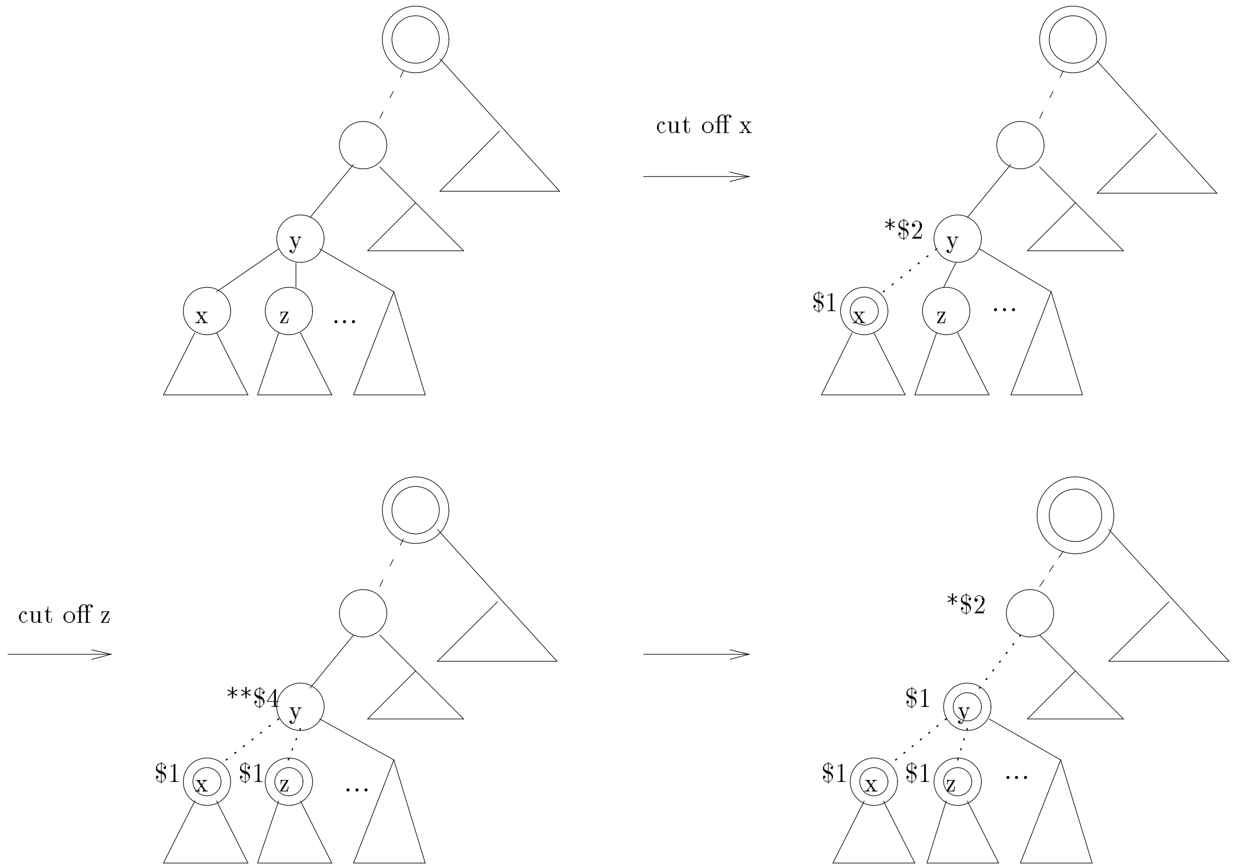


Figure 2: Marking Strategy

The cost of Decrease-Key operation = cost of cutting link + \$3. We can see that no extra dollars are needed when the parent is cut off recursively since when the cut off is done, that parent must have \$4 in hand (\$2 from each cut child and there must be 2 children have been cut), then we use those \$4 dollars to pay for cutting link cost (\$1), giving to its parent (\$2), and for itself as a new root (\$1).

Example: How does Decrease-Key work (see CLR also)?

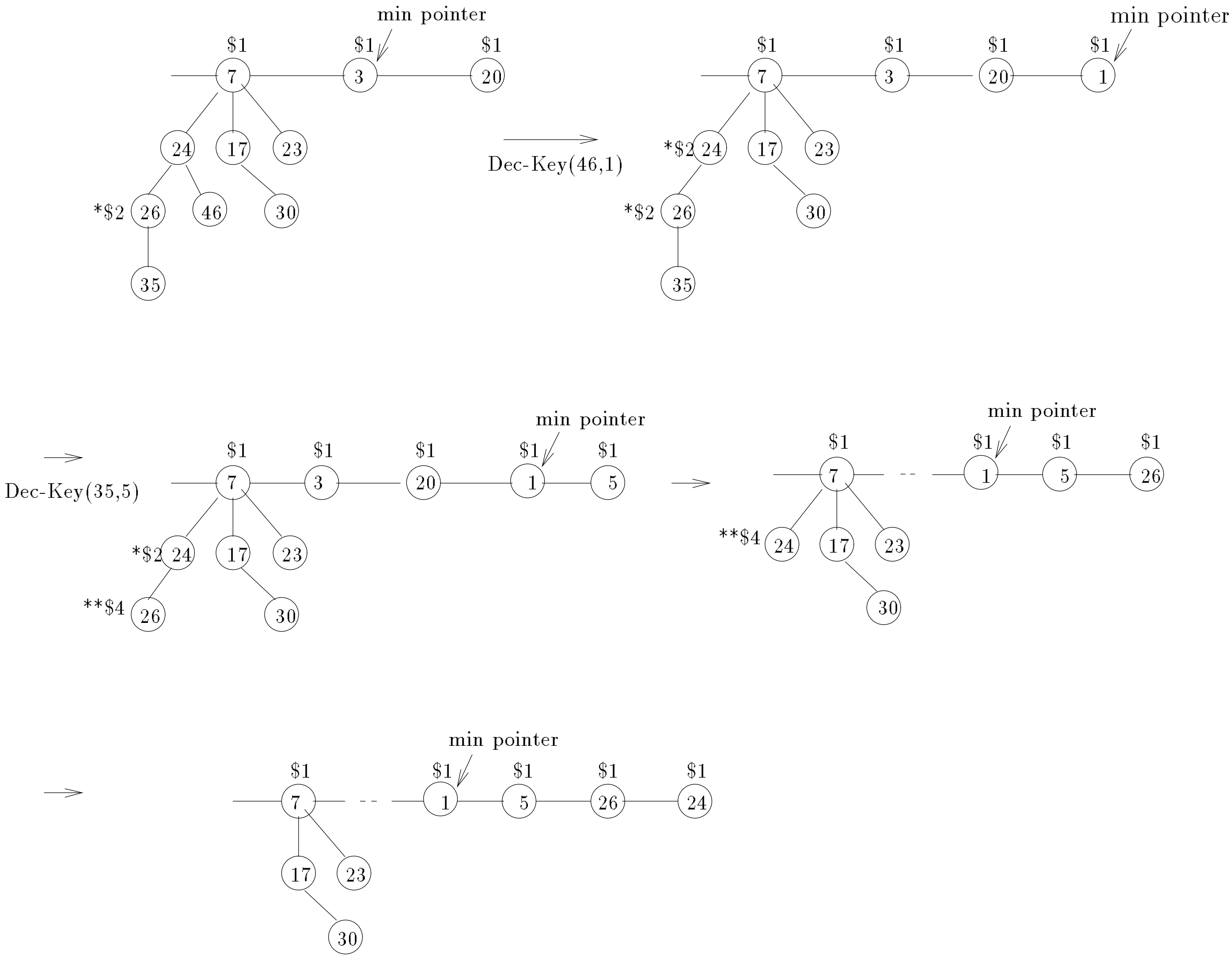


Figure 3: Example

The property we need when two trees are merged, is the degree of the roots of both trees should be the same.

Let y_1 be the oldest child and y_k be the youngest child. Consider y_i, at the point y_i was made a child of the root x, x had degree at least $i-1$ and so does y_i. Since y_i has lost at most 1 child, now y_i has at least degree $i-2$.

We now study some properties about Fibonacci numbers, and see how they relate to the sizes of the trees with the decrease-key operation. Define

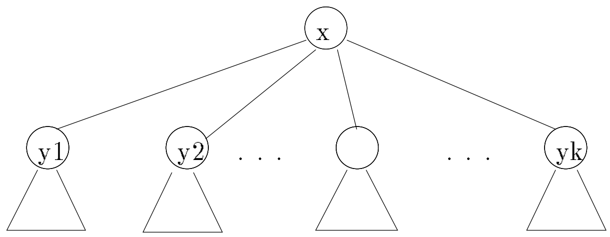


Figure 4: Example

$$F_k = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

These numbers are called Fibonacci numbers.

Property 1.1 $F_{k+2} = 1 + \sum_{i=1}^{k} F_i$

Proof:
[By induction]

$$\begin{aligned} k = 1: \qquad F_3 &= 1 + F1 \\ &= 1 + 1 \\ &= F_1 + F_2 \quad (\text{where } F_2 = F_1 + F_0 = 1 + 0) \end{aligned}$$

Assume $F_{k+2} = 1 + \sum_{i=1}^{k} F_i$ for all $k \leq n$

$$\begin{aligned} k = n + 1: \qquad F_{n+3} &= 1 + \sum_{i=1}^{n+1} F_i \\ &= 1 + \sum_{i=1}^{n} F_i + F_{n+1} \\ &= F_{n+2} + F_{n+1} \end{aligned}$$

□

Property 1.2 $F_{k+2} \geq \phi^k$

Proof:
[By induction]

$$\begin{aligned} k = 0: \qquad F_2 &= 1 \\ &\geq \phi^0 \\ k = 1: \qquad F_3 &= 2 \\ &\geq \phi = \frac{1+\sqrt{5}}{2} = 1.618.. \end{aligned}$$

Assume $F_{k+2} \geq \phi^k$ for all $k < n$

$$
\begin{aligned}
k = n: \qquad F_{n+2} &= F_{n+1} + F_n \\
&\geq \phi^{n-1} + \phi^{n-2} \\
&\geq \frac{1+\phi}{\phi^2} \cdot \phi^n \\
&\geq \phi^n
\end{aligned}
$$

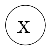
□

Theorem 1.1 *x is a root of any subtree. $size(x) \geq F_{k+2} \geq \phi^k$ where k is a degree of x*

Proof:

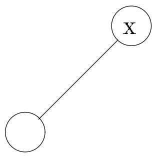
[By induction]

$k = 0$:



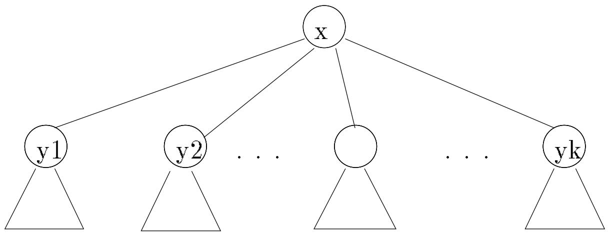
$size(x) = 1 \geq 1 \geq 1$

$k = 1$:



$size(x) = 2 \geq 2 \geq \frac{1+\sqrt{5}}{2}$

Assume $size(x) \geq F_{k+2} \geq \phi^k$ for any $k \leq n$



$k = n + 1$:

$$
\begin{aligned}
size(x) &= 1 + size(y_1) + ... + size(y_i) + ... + size(y_k) \\
&\geq 1 + 1 + 1 + F_3 + ... + F_k \ \ (from \ \ assumption) \\
&\geq 1 + F_1 + F_2 + ... + F_k \\
&\geq F_{k+2} \ \ (from \ \ property1) \\
&\geq \phi^k \ \ (from \ \ property2) \\
\log_\phi size(x) &\geq k
\end{aligned}
$$

So the number of children of node x is bounded by $\log_\phi size(x)$. □